Salih Durhan (10.10.2013)

Valued Fields

$v(0) := \infty > \Gamma$

(K, Γ, k, v)

field — ordered ab. group — field — "valuation" $v : K^\times \to \Gamma$

$v(a \cdot b) = v(a) + v(b)$

$v(a+b) \geq \min\{v(a), v(b)\}$

$\mathcal{O}_v = \{a \in K \text{ w/ } v(a) \geq 0\}$

$m_v = \{a \in K \text{ w/ } v(a) > 0\}$

$\Rightarrow k = \mathcal{O}_v / m_v$

Ex: $k((t))$ and $v(\sum_i a_i t^i) = \min_i (a_i \neq 0)$

coeff of lowest power

$\rightarrow$ In some sense, $v \simeq \log_t$

Remark: If $v(a) \neq v(b)$ then $v(a+b) = \min\{v(a), v(b)\}$

($>$ only if $v(a) = v(b)$)

Connection to Tropical Geom:

Define
- $a \odot b = v(a) \cdot v(b)$
- $a \oplus b = \min\{v(a), v(b)\}$

Hahn fields

$\boxed{\text{support} = \{\gamma \text{ w/ } a_\gamma \neq 0\}}$

k any field, Γ any ordered group $\Big\}$ $k((t^\Gamma)) = \left\{ \sum_i a_\gamma t^\gamma \text{ w/ support } (\sum a_\gamma t^\gamma) \text{ well-ordered} \right\}$

$v(\sum_i a_\gamma t^\gamma) = \min(\text{support})$

Ex: $K = \mathbb{C}((t^{\mathbb{Q}}))$ is an algebraically closed field.

Note: $t^{1/2} + t^{2/3} + t^{3/4} + \dots \in K$

$t^{1/2} + t^{1/3} + t^{1/4} + \dots \notin K$

"For all practical purposes, this is equivalent to pezuoseries"

$\mathcal{P} = \bigcup_n \mathbb{C}((t^{1/n})) \subsetneq \mathbb{C}((t^{\mathbb{Q}}))$

Polynomials (in one variable) over K:

Let $f(x) = a_0 + a_1 x + \dots + a_n x^n \,/\, K$

$c \in K$ be such that w/ $\gamma = v(c)$ we get $v(a_i c^i) \neq v(a_j c^j)$ for $i \neq j$ $\Big\}$ Then $v(f(c)) = \min_i \{v(a_i c^i)\} < \infty$!!

→ Conclusion: If $f(c)=0$ then $\exists\ i\neq j$ w/
$$v(a_i c^i) = v(a_j c^j) \quad !!$$

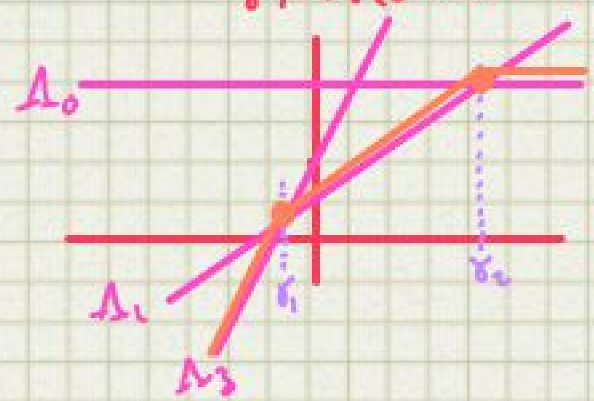
Let $\Lambda_i : T \to T$ be the map $\quad \left(\text{So } \Lambda_i(\gamma) = v(a_i b^i) \text{ when } v(b)=\gamma \right)$
$$\gamma \longmapsto v(a_i) + i\gamma$$

Define $f_v(\gamma) = \min\{\Lambda_i(\gamma)\}$ "If this minimum is attained at a single index then $v(f(b)) = f_v(\gamma)$"

