① Salih Azgin (Model Theory) $\{R_i\}_I$ $\{f_j\}_J$

Def: A First-order language L is a set of relation & function symbols.

- An L-structure is a set M w/ relations $R_i^M \subseteq M^n$
  functions $f_j^M : M^k \to M$

EX $L_G = \{\cdot, 1\}$ group language
Groups are L_G-structures
$L_R = \{+, \cdot, 0, 1\}$ ring language
Rings are L_R-structures
ordered rings, etc...

Def: A term in L is inductively obtained from:
1) Variable symbols $x, y, z, \ldots$
2) For $f \in L$ (arity k) $f(t_1, \ldots, t_k)$ also a term.

EX In ring language,
- $x+y$ is a term
- any polynom. is a term.

(Note: $=$ is formally a relation in each language
→ suppressed in definitions.)

Def: Atomic Formulas are equivalences of terms
$$t_1 = t_2 \qquad f(x) = g(y)$$
or $R(t_1, \ldots, t_n)$
(ex. $x^2+y^2>0$)

Def: L-formulas are

Quantifier free formulas:
1) Atomic Formulas
2) Conjunctions ϕ_1 and ϕ_2
Disjunctions ϕ_1 or ϕ_2
Negations Not ϕ_1

3) $\forall x \ \phi(x)$
4) $\exists x \ \phi(x)$

Def An L-sentence σ is an L-formula with no free variables.

EX.
- $1<2$
- $\exists x \quad x>1$
- $\forall y \, \exists x \quad x^2+y^2>0$

(Note: L-sentences can be true or false

$M \models \sigma$ means σ is true in M.)

Def: $M \equiv N$ if for all σ
$M \models \sigma \iff N \models \sigma$.

(This is weaker than isomorphism!

Def $X \subseteq M^n$ is definable if there is an L-formula ϕ so that

$$X = \{a \in M^n \text{ w/ } M \models \phi(a)\}$$

EX $X = \{(a,b) : a^2 - b^3 + \pi > 0\}$ is definable.

Thm: If $X \subseteq \mathbb{R}$ is definable then X is a finite union of intervals and points.

To show that $M \equiv N$

Idea:
1. Find a "small" set of sentences which imply all true ones in M & N
2. Show that these are simultaneously true/false

Def: An L-theory is a set of L-sentences Σ

Σ is complete if $\forall \sigma$
$\Sigma \models \sigma$ or $\Sigma \models \text{not}(\sigma)$

Note: L-theory for $(\mathbb{Z}, \cdot, +, <)$ is terrible
→ No finite computable L-theory is complete!!

EX Let ACF be the theory of algebraically closed fields

$$\{\forall a_0, \dots, a_n \, \exists x \quad a_0 + a_1 x + \dots + a_n x^n = 0\}_{n \in \mathbb{N}}$$

Thm ACF

3 ~~Thm~~ **Cor:** $\mathbb{Q}^{ac} \equiv \mathbb{C}$

algebraic closure of $\mathbb{Q}$ is elementarily equivalent to $\mathbb{C}$.

Note: elementary equivalence cannot distinguish cardinality

EX Let RCF be the theory

- M is an ordered field
- M satisfies Interm. Value Thm. for all polynomials

Thm RCF is complete.

Cor $\mathbb{Q}^{rc} \equiv \mathbb{R}$

(Idea: Do stuff in a field which is simple & pull results across $\equiv$.)

Compactness

Let Σ be a set of ~~formulas~~ sentences such that every finite ~~part~~ subset of Σ holds in M

$\Longrightarrow \exists$ extension N of M $\ni$ $N \models \Sigma$

Notation: N extension of M
$M \prec N$

$M \equiv N$ w/ $i: M \hookrightarrow N$ respecting truth of formulas

EX $\mathbb{R}$, ordered ~~fields~~ rings

$\Sigma = \{0<x<1,\ 0<x<\frac{1}{2},\ 0<x<\frac{1}{3}, \dots\}$

$\Longrightarrow \mathbb{R} \prec \mathbb{R}^* \ni a \in \mathbb{R}^*$ w/ $0<a<\frac{1}{n}\ \forall n$

[Allows us to do calculus proofs w/ infinitesimals and then trasport ~~proofs~~ things back to $\mathbb{R}$ (instead of using limits inside of $\mathbb{R}$)]

(4) · Salih:

"Theory of p-adics?" $Th(\mathbb{Q}_p)$

Ax-Kocher Thm: $\underbrace{\prod_p \mathbb{Q}_p / \mathcal{U}}_{\text{Limit theory of } \mathbb{Q}_p} \equiv \underbrace{\prod_p \mathbb{F}_p((t)) / \mathcal{U}}_{\text{Limit theory of } \mathbb{F}_p((t))}$

{sentences true in almost all $\mathbb{Q}_p$} = {sentences true in almost all $\mathbb{F}_p((t))$}

Used to prove: Artin's conj: Every ~~form~~ formula of degree d in d^2+1 variables of $\mathbb{Q}_p$ has a soln. in $\mathbb{Q}_p$.

$\leadsto$ True in $\mathbb{F}_p((t)) \Longrightarrow$ True in almost all $\mathbb{Q}_p$

(For every d $\exists N \ni$ every formula in deg d & (d^2+1) var in $\mathbb{Q}_p$ has soln in $\mathbb{Q}_p$ $\forall p > N$.)