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Contact Structures

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Today: dim = 3 Overview.

Next Time: dim = 5

Def: Let N be an orientable $(2n+1)$-dim'l closed compact manifold.

- A contact structure on N is an oriented, codim 1 hyperplane field $\xi \subset TN$

  aka "distribution"

- A contact form is a 1-form α of N w/
  - $\xi = \ker \alpha$
  - $\alpha \wedge (d\alpha)^n > 0$ ← "co-orientation condition." TN/ξ is trivial

EX $\mathbb{R}^{2n+1}$ or $S^{2n+1} \setminus \{pt\}$

$$\xi_{st} = \ker \left(dz + \sum_{j=1}^{n} x_j \, dy_j \right)$$ ("standard contact structure")

Thm: (Darboux) Locally, all contact structures are contactomorphic to $(\mathbb{R}^{2n+1}, \xi_{st})$

$\left(X \xrightarrow{f} Y \text{ w/ } f^* \alpha_Y = \alpha_X \right)$

(see) pictures

EX: $\mathbb{R}^3$

$$\xi_{ot} = \ker \left(\cos(2\pi r)\, dz + r \sin(2\pi r)\, d\theta \right)$$ ("over twisted contact structure")

EX: $T^3 = S^1_r \times S^1_\theta \times S^1_t$

$$\xi_k = \ker \left(\cos(2\pi k r)\, d\theta + \sin(2\pi k r)\, dt \right)$$ twists k times on $[0,1] \times T^2$

(Martinet '71)
Thm: All compact 3-mflds have a (overtwisted) contact structure.

Def: (N^3, ξ) is "overtwisted" if there is a disk $D^2 \subset N^3$ w/ ∂D tangent to ξ.
→ "a full twist happens in finite time"

↳ If no such D^2 exists, (N^3, ξ) is called "tight"

(Eliashberg '92)
Thm: The only tight contact structure on S^3 is ξ_{st}!

Q: (1) Which manifolds have tight structures? (Not all...)
(2) Classify tight/overtwisted structures on M. (solved by E.)

→ Classification Schemes:

- $\phi: (N_1, \alpha_1) \longrightarrow (N_2, \alpha_2)$ w/ $\xi_i = \ker(\alpha_i)$
contactomorphism if $\phi^* \alpha_2 = g\alpha_1$ $(g>0)$ (and ϕ diffeo)

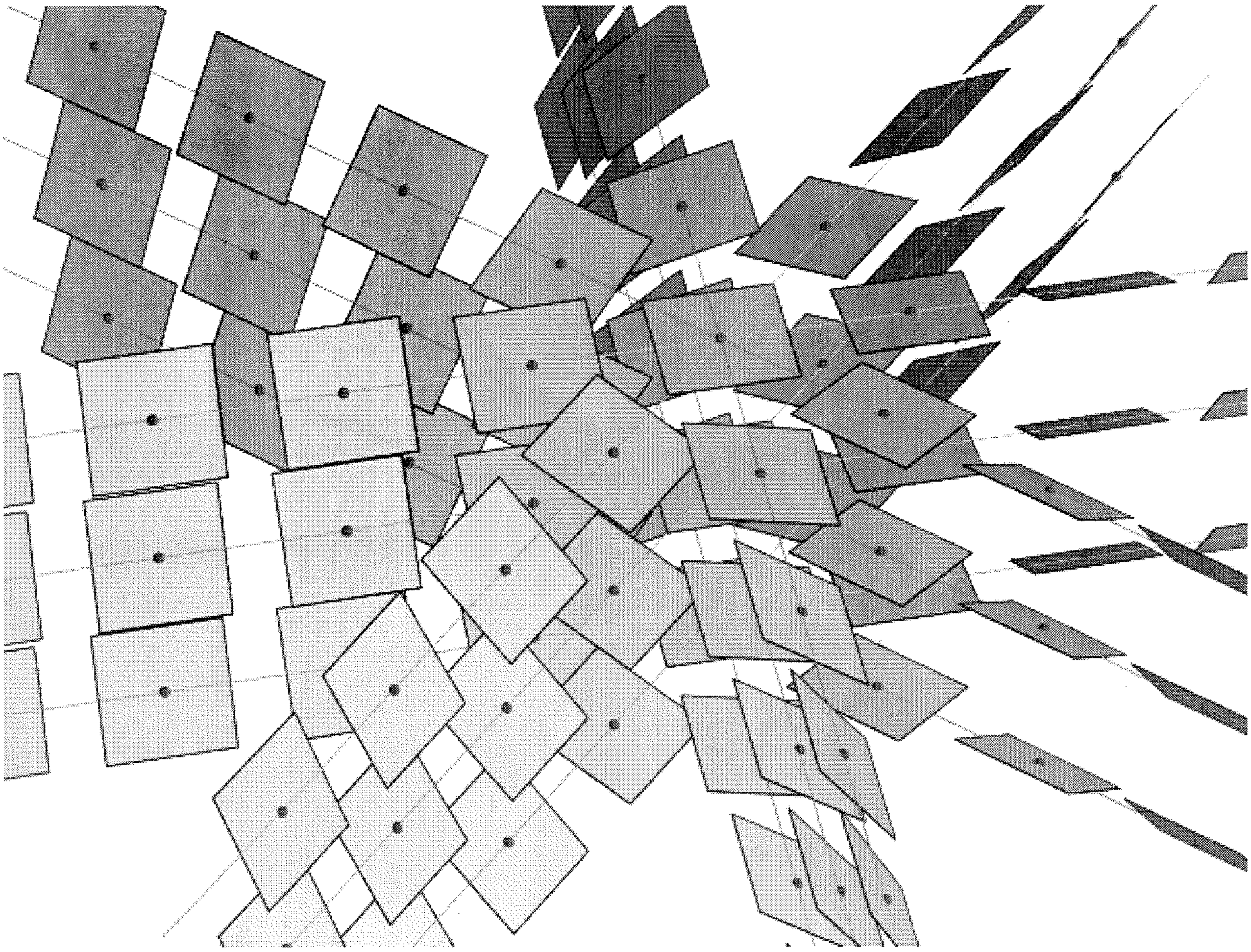
- $\phi: (N_1, \alpha_1) \longrightarrow (N_2, \alpha_2)$ isotopy if ϕ isotopic to id.

