1 Salih Azgin — Non Extremal Henselian Valued Fields

Old conj: (If char $k=0$) K Henselian $\Longleftrightarrow$ K extremal.

FALSE!!!

Thm: If $\Gamma = \mathbb{Q}$ & $k=\mathbb{R}$, K Henselian $\Longrightarrow$ K extremal.

- **Cor:** Γ divisible & k (real, closed field) large, K Henselian $\Longrightarrow$ K extremal.

Def: A field k is "large" if
- k is existentially closed in $k((t))$

Ex: Let $F(x,y) = x^2 - (xy-1)^2$ over $\mathbb{R}((t))$

Note that $\{v(F(x,y))\}$ is not ∞, but has v unbounded

$F(a,b) \neq 0$ $\quad v(F(t^n, t^{-n})) = v(t^{2n}) = 2n$.

Ex: Let $\Gamma = \mathbb{Z} \oplus \gamma\mathbb{Z}$ where $\gamma >> 0$ (make γ be infinite).

$K = k((t^\Gamma))$

let $a_n = t^{\gamma+n}$ and $b_n = t^{\gamma-n}$ $\}$ $a_n, b_n \in \mathcal{O}_v$

So $v(a_n) = \gamma + n$ $\quad v(b_n) = \gamma - n$

$v(a_n b_n) = 2\gamma$

Let $F(x,y) = x^8 + t^\gamma (xy - t^{2\gamma})^2 + t^{2\gamma} y^8$

$F(a_n, b_n) = t^{8\gamma+8n} + t^\gamma \cdot 0 + t^{2\gamma} \cdot t^{8\gamma-8n}$

$v(F(a_n,b_n)) = \min(8\gamma + 8n, 10\gamma - 8n)$

$= 8\gamma + 8n \quad < 9\gamma$

BUT $F(x,y) < 9\gamma + n$ all x,y!! (below $9\gamma + \mathbb{Z}$)

any $n \in \mathbb{Z}$.

$v(x^8) = 8v(x)$

$v(t^\gamma (ab - t^{2\gamma})^2) = \gamma + 2v(ab - t^{2\gamma})$

$v(t^{2\gamma} y^8) = 2\gamma + 8v(y)$

$\}$ Distinct so $vF = \min$ of v(terms)

$\to$ look at cases.

② **Thm**: If $\Gamma \neq \mathbb{Z}$ and $\Gamma \neq \mathbb{Q}$ then K not extremal.

Pf: (Sketch)

In $K \neq \mathbb{Z}, \mathbb{Q}$ there is a $\gamma >> 0$ w/ γ not divisible by any n or something even worse happens. ∎

Thm: If $\Gamma = \mathbb{Q}$ and k is not large then K not extremal.

EX $K = \mathbb{F}_5((t^{\mathbb{Q}}))$

let $F(x,y,z) = x^4 + (xy - t)^4 + (z^5 - z - y)^4$

$a_n = t^{1 + 1/n}$

$b_n = t^{1 - 1/n}$