Ozgur Kisisel

Toric Varieties II

(Work over an algebraically closed field.)

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Def: An irred. variety over k is a toric variety X if

(a) X is normal (local rings are integrally closed)

(b) $\exists$ embedding $i : (k^*)^n \hookrightarrow X$ (as Zariski-open subset) ($\dim X = n$)

(c) $\exists$ action of $(k^*)^n$ on X restricting to std. action on embedded torus

FACT: Integrally closed $\Rightarrow$ singularities of X are at least codim 2.

Remark: • Any toric variety is rational

• Ex $(k^*)^n$, k^n, $\mathbb{P}^n(k)$ are toric varieties
$k^n = \mathbb{A}^n(k)$

• Toric varieties are closed under products

Let's try to classify affine toric varieties

Recall: Affine irred. varieties are Spec (ring of regular functions on var.)

reduced f.g. k-alg.

In fact there is an equivalence of categories here.

$$\begin{array}{ccc} X = \mathrm{Spec}(A) & & A \\ \psi \downarrow & \rightsquigarrow & \uparrow \psi^* \\ Y = \mathrm{Spec}(B) & & B \end{array}$$

$\psi^*(f) = f \circ \psi$

$$\left(\begin{array}{c} X \xrightarrow{\psi} Y \\ \psi^*(f) \searrow \quad \swarrow f \\ k \end{array} \right)$$

$$\begin{array}{ccc} \mathrm{Spec}(A) & & A \\ \Phi^* \downarrow & \leftsquigarrow & \uparrow \Phi \\ \mathrm{Spec}(B) & & B \end{array}$$

(pull back maximal ideals)

(2)

X affine, toric

so $X = \mathrm{Spec}(A)$

$i : (k^*)^n \hookrightarrow X$

$i^* : A \longrightarrow k[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}]$

Notation: $k[\mathbb{Z}^n]$

Prop: i^* is monomorphism. !

Proof:

$i^*(f) = 0 \implies f(i(x)) = 0 \quad \forall x \in (k^*)^n$

$\implies f = 0$ on Zariski open set

X irred $\implies f = 0$ on dense set

$\implies f = 0$ on X ∎

$\to A$ is a k-subalgebra of $k[\mathbb{Z}^n]$

Prop: $i^*(A) \subset k[\mathbb{Z}^n]$ has monomial generators. !

Proof:

Given $f \in i^*(A)$ wts all monomial summands of f in $i^*(A)$

Write $f = z_1^{a_1} f_{a_1}(z_2, \ldots, z_n) + \cdots + z_1^{a_k} f_{a_k}(z_2, \ldots, z_n)$

$f(\bar{z})$ regular $\implies f(\lambda^{-1}\bar{z})$ regular $\forall \lambda \in (k^*)^n$

take $\lambda = (\lambda_1, 1, \ldots, 1)$

$f(\lambda^{-1}\bar{z}) = \lambda^{-a_1} z^{a_1} f_{a_1}(z_2, \ldots, z_n) + \cdots + \lambda^{-a_k} z^{a_k} f_{a_k}$

$f(\lambda^{-2}\bar{z}) = \cdots$

$f(\lambda^{-3}\bar{z}) = \cdots$

$\implies$ each row of $\begin{bmatrix} \lambda^{-a_1} & \cdots & \lambda^{-a_k} \\ \vdots & & \vdots \\ \lambda^{-ka_1} & \cdots & \lambda^{-ka_k} \end{bmatrix} \begin{bmatrix} z_1^{a_1} f_1 \\ \vdots \\ z_k^{a_k} f_k \end{bmatrix}$ in $i^*(A)$

Van der Monde matrix $\implies$ invertible.

$\implies$ each $z_i^{a_i} f_i$ ~~are~~ is in $i^*(A)$. ∎

Induct.

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Prop: $A = \mathcal{O}(X)$ is generated by finitely many monomials

Proof

Hilbert's Basis Thm. ∎

Def: $B \subset k[\mathbb{Z}^n]$ a k-alg generated by monomials.

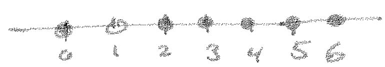
Let $S_B = \{(a_1, \dots, a_n) \mid z_1^{a_1} \cdots z_n^{a_n} \in B\}$

(S_B = subsemi-group of $\mathbb{Z}^n$)

Def: Let $S \subset \mathbb{Z}^n$ be a semi-group. S is saturated if

$m \cdot s \in S \Rightarrow s \in S$ (for $m \in \mathbb{Z}_{>0}$)

Ex $S = \langle 2, 3 \rangle$



$k[t^2, t^3] = k[x,y]/(x^3 - y^2)$

singularity