(Eliashberg)
Thm: Isotopy classes of overtwisted structures on N^3 are determined by htopy classes of underlying 2-plane fields.

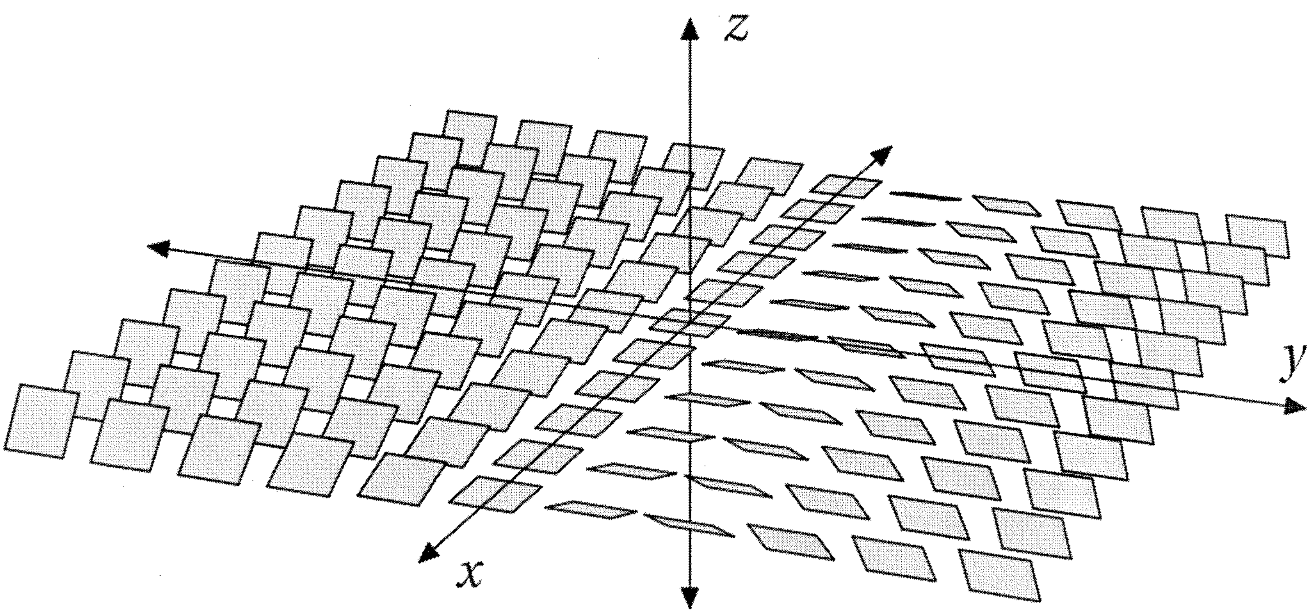
(Giroux '02)

Thm: $\left\{\begin{array}{l}\text{Isotopy classes of}\\ \text{contact str. on } N\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{open books on } N\\ \text{up to pos. stabilizations}\end{array}\right\}$

