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Ahmet Beyaz — Symplectic Topology — 21. 2. 2013

(see chapter 10 of Gompf and Stipsicz. 4-Mflds & Kirby Calculus)

1- Intersection Forms of 4-Manifolds

Let X be smooth, closed, oriented 4-mfld

Homology

$$\begin{array}{ccccc} H_0 & H_1 & H_2 & H_3 & H_4 \\ \| & \| & \| \boxed{P.D.} & \| & \| \\ \mathbb{Z} & 0 & H^2 & 0 & \mathbb{Z} \end{array}$$

simply connected

Torsion free $\Rightarrow$ $H_2 \cong H^2$ lattices over $\mathbb{Z}$

Poincare Duality gives symmetric bilinear form

$$Q_X : H_2 \times H_2 \longrightarrow \mathbb{Z}$$

$$\updownarrow$$

$$H^2 \times H^2 \longrightarrow \mathbb{Z}$$

$$\alpha, \beta \longmapsto (\alpha \cup \beta)[X]$$

Intersection Form

(Thom): The map Q_X takes representing subsurfaces and does transverse intersection in X

$$Q_X([A],[B]) = A \pitchfork B$$

'82 (Freedman): X_1 is homeom. to X_2 (given $\pi_1(X_1) \cong \pi_1(X_2)$) iff Q_{X_1} equivalent to Q_{X_2}

EX: $\mathbb{CP}^2$ has $H_2 = \langle \mathbb{CP}^1 \rangle$, $Q_{\mathbb{CP}^2}([\mathbb{CP}^1],[\mathbb{CP}^1]) = 1$

$$Q_{\mathbb{CP}^2} = \langle 1 \rangle$$

1x1 matrix

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EX: $\overline{\mathbb{C}P}^2$ ($\mathbb{C}P^2$ w/ reverse orientation)

has $H_2 = \langle e \rangle$ w/ $Q_{\overline{\mathbb{C}P}^2}(e,e) = -1$

$$Q_{\overline{\mathbb{C}P}^2} = (-1) \quad \text{(1x1 matrix)}$$

EX $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ (connect sum)

has $H_2 = \langle \mathbb{C}P^1, e \rangle$

$$Q_{\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

EX $S^2 \times S^2$ has $H_2 = \langle * \times S^2, S^2 \times * \rangle$

$$Q_{S^2 \times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

If we diagonalize the matrix for Q_X over $\mathbb{R}$,

- $\#(+ \text{ entries}) = b_2^+(X)$
- $\#(- \text{ entries}) = b_2^-(X)$

2nd Betti # $b_2(X) = b_2^+ + b_2^-$

signature $\sigma(X) = b_2^- - b_2^-$

(Freeman-Davis): Homeomorphism type of X is completely determined by:

- π_1
- $b_2(X)$
- $\sigma(X)$
- parity of Q_X

(3) 2 - Symplectic Manifolds

Let M be smooth, closed $2n$-dim'l manifold

Def: A symplectic structure on M is a 2-form ω which is
- closed $(d\omega = 0)$
- nondegenerate $(\underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_{n \text{ times}} > 0)$

Def: An almost complex structure on M is a linear map
$$J : TM \longrightarrow TM \quad w/ \quad J^2 \big|_{TM_x} = -id.$$

Furthermore J and ω are compatible if
$\omega(\cdot, J\cdot)$ is a Riemannian metric.

EX S^2
Σ_g Reimannian Surf. w/ ω = volume form

EX $\mathbb{R}^{2n}$
$\mathbb{D}^{2n}$ w/ $\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$

EX $\mathbb{C}P^n$, any Kahler manifold

EX: X^4 symplectic $\Longrightarrow X \# \overline{\mathbb{C}P^2}$ (blow-up) is symplectic.

EX: (X_1, ω_1) and (X_2, ω_2) symplectic $\Longrightarrow \left(X_1 \times X_2, \pi_1^* \omega_1 + \pi_2^* \omega_2\right)$ symplectic.

Are all $2n$-dim'l manifolds symplectic??

→ No. Try S^{2n} $(n>1)$.

Not symplectic b/c $H_2 = 0$.

Ha! There can't be a symplectic form.

(4) Even w/ $H^2 \neq 0$ mfld can fail to be symplectic

→ $S^2 \times S^{2n}$ $(n > 1)$

→ $S^1 \times S^3$

→ $2\mathbb{C}P^2 = \mathbb{C}P^2 \# \mathbb{C}P^2$ (next time)