

Valued Fields

Evaluation on a field is $v: K^* \rightarrow T$

w/ $v(a \cdot b) = v(a) + v(b)$ ^{ordered ab. gr}
 $v(a+b) \geq \min\{v(a), v(b)\}$

convention: $v(0) := \infty > T$

- Note: $v(1) = v(1 \cdot 1) = v(1) + v(1)$
- $\implies v(1) = 0$
 - also $v(-1) = 0$
 - $v(a) = v(-a)$

Given $O_v = \{a \in K : v(a) \geq 0\}$ local ring for valuation } $v(a) = 1 \implies a$ is unit.
 $M_v = \{a \in K : v(a) > 0\}$ max ideal.

Residue Field

$O_v \twoheadrightarrow O_v/M_v =: k$ $\leftarrow k = \underbrace{(O_v \setminus M_v)}_{v < 0} \coprod \underbrace{M_v}_{v = 0} \coprod \underbrace{M_v^{-1}}_{v < 0}$
 $a \mapsto a + M_v =: \bar{a}$

Ex: $K = \mathbb{R}((t))$ Laurent Series (ordered below)
 $v(\sum a_i t^i) = \min\{i \mid a_i \neq 0\}$

$O_v = \mathbb{R}[[t]]$
 $M_v = t \cdot O_v$
 $O_v/M_v = \mathbb{R}$

Ex \mathbb{Q}_p p-adics
 $v(n) = k$ if $n = p^k m$ w/ $p \nmid m$ on integers
 $v(\frac{n}{m}) = v_p(n) - v_p(m)$ on fractions.

$O_v/M_v = \mathbb{F}_p$

Hahn-Fields

Let k be any field $\cong T$ any ordered ab. gp.

Def: $k((t^T)) = \{ \sum_{r \in T} a_r t^r \mid \text{supp}(\sum a_r t^r) = \{r \in T : a_r \neq 0\} \text{ well ordered} \}$

\rightarrow This is a field (use ordering)

$v(\sum a_r t^r) = \min \text{ supp.}$

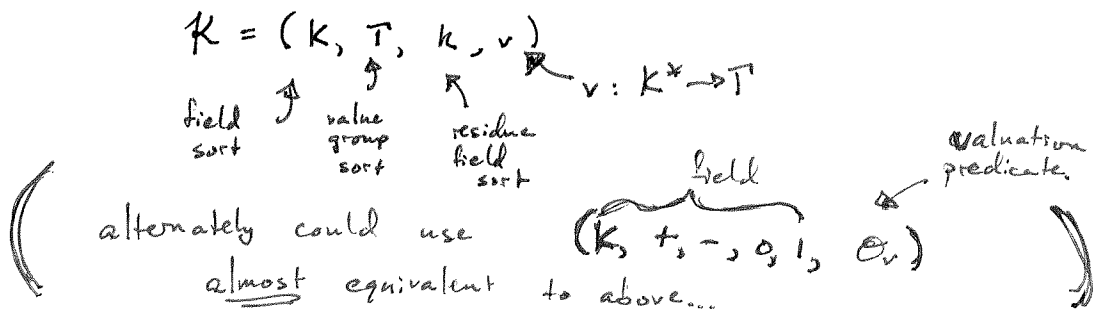
Residue field $O_v/M_v = k$

Ex $\mathbb{R}((t^{\mathbb{Q}}))$

Ex $\mathbb{R}((t^{\mathbb{Q} \otimes \mathbb{Q}}))$

(order τ above \mathbb{Q} .)

As a first-order structure, we may view valued field as:



Thm (Ax-Kochen, Erster): Suppose K, K' are Henselian valued fields w/ residue char. 0.

$$\text{Then } (K, T, k, v) \equiv (K', T', k', v')$$



$$K \equiv k' \text{ and } T \equiv T'$$

\mathbb{R} like: "complete enough"
 \Leftrightarrow Newton's method works.

Def: Given $f \in \mathcal{O}_v[x]$ w/ $a \in \mathcal{O}_v$

w/ $v(f(a)) > 0$ and $v(f'(a)) < 0$ \Rightarrow there is $\varepsilon \in \mathcal{O}_v$ w/ $f(a+\varepsilon) = 0$

\downarrow \downarrow
 $f(a)$ close to 0 $f'(a)$ is big

then field is Henselian.

Note: Complete \Rightarrow Henselian.

Ex $K = \bigcup_n \mathbb{F}_p((t^{\frac{1}{p^n}}\mathbb{Z}))$ \leftarrow piecewise power series w/ powers have common denom.

$L = \mathbb{F}_p((t^{\mathbb{Q}}\mathbb{Z}))$ \leftarrow do not need common denom.
 $t^{-1/p} + t^{-1/p^2} + t^{-1/p^3} + \dots \in L$

\circ K is not an elementary substructure of L .

b/c $x^p - x - t^{-1} = 0$ has soln in L but not in K .

Open problem: Is there $T \equiv \mathbb{Z}$ so that

$$\mathbb{F}_p((t^T)) \not\equiv \mathbb{F}_p((t)) ?$$