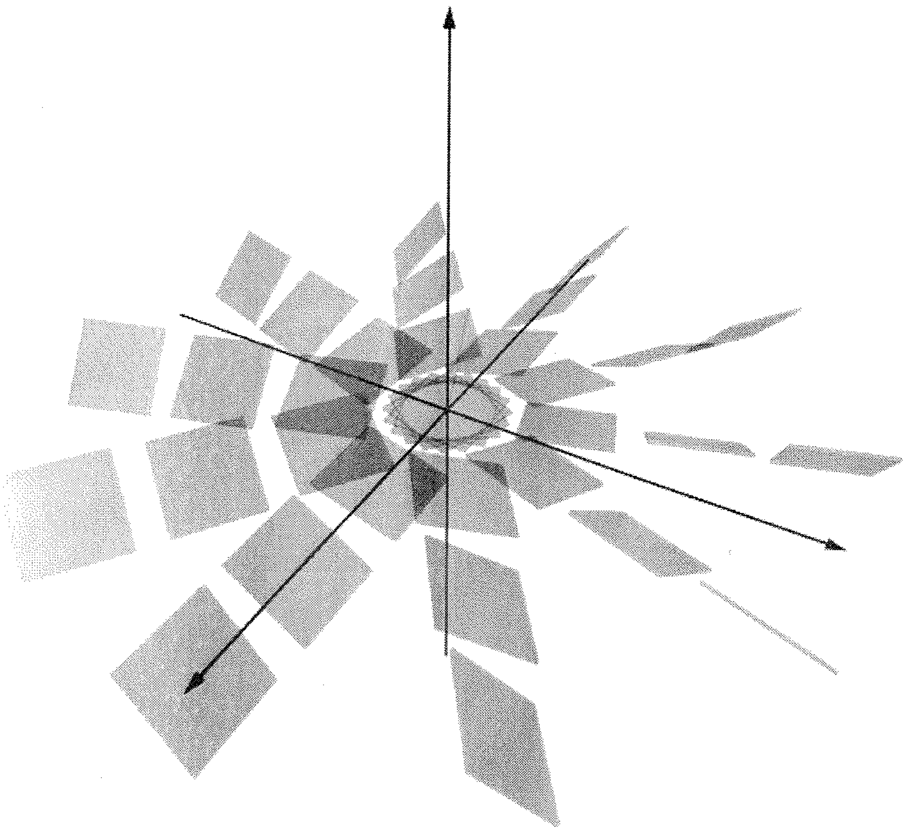
Next: "Fillability" then 5 manifolds.



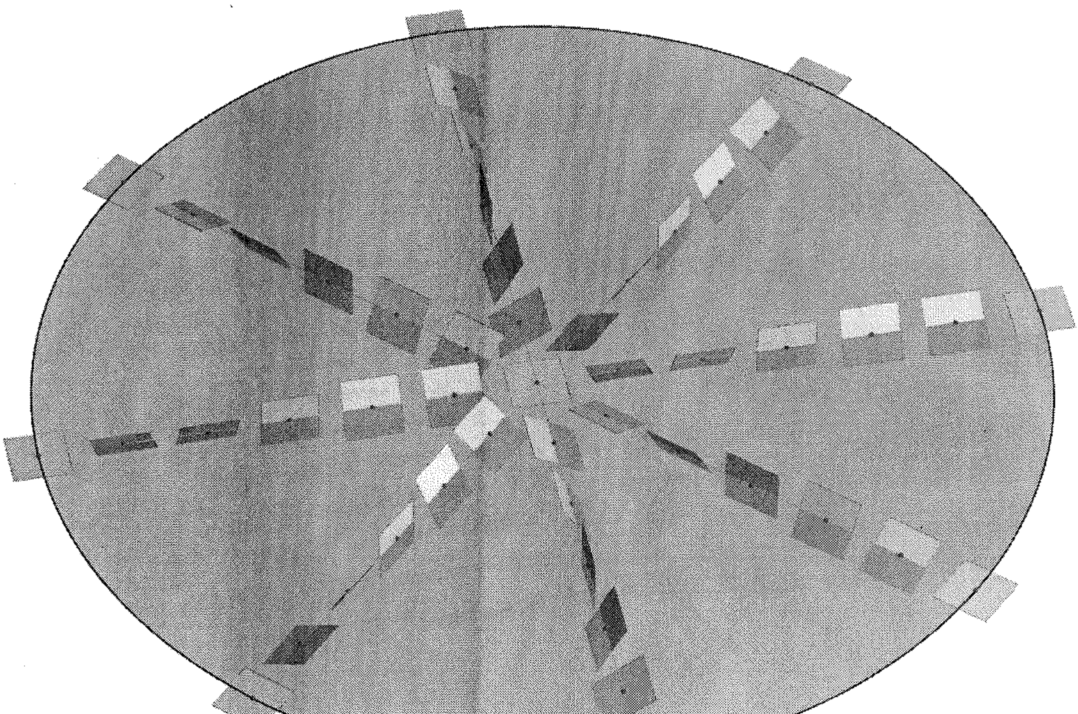
kernel(dz + r^2dθ) Standard contact structure



kernel(dz-ydx) Standard contact structure



Standard contact structure (tight)



kernel(cos(r)dz + rsin(r)dθ) (overtwisted)