

Previously: 4-manifold topology \ncong SW invariants

For 1-ctd, symplectic 4-mflds ($b_2^+ > 1$) of "simple type"
w/ no 2-torsion (\Rightarrow SW defined on basic classes (cohomology))

Seiberg-Witten Invariants are a function

$$SW_x : H^2(X, \mathbb{Z}) \rightarrow \mathbb{Z}_{\pm}$$

nonzero at finitely many classes ("basic classes")

Note: $SW_x(\alpha) = \mp SW_x(-\alpha)$

Example: $SW_{E(2)}(c_1(E(2))) = \mp 1$

(this will be true for any 4-mfd)

→ Now consider SW_x as a Laurent polynomial in $\mathbb{Z}[H^2(X, \mathbb{Z})]$

$$SW_x = \sum_{\alpha \in H^2(X, \mathbb{Z})} SW_x(\alpha) \cdot t_\alpha \quad \left\{ \begin{array}{l} \text{If } \alpha = k\beta \text{ then} \\ t_\alpha = (t_\beta)^k \end{array} \right.$$

(Note: This is a Laurent polynomial b/c
basic classes come in \pm pairs: $t_\alpha = 1/t_{-\alpha} = t_\alpha^{-1}$)

Example: • $SW_{E(2)} = 1$

• $SW_{E(n)} = (t_F - t_F^{-1})^{n-2}$

• $SW_{E(6)} = t_{4F} + SW_{E(6)}(2F)t_{2F} + SW_{E(6)}(0)t_0 + SW_{E(6)}(-2F)t_{2F}^{-1} + \dots$

(Note: $E(6)$ has basic classes $\{\pm 4F, \pm 2F, 0\}$)

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Knot Surgery

Let X be as above w/ essential self-intersections \cap torus, F
 Suppose still simply conn. after removing nbhd of torus.

→ $E(n)$ are examples like this

Take a knot $k: S^1 \hookrightarrow S^3$.

Removing a neighbourhood of knot in S^3 and crossing w/ S^1

$$[S^3 \setminus N(k)] \times S^1$$

gives a 4-mfld as above (self int \cap torus, 1-ctd)

Idea: Glue this kn object into hole left in X by removing F .

$$X_k = (X \setminus N(F)) \coprod ([S^3 \setminus N(k)] \times S^1)$$

Note: $\circ X_k$ is well-defined 1-ctd mfd

$\circ X_k$ is homeom to X

$\circ X_k$ is not diffeo to X

$$SW_{X_k} = SW_X \cdot \underbrace{\Delta_k}_{\text{the Alexander polynom. of } k}$$

the Alexander polynom. of k .

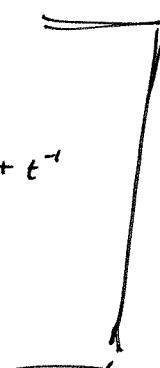


Ex: Let $X = E(2)$

$k =$ trefoil knot $\Rightarrow \Delta_k = t - 1 + t^{-1}$

$$SW_{E(2)} = 1$$

$$\text{but } SW_{E(2)_k} = t - 1 + t^{-1}$$



Remark: If Δ_k is not a monic polynomial,
 then X_k is not symplectic

Cor: X_k is not symplectic if k is not fibered.

J-holomorphic Curves and Gromov-Witten Invariants

Recall: If M is symplectic, then it has a family of compatible almost- \mathbb{C} structures
 → contractible family, so we can take a "generic structure" and call it " J ".

Def: A J-holomorphic curve in M is a map

$$u: \Sigma_g \longrightarrow M$$

from Riemann surf. to M w/
 $J \circ du = du \circ i$
 $(i$ is \mathbb{C} -structure on Σ_g)

Let $[A] \in H_2(M, \mathbb{Z})$ and $k \in \mathbb{Z}^+$.

Write $\mathcal{M}_{[A], g, k}^M := \left\{ \begin{array}{l} \text{Moduli space of genus } g \text{ J-holom. curves } \simeq [A] \\ \text{with } k \text{ fixed points} \end{array} \right\}$

Note: This moduli space may not be compact

→ compactify! (only need $\dim M = 6$ of curves)

For now let $g=0$. Assume $\dim M \leq 6$.

Then $\mathcal{M}_{[A], 0, k}^M$ is a "pseudo-cycle"

and Gromov-Witten invariants are

$$GW_{[A], 0, k}^M(\alpha_1, \dots, \alpha_k) = [\mathcal{M}_{[A], 0, k}^M] \cdot ev_1^*\alpha_1 \cup \dots \cup ev_k^*\alpha_k$$

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It behaves like a cycle
 → because of semi-positivity of manifold

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where $\alpha_j \in H^*(M, \mathbb{Z})$ and

$$ev = (ev_1, \dots, ev_k) : \overline{\mathcal{M}}_{[A] \text{ or } k}^M \longrightarrow \underbrace{M \times M \times \dots \times M}_{k \text{ times}}$$

by

$$(u, x_1, \dots, x_k) \mapsto (u(x_1), \dots, u(x_k))$$

Note: $\dim \overline{\mathcal{M}}_{[A] \text{ or } k}^M = 2n - 6 + c_1(M)[A] + 2k$

Dimension condition: If $GW \neq 0$ then

$$\sum_{j=1}^k e_j = \dim \overline{\mathcal{M}}_{[A] \text{ or } k}^M.$$