Previously: 4-manifold topology & SW invariants

For 1-ctd, symplectic 4-mflds (w/ $b_2^+ > 1$) of "simple type" w/ no 2-torsion ($\Rightarrow$ SW defined on basic classes (cohomology))

Seiberg-Witten Invariants are a function
$$SW_X : H^2(X, \mathbb{Z}) \longrightarrow \mathbb{Z}/_{\pm}$$
nonzero at finitely many classes ("basic classes")

Note: $SW_X(\alpha) = \mp SW_X(-\alpha)$

Example: $SW_{E(2)}(c_1(E(2))) = \mp 1$
(this will be true for any 4-mfld)

$\rightarrow$ Now consider SW_X as a Lorentz polynomial in $\mathbb{Z}H^2(X, \mathbb{Z})$

$$SW_X = \sum_{\alpha \in H^2(X,\mathbb{Z})}' SW_X(\alpha) \cdot t_\alpha \qquad \begin{cases} \text{If } \alpha = k\beta \text{ then} \\ t_\alpha = (t_\beta)^k \end{cases}$$

(Note: This is a Lorentz polynomial b/c basic classes come in $\pm$ pairs: $t_{-\alpha} = 1/t_\alpha = t_\alpha^{-1}$)

Example:
- $SW_{E(2)} = 1$
- $SW_{E(n)} = (t_F - t_F^{-1})^{n-2}$
- $SW_{E(6)} = t_{4F} + SW_{E(6)}(2F)\, t_{2F} + SW_{E(6)}(0)\, t_0 + SW_{E(6)}(-2F)\, t_{2F}^{-1} + \cdots$

(Note: $E(6)$ has basic classes $\{\pm 4F, \pm 2F, 0\}$)

Knot Surgery

Let X be as above w/ essential self-intersection 0 torus, F
Suppose still simply conn. after removing nbhood of torus.

→ $E(n)$ are examples like this

Take a knot $k: S^1 \hookrightarrow S^3$.
Removing a neighborhood of knot in S^3 and crossing w/ S^1
$$[S^3 \setminus N(k)] \times S^1$$
gives a 4-mfld as above (self int 0 torus, 1-ctd)

Idea: Glue this object into hole left in X by removing F.

$$X_k = \left(X \setminus N(F)\right) \coprod_{\partial} \left([S^3 \setminus N(k)] \times S^1\right)$$

Note:
- X_k is well-defined 1-ctd mfld
- X_k is homeom to X
- X_k is not diffeo to X

$$SW_{X_k} = SW_X \cdot \underbrace{\Delta_k}_{\text{the Alexander polynom. of } K}$$

EX: Let $X = E(2)$
$k =$ trefoil knot $\rightsquigarrow \Delta_k = t - 1 + t^{-1}$

$SW_{E(2)} = 1$

but $SW_{E(2)_k} = t - 1 + t^{-1}$

Remark: If Δ_k is not a monic polynom,
then X_k is not symplectic

Cor: X_k is <u>not</u> symplectic if k is <u>not</u> fibered.

J-holomorphic Curves and Gromov-Witten Invariants

Recall: If M is symplectic, then it has a family of compatible almost-$\mathbb{C}$ structures
→ contractible family, so we can take a "generic structure" and call it "J".

Def: A <u>J-holomorphic curve</u> in M is a map
$$u: \Sigma_g \longrightarrow M$$
from Riemann surf. to M w/
$$J \circ du = du \circ i$$
(i is $\mathbb{C}$-structure on Σ_g)

Let $[A] \in H_2(M, \mathbb{Z})$ and $k \in \mathbb{Z}^+$.

Write $\mathcal{M}^M_{[A],g,k} := \left\{ \begin{array}{l} \text{Moduli space of genus } g \text{ J-holom curves} \simeq [A] \\ \text{with } k \text{ fixed points} \end{array} \right\}$

Note: This moduli space may not be compact
→ compactify! (only need dim = 1 of curves)

- For now let $g=0$. Assume $\dim M \le 6$.

→ Then $\overline{\mathcal{M}}^M_{[A],0,k}$ is a "pseudo-cycle"

It behaves like a cycle → because of semi-positivity of manifold

and Gromov-Witten invariants are
$$GW^M_{[A],0,k}(\alpha_1, \dots, \alpha_k) = [\overline{\mathcal{M}}^M_{[A],0,k}]\, ev_1^*\alpha_1 \cup \dots \cup ev_k^*\alpha_k$$

(continued on next page)

where $\alpha_j \in H^*(M, \mathbb{Z})$ and

$$ev = (ev_1, \ldots, ev_k) : \overline{\mathcal{M}}^M_{[A]0k} \longrightarrow \overbrace{M \times M \times \cdots \times M}^{k \text{ times}}$$

by $\quad (u, x_1, \ldots, x_k) \longmapsto (u(x_1), \ldots, u(x_k))$

<u>Note</u>: $\dim \overline{\mathcal{M}}^M_{[A]0k} = 2n - 6 + c_1(M)[A] + 2k$

<u>Dimension condition</u>: If $GW \neq 0$ then

$$\sum_{j=1}^{k} e_j = \dim \overline{\mathcal{M}}^M_{[A]0k} .$$