↳ If $f(c)=0$ then the points w/ $v(a_i c^i) = v(a_j c^j)$ must be at minimum!
$$f_v(\gamma) = \Lambda_i(\gamma) = \Lambda_j(\gamma) \quad \text{w/ } \gamma = v(c)$$

Ex: $f(x) = t^3 + tx + t^2x^3$

$\Lambda_0 : T \to T$, $\gamma \longmapsto v(t^3) = 3$ $\quad$ $\Lambda_1 : T \to T$, $\gamma \longmapsto v(t) + \gamma = 1+\gamma$ $\quad$ $\Lambda_3 : T \to T$, $\gamma \longmapsto v(t^2) + 3\gamma = 2+3\gamma$



"In multivariable setting, this becomes tropical lines, planes, etc!"

Multivariable polynomials / K

$$f(\underline{x}) = \sum_{\underline{i}\in\mathbb{N}^n} a_{\underline{i}} \underline{x}^{\underline{i}} \quad \text{where } \underline{x} = (x_1,\dots,x_n),\ \underline{i} = (i_1,\dots,i_n),\ a_{\underline{i}}\underline{x}^{\underline{i}} = a_{\underline{i}} x_1^{i_1}\cdots x_n^{i_n}$$

For each $\underline{i}$ w/ $a_{\underline{i}} \neq 0$ we have
$$\Lambda_{\underline{i}} : T^n \to T \quad \text{by}$$
$$(\gamma_1,\dots,\gamma_n) \longmapsto v(a_{\underline{i}}) + i_1\gamma_1 + \cdots + i_n\gamma_n$$

$f_v : T^n \to T$
$\underline{\gamma} \mapsto \min\{\Lambda_{\underline{i}}(\underline{\gamma})\}$

If $f(\underline{c}) = 0$ then there are $\underline{i} \neq \underline{j}$ w/
$f_v(\underline{\gamma}) = \Lambda_{\underline{i}}(\underline{\gamma})$ where $v(c) = \underline{\gamma}$
$= \Lambda_{\underline{j}}(\underline{\gamma})$

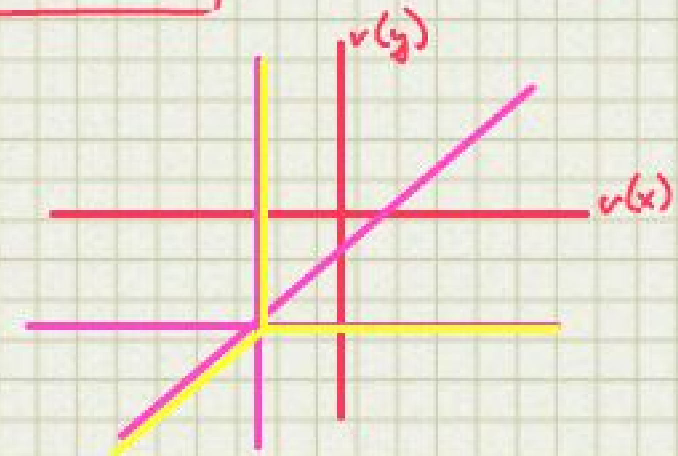
Def: $\text{Trop}(f) = \{(\gamma_1, \dots, \gamma_n)$ w/ $f_v(\underline{\gamma}) = \Lambda_{\underline{i}}(\underline{\gamma}) = \Lambda_{\underline{j}}(\underline{\gamma})$ some $\underline{i} \neq \underline{j}\}$

"Tropical Locus"

Ex: Let $f(x,y) = 1 + tx + t^2 y$

$\Lambda_0 : T^2 \to T$ $[v(x), v(y)] \mapsto v(1) = 0$	$\Lambda_1 : T^2 \to T$ $[v(x), v(y)] \mapsto v(t) + v(x) = 1 + v(x)$
$\Lambda_2 : T^2 \to T$ $[v(x), v(y)] \mapsto v(t^2) + v(y) = 2 + v(y)$	$f_v : T^2 \to T$ $[v(x), v(y)] \mapsto \min_i \Lambda_i$

- $\Lambda_0 \cap \Lambda_2$ is $v(y) = -2$
- $\Lambda_1 \cap \Lambda_2$ is $1 + v(x) = 2 + v(y)$
- $\Lambda_0 \cap \Lambda_1$ is $v(x) = -1$



This is a (flipped) tropical line !!!
→ The flip is b/c used min, not max.

