Valued Fields

Evaluation on a field is $v: K^* \to \Gamma$ (ordered ab. gr)

w/ $v(a \cdot b) = v(a) + v(b)$

$v(a+b) \geq \min\{v(a), v(b)\}$

convention: $v(0) := \infty > \Gamma$

Note: $v(1) = v(1 \cdot 1) = v(1) + v(1)$
- so $v(1) = 0$
- also $v(-1) = 0$
- $v(a) = v(-a)$

Given $\mathcal{O}_v = \{a \in K : v(a) \geq 0\}$ local ring for valuation
$M_v = \{a \in K : v(a) > 0\}$ max ideal.
$\Big\}$ $v(a) = 1 \Rightarrow a$ is unit.

Residue Field $\quad \mathcal{O}_v \to \mathcal{O}_v / M_v =: k \quad \leftarrow \quad K = \underbrace{(\mathcal{O}_v \setminus M_v)}_{v=0} \amalg \underbrace{M_v}_{v>0} \amalg \underbrace{M_v^{-1}}_{v<0}$

$a \mapsto a + m_v =: \bar{a}$

...ual characteristic case

EX: $K = k((t))$ Laurent Series (bdd below)

$v(\sum a_i t^i) = \min\{i \text{ w/ } a_i \neq 0\}$

$\mathcal{O}_v = k[[t]]$
$M_v = t \cdot \mathcal{O}_v$
$\mathcal{O}_v / M_v = k$

...ixed characteristic case

EX $\mathbb{Q}_p$ p-adics

$v(n) = k$ if $n = p^k m$ w/ $p \nmid m$ on integers

$v(n/m) = v_p(n) - v_p(m)$ on fractions.

$\mathcal{O}_v / M_v = \mathbb{F}_p$

Hahn-Fields

Let k be any field & Γ any ordered ab. gp.

...$((\Gamma))$ ...ahn-Field

Def: $k((t^\Gamma)) = \{ \sum_{\gamma \in \Gamma} a_\gamma t^\gamma$ w/ $\mathrm{supp}(\sum a_\gamma t^\gamma) = \{\gamma \in \Gamma : a_\gamma = 0\}$ well ordered $\}$

→ This is a field (use ordering)

$V(\sum a_\gamma t^\gamma) = \min \mathrm{supp}.$

Residue field $\mathcal{O}_v / M_v = k$

EX $\mathbb{R}((t^{\mathbb{Q}}))$

EX $\mathbb{R}((t^{\mathbb{Q} \oplus \tau \mathbb{Q}}))$

(order τ above $\mathbb{Q}$.)

As a first-order structure, we may view valued field as:

$$\mathcal{K} = (\underset{\substack{\text{field}\\\text{sort}}}{K}, \underset{\substack{\text{value}\\\text{group}\\\text{sort}}}{\Gamma}, \underset{\substack{\text{residue}\\\text{field}\\\text{sort}}}{k}, v) \quad v : K^* \to \Gamma$$

(alternately could use $(\underbrace{K, +, -, 0, 1}_{\text{field}}, \mathcal{O}_v)$ ← valuation predicate.
almost equivalent to above...)

Thm (Ax-Kochen, Ersher): Suppose K, K' are Henselian valued fields w/ residue char. 0.

↖ like: "complete enough" ⟺ Newton's method works.

Then $(K, \Gamma, k, v) \equiv (K', \Gamma', k', v')$

$$\Updownarrow$$

$k \equiv k'$ and $\Gamma \equiv \Gamma'$

Def: ~~Any time~~ Given f & a w/ $f(x) \in \mathcal{O}_v[x]$ & $a \in \mathcal{O}_v$

w/ $v(f(a)) > 0$ and $f'(a) = 0$ ⟹ there is $\varepsilon \in \mathcal{O}_v$ w/ $f(a+\varepsilon) = 0$

$f(a)$ close to 0 $\qquad$ $f'(a)$ is big

then field is Henselian.

Note: Complete ⟹ Henselian.

Ex. $K = \bigcup_n \mathbb{F}_p((t^{\frac{1}{p^n}\mathbb{Z}}))$ ← ~~piez~~ peeroseries
power series w/ powers have common denom.

$\qquad +\cap$

$L = \mathbb{F}_p((t^{\bigcup_n 1/p^n \mathbb{Z}}))$ ← do not need common denom.

$t^{-1/p} + t^{-1/p^2} + t^{-1/p^3} + \cdots \in L.$

- K is not an elementary substructure of L.
b/c $\underline{x^p - x - t^{-1} = 0}$ has soln in L but not in K.

Open problem: Is there $\Gamma \equiv \mathbb{Z}$ so that
$\mathbb{F}_p((t^\Gamma)) \not\equiv \mathbb{F}_p((t))$?