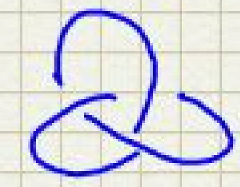
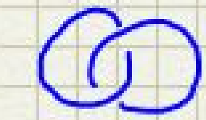
Quantum Groups and Links I

24.9.2013

(Özgür Kişisel)

Today: Summary of geometric side

Knot: Smooth embedding of $S^1 \hookrightarrow S^3$ 
(alternately S^3 can be replaced by M^3)

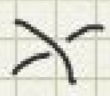
Link: Smooth embedding of $\coprod_k S^1 \hookrightarrow S^3$ 

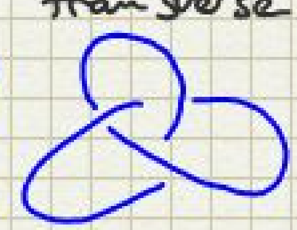
Knot equivalence $K_1 \sim K_2$ if there is an ambient isotopy taking K_1 to K_2

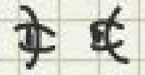
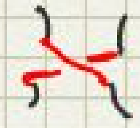
Link equivalence defined similarly

Goal of knot theory: Classify all knots/links up to knot equiv.

Useful definitions:

Crossing #: the minimum number of crossings  in any transverse diagram of a knot

ex Trefoil  has crossing # = 3

Connect Sum: $K_1 \# K_2$ is knot produced by cutting  from each knot and connecting  or 

Prime Knot: K is prime if $K = K_1 \# K_2 \Longleftrightarrow \{ K_1$ or K_2 is unknot O

Thm: All knots have unique prime decomposition! ← Use genus of knots

Current status: Prime knots w/ crossing # ≤ 16 are classified!
⇝ Look up list on internet.

To distinguish knots people use knot invariants

Knot invariant is a function taking constant values on knot equivalence classes.

Examples: (1) Homeomorphism class of knot complement $S^3 \setminus K$ ↳ Basically impossible to compute.

(2) Fundamental group of complement $\pi_1(S^3 \setminus K)$ ↳ "Computable" using knot diagram
→ But reduces to classifying nonabelian groups.

Note: $H_1(S^3 \setminus K) = \mathbb{Z}$ for all knots... not useful

(3) Alexander Invariant (1930's)

Use "universal cyclic cover" of $S^3 \setminus K$.

$H_1(\widehat{S^3 \setminus K})$ is a module ⇝ Alexander polynomial.

[In 1960's J. Conway found a way to get Alexander poly. using diagrams]

↳ Also he found knots w/ $K_1 \not\sim K_2$ having same Alexander polyn.

(4) Jones Polynomial (1984)

(stems from work on algebraic properties of subfactors)

[Soon after Kaufmann found a way to get Jones polynom w/ diagrams]

(5) HOMFLY-PT polynomial (1988)

⇝ Two variable polynomial generalizing both Jones & Alexander polyn.

Note: There are nontrivial knots w/ Alexander polynomial trivial...
But there are no known nontrivial knots w/ Jones polynomial trivial!!

Knots and 3-Manifolds

Dehn surgery: Take a link $L \subset S^3$ and consider a tubular nbhood.

Topologically, each component is a solid torus
→ Take the tubular nbhood out of S^3 and glue back in a solid torus w/ nontrivial attatching maps

Dehn-Lickorish Thm: Any 3 manifold can be obtained from S^3 by Dehn surgery along a link.

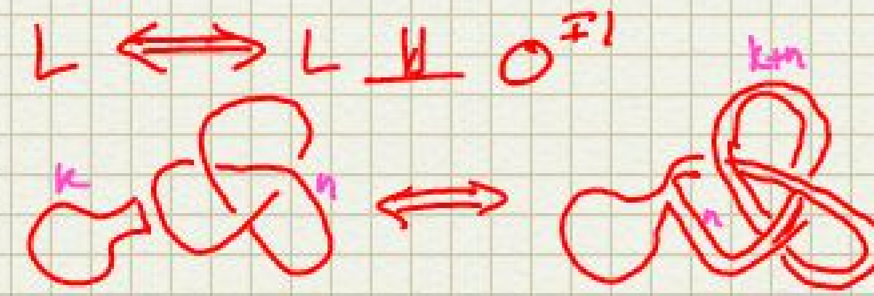
Note: Information about each surgery → framing on each component of link

↳ 

Problem: Two different framed links could give the same 3-manifold!