"Kapranov's Thm": If K is algebraically closed and $(\gamma_1, \dots, \gamma_n) \in \text{Trop}(f)$ then there is $(a_1, \dots, a_n) \in K^n$ with $v(a_i) = \gamma_i$ and $f(a_1, \dots, a_n) = 0$

→ In the next talk we will discuss the proof of this.

(24.10.13) Model Theory II

Recall: Valued Difference Fields

Def: A VDF is a valued field w/ distinguished automorphism σ such that $\sigma(\mathcal{O}_v) = \mathcal{O}_v$

Fixes valuation ring setwise

(Note: Do not require $v(\sigma(a)) = v(a)$)

In this case, σ induces a map on the residue field

$$\bar{\sigma} : \mathcal{O}_v/m_v \longrightarrow \mathcal{O}_v/m_v \quad \text{w/ } \bar{\sigma} : k \to k$$

an automorphism

and

$$\sigma : \Gamma \longrightarrow \Gamma$$
$$\gamma \longmapsto \sigma(\gamma) = v(\sigma(x))$$

where $v(x) = \gamma$.

FACT: If $v(x) = v(y)$ then $v(\sigma(x)) = v(\sigma(y))$

$\to v(x/y) = 0 \Rightarrow x/y \in \mathcal{O}_v \setminus m_v$

$\Rightarrow \sigma(x/y) \in \mathcal{O}_v \setminus m_v$

Ex: Let $(k, \bar{\sigma})$ be a valued difference field &

(Γ, σ) be a valued difference ordered group.

then $\sigma : k((t^\Gamma)) \to k((t^\Gamma))$ satisfies $\sigma(\mathcal{O}_v) = \mathcal{O}_v$

by $\sum a_\gamma t^\gamma \longmapsto \sum \bar{\sigma}(a_\gamma) t^{\sigma(\gamma)}$

Ex: $\Gamma = \mathbb{Q}$ $\quad \sigma : \mathbb{Q} \to \mathbb{Q}$, $\gamma \mapsto 2\gamma$ $\quad$ If $\Gamma = \bigcup_{\mathbb{Z}} \Gamma_i$ where Γ_i are convex subgroups we may have $\sigma(\Gamma_i) \subseteq \Gamma_{i+1}$.

(AKE) **Thm:** If $\text{char}(k) = 0$ and K is Henselian then $Th(k) \cup Th(\Gamma)$ determines $Th(K)$.

$\to$ Apply same idea to valued difference fields.

Thm: If $\text{char}(k) = 0$ and K is "σ-Henselian" and either

1) $v(\sigma(a)) = v(a)$ all a
2) $v(\sigma(a)) > n v(a)$ all n and a (w/ $v(a) > 0$)
3) $v(\sigma(a)) = n v(a)$ all a, some n

These can all be removed [Durhan]

and $(k, \bar{\sigma})$ is "linear difference closed"(*)

then $Th(K)$ is determined by $Th(k, \bar{\sigma})$ and $Th(\Gamma, \sigma)$.

(*) If $a_0, \dots, a_n \in k$ not all zero then
$$1 + a_0 x + a_1 \sigma(x) + a_2 \sigma^2(x) + \cdots + a_n \sigma^n(x) = 0$$
has a solution.

Let $F(x) = \sum_{\underline{i} \in \mathbb{N}^{n+1}} a_{\underline{i}} \sigma^{\underline{i}}(x)$ where $\sigma^{\underline{i}}(x) = x^{i_0} \sigma(x)^{i_1} \sigma^2(x)^{i_2} \cdots \sigma^n(x)^{i_n}$

define $F_v : \Gamma \to \Gamma$
by $\gamma \mapsto \min_{\underline{i}} \{ v(a_{\underline{i}}) + v(\sigma^{\underline{i}}(\gamma)) \}$ "Connection to Kapranov's Thm..."

