Quantum Calculus

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History:

E. Effros
Z.J. Ruan
C. Webster
S. Winkler
} 1995 – 1999 UCLA

"Local operator spaces"

W. Arveson } 1990 — Berkeley

D. Blecher
V. Paulsen
I. Todorov
} U. Houston

"Quantum Functional Analysis"

A. Helemskii
O. Aristov
A. Dosiev
} Russians

Now → "Quantum Information Theory"

→ Fix linear space V.

$M_{mn}(V)$ = $m\times n$ matrices over V

$M_n(V)$ = $n\times n$ " " "

$M(V)$ = finite (square) matrices over V

(Idea: Replace V by $M(V)$)

→ Quantum operations

- $\oplus : M(V)\times M(V) \longrightarrow M(V)$ $\qquad V\oplus W = \begin{bmatrix} V & 0 \\ 0 & W \end{bmatrix}$

Let $M = M(\mathbb{C})$

M-bimodule operation

- $\cdot : M\times M(V)\times M \longrightarrow M(V)$

$a\in M_{m,s}$ $\quad v\in M_s(V)$ $\quad b\in M_{s,n}$

$a\cdot v\cdot b = \left[\sum_{k,t} a_{ik}v_{kt}b_{tj}\right]_{ij}$

Def: $\sum_s a_s v_s b_s$ is a quantum combination

$b_s, a_s \in M_s \quad v_s\in M_s(V)$

$\begin{bmatrix} \cdots a_s \cdots \end{bmatrix} \begin{bmatrix} v_1 & & 0 \\ & v_s & \\ 0 & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ b_s \\ \vdots \end{bmatrix}$

" $a\cdot v\cdot b$ "

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② **Morphisms**

- Any linear map $V \to W$ has canonical extension
$$\phi^{(\infty)} : M(V) \longrightarrow M(W)$$

- $\boxed{\text{Lemma: } \phi^{(\infty)} \text{ preserves quantum operations.}}$

Quantum Sets

Notation: $\mathcal{B} \subseteq M(V)$ w/ $\mathcal{B} = (b_n)$ $\quad b_n \subseteq M_n(V)$

- $\mathcal{B}$ is an abs. matrix convex set if $\forall n$
  ① $\mathcal{B} \oplus \mathcal{B} \subseteq \mathcal{B}$
  ② $a \cdot \mathcal{B} \cdot b \subseteq \mathcal{B}$ for $a, b \in \text{ball}(M)$ (closed unit)

$= \{a \mid \|a\| \leq 1\}$

Norm is operator norm
$a \in M$ is $\mathbb{C}^n \to \mathbb{C}^n$
$\|a\| = \sup\{|\langle a\eta, \xi\rangle| \text{ w/ } \eta, \xi \in B(\mathbb{C}^n)\}$

Lemma: If $\mathcal{B}$ is abs. matrix convex then $\mathcal{B}$ is abs. convex.

(i.e. b_n abs. convex for all n $\iff$ $\lambda b_n + \mu b_n \subseteq b_n$ $\forall\, |\lambda| + |\mu| \leq 1$)

Norms

Let $\mathcal{B}$ be an abs. convex matrix set.

matrix semi-norm

Notation: $P_{\mathcal{B}}(v) = \inf\{t > 0 \text{ w/ } t^{-1}v \in \mathcal{B} \text{ all } v \in M(V)\}$

Lemma: $p : M(V) \longrightarrow \mathbb{R}_+$ is a matrix seminorm $\iff$
- $p(v \oplus w) \leq \max\{p(v), p(w)\}$
- $p(a v b) \leq \|a\|\, p(v)\, \|b\|$ all $a, b \in M$

$\}$ Ruan's axioms

③ Examples :

① $V = \mathcal{B}(H)$ the C^*-algebra of bdd linear operators on H, a Hilbert space.

$$M_n(\mathcal{B}(H)) = \mathcal{B}(H^n)$$

→ Gives new norms $\|\cdot\|_n$ on $M_n(\mathcal{B}(H))$

Combine these for $\|\cdot\| : M(\mathcal{B}(H)) \to \mathbb{R}$

Then
- $\|T \oplus S\| = \max\{\|T\|, \|S\|\}$
- $\|a \cdot T \cdot b\| \le \|a\| \cdot \|T\| \cdot \|b\|$

Thm (Ruan c.'90): If (V, p) is a linear space w/ matrix norm then $(V, p) \hookrightarrow \mathcal{B}(H)$ up to matrix isometry.

② ~~Hilbert space~~
Given Hilbert space H,

Quantum Domain

$$\mathcal{E} = \{p_\alpha : \alpha \in \Lambda\} \subseteq \mathcal{B}(H)$$

↖ upward filtered (i.e. poset filtered)

$$C^*_{\mathcal{E}}(D) = \ldots\ldots$$

→ Quantum Space

$p = \{\mathcal{B}\}$ family of abs. matrix convex sets of $M(V)$

↖ filter base. $\left(\text{so } \bigcap_{\varepsilon > 0,\, \mathcal{B} \in p} \varepsilon \mathcal{B} = \{0\}\right)$

- p defines a polynormed topology in $M(V)$
- (V, p) is a quantum space.

Hausdorff, locally convex,

thm (Anar): $(V, p) \hookrightarrow C^*_{\mathcal{E}}(D)$

- Morphism $(V, p) \xrightarrow{\phi} (W, q)$ "quantum-continuous"
  iff $\phi^{(\infty)} : M(V) \to M(W)$ is continuous

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Note: Quantum topologies restrict to topologies on V

→ Restriction is locally convex topology (classical).

Def: A quantum topology which restricts to a classical topology is called a "quantization" of the classical topology.