Solution: Kirby moves

① $L \Longleftrightarrow L \sqcup O^{\pm 1}$

② 

Thm: Two framed links describe the same 3-mfld $\Longleftrightarrow \exists$ finite sequence of moves ① & ② between the links.

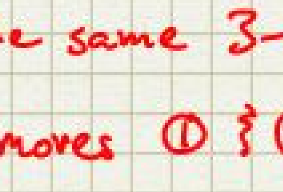
Cor: Any link invariant which is not changed by Kirby moves yields a 3-mfld invariant!!

(Before going further we must discuss braids.)

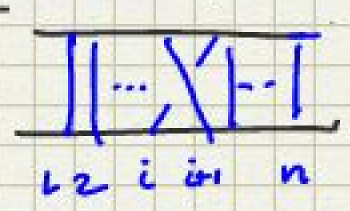
Braids

B_4

Considered up to isotopy equiv.

→ Group w/ 

Presentation (E. Artin)

Generators 

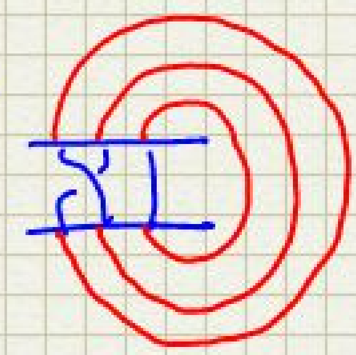
σ_i

Relations ① $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|j-i| \geq 2$

② $\sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i$

Connection to Knots.

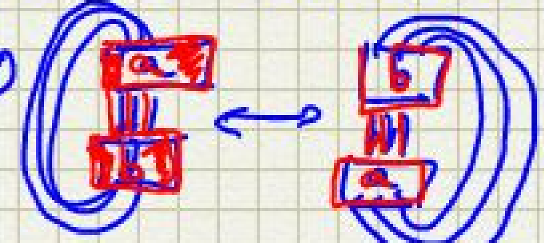
Braids $\xrightarrow{\text{Braid Closure}}$ Links



Alexander's Thm Every link can be written as a braid closure

Markov's Thm Two braids have the same closure

$\Longleftrightarrow$ there is a series of moves between them

①


Similar to $tr(ab) = tr(ba)$

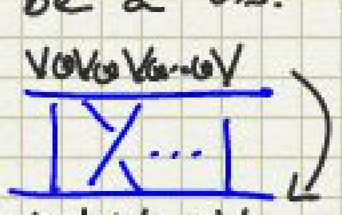
and ② (Stabilization)


Cor: Any quantity obtained from B_n which is invariant wrt these is a link invariant.

Representations of B_n:

R-matrix representations.

Let V be a v.s. then braids give elements of $End(V^{\otimes n})$

by 

$\sigma_i \longmapsto R_{i,i+1}$ $\left\{ \begin{array}{c} R \text{ is a matrix in} \\ \text{End}(V \otimes V) \end{array} \right]$

→ What relations should R satisfy???

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rightsquigarrow R_{12} R_{23} R_{12} = R_{23} R_{12} R_{23}$$

Yang-Baxter Equation

$$\begin{pmatrix} R_{12} = R \otimes id \\ R_{23} = id \otimes R \end{pmatrix}$$

[There are specific 4x4 matrices R ($\dim V = 2$) yielding Alexander and Jones polynomials!!]

Lecture 2 (1.10.2013)

Last Time: Knots, Links, Invariants
Braids ⇝ Braid closures
Braid group representations
— "R matrices" Satisfy Yang-Baxter
$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}$
— How do we get solutions to YBE ??

Quantum Groups

⇝ Start from noncommutative geometry

Idea: Space is determined by algebra of functions on it.