<u>Note</u>: "Linear Difference closed" is still pretty strong.

<u>EX</u>: <u>"Trans-series"</u>

$\mathbb{R} \subseteq \mathbb{R}((x^{-1}))$ (Laraunt series on x^{-1})

w/ $\sum_{i=n}^{\infty} a_i (x^{-1})^i > 0$ if $a_n > 0$ (i.e. just look at first term $\rightsquigarrow$ lexicographic ordering) coeff of highest power of x

<u>Note</u>: $x >> \mathbb{R}$ b/c $1 \cdot x - r > 0$ for all $r \in \mathbb{R}$.

$(\mathbb{R}, <, \exp, \log)$

$\mathbb{R} \subseteq \mathbb{R}((\,(x^{-1})^{\mathbb{R}}))$ If $a \in \mathbb{R}((\,(x^{-1})^{\mathbb{R}}))$ w/ $v(a) \geq 0$ then
$$a = s + \sum_{\gamma > 0} (x^{-1})^{\gamma} \text{ for some } s \in \mathbb{R}$$
and
$$\exp(a) = e^s \cdot \left(e^{\sum (x^{-1})^{\gamma}}\right)$$

$\to$ <u>Note</u>: $e^{(x^{-1})} = \sum \frac{(x^{-1})^n}{n!} \in \mathbb{R}((\,(x^{-1})^{\mathbb{R}}))$

Then $\exp : K_0 \to K_0$ is a <u>partial exponential</u>

Define $e^x = \exp(x)$ a <u>new</u> monomial w/ $v(e^x) < v(x^n)$ all n.

Obtain
$$K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \quad K = \bigcup_n K_n$$
x $\quad e^x \quad e^{(e^x)}$ etc...

K has exp, but not log... do the same trick again:

$$K = L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots \quad \mathbb{T} = \bigcup_n L_n$$

$x \quad \ln x \quad \ln(\ln x)$ etc...

Note: K has $\ln x^{-1}$ but not $\ln x$

$(\mathbb{T}, <, \exp, \ln)$ is big valued field

Moreover $\exists\ \partial : \mathbb{T} \to \mathbb{T}$ w/ $\partial(x) = 1$ & $\partial(e^x) = e^x$ $\quad \int : \mathbb{T} \to \mathbb{T}$, $a \mapsto \int a$ inverse.

This is like the "Differential Closure"

Everything that should be solvable, is solvable.

(Note: $v(e^x) < v(x^n) < v(\ln x)$)

Moreover if $g \in \mathbb{T}$ with $g > \mathbb{R}$ then we have

$\sigma_g : \mathbb{T} \to \mathbb{T}$ is an automorphism.

by $f(x) \mapsto f(g(x))$

Important examples: $g(x) = e^x$

$g(x) = \ln x$

Open Question: $Th((\mathbb{T}, \sigma_{e^x}, v)) = ???$

Note: $\sigma_{e^x} = Id/\mathbb{R}$, $\sigma(f) - f + 1 = 0$

$f(e^x) - f(x) + 1 = 0$

has no soln in $\mathbb{T}$

$\Rightarrow$ Stuff from before doesn't apply !!

Early def of "Difference Equation":

$f(x+1) - f(x) = g(x)$ Given $g(x)$ can you find $f(x)$ so that this is true??

Let $\sigma : \mathbb{T} \to \mathbb{T}$ by $f \mapsto f(x+1)$

Thm: There is a subfield $\mathbb{T}^\# \leq \mathbb{T}$ such that $(\mathbb{T}^\#, \sigma)$ is linear difference closed.

Big Thm $\Longrightarrow Th(\mathbb{T}, \sigma, v)$ is determined by $Th(\mathbb{T}^\#, \sigma)$ and $Th(T, \sigma)$

This is basically $K_0 = \mathbb{R}((x^{-1})^{\mathbb{R}})$ from before.