EX: V an affine alg variety $/k$ (w/ $k = \bar{k}$)
Then $k[V]$ (the ring of regular functions on V) has $V = \mathrm{Spec}(k[V])$

EX: X a smooth cpt mfld then X can be recovered from $C^\infty(X)$

Plan: Replace "commutative function algebra" of a space by something non-commutative and do geometry as if it had a spectrum

↳ **Hopf algebra:** "like a group with deformed function space"

$(H, m, \eta, \Delta, \varepsilon, S)$

- H: k-vector space
- m, η: product and unit
- Δ, ε: coproduct and counit
- S: antipode

$m: H \otimes H \to H$ $\quad \Delta: H \to H \otimes H$ $\quad S: H \to H$
$\eta: k \to H$ $\quad \varepsilon: H \to k$

w/ compatibility axioms.

EX: If G is an algebraic group, define

$H = k[G]$

$$\begin{cases} m : H \otimes H \to H & \text{by } f \otimes g \mapsto fg \\ \eta : k \to H & \text{by } 1_k \mapsto \text{const}_1 \end{cases}$$

$$\begin{cases} \Delta : H \to H \otimes H & \text{by } \Delta(f) : x \otimes y \mapsto f(xy) \\ \varepsilon : H \to k & \text{by } f \mapsto f(e) \end{cases}$$

$$S : H \to H \quad \text{by } S(f) : x \mapsto f(x^{-1})$$

→ This is a commutative, not co-commutative Hopf algebra.

More interesting examples are non-commutative, non-cocommutative Hopf algebras ⇝ "**Quantum Groups**"

To get R-matrices into the picture, need

Quasi-Triangular Hopf Algebra Structure

Def: A Hopf algebra is *quasi-triangular* if there is an element $R \in H \otimes H$ w/

⓪ R is invertible

① $R\Delta = \Delta^{op} R$

② $(\Delta \otimes id)R = R_{13} R_{23}$

③ $(id \otimes \Delta)R = R_{13} R_{12}$

$R_{12} = id \otimes R$
$R_{23} = R \otimes id$
etc

Lemma: Such an R satisfies the Yang-Baxter Equation!

⟹ Gives a representation of Braid group!!

⟹ Gives a knot invariant!!

→ Methods exist for constructing quasitriangular Hopf algebras from Hopf algebras

— "Drinfeld's Quantum Double" —

⇝ Given a Hopf algebra H, we can construct $D(H)$ a quasi-triangular Hopf alg.

(Munevver will give more details next week)

Basic idea: Given $(H, m, \eta, \Delta, \epsilon, S)$ w/ S invertible,

- Dual Hopf algebra is $(H^*, \Delta^*, \epsilon^*, (m^{op})^*, \eta^*, S^{-1}) = (H^{op})^*$
- $D(H)$ is "bicrossed product"

$$D(H) = (H^{op})^* \bowtie H$$

- as a vector space this is $H^* \otimes H$
- operations are more complicated...

unit: $1 \otimes 1$

counit: $\epsilon(f \otimes a) = \epsilon(a) f(1)$

coprod: $\Delta(f \otimes a) = \sum (f' \otimes a') \otimes (f'' \otimes a'')$

where $\Delta f = \sum f' \otimes f''$, $\Delta a = \sum a' \otimes a''$

prod: $(f \otimes a) \cdot (g \otimes b) = \sum f g(S^{-1}(a''') x a') \otimes a'' b$

where $\Delta^2 a = \sum a' \otimes a'' \otimes a'''$

(x is the variable making the first part a function)

What about R?

$$H \xrightarrow{i_H} D(H) \xleftarrow{i_{H^*}} (H^{op})^*$$

w/ $i_H(a) = 1 \otimes a$, $i_{H^*}(f) = f \otimes 1$

R-matrix: $\lambda_{H,H} : H \otimes H^* \longrightarrow End(H)$

by $\lambda_{H,H}(a \otimes f)(b) = f(b) \cdot a$

(isomorphism b/c H is finite dim'l)

$$\rho = \lambda_{H,H}^{-1}(id_H) \in H \otimes H^*$$

$$\hookrightarrow R = (i_H \otimes i_{H^*})(\rho) \in D(H) \otimes D(H)$$

(Drinfeld 80's)

Thm: $D(H)$ is quasi-triangular with this R-matrix.

(Next week, Munevver will continue w/ more description of Drinfeld's Double and R-matrices.)