Recall: $\mathfrak{sl}_2$ is 2×2 matrices w/ trace $= 0$

↳ Lie algebra under usual commutator

↳ generators are:

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$X_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$X_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

↳ relations are:

$$[H, X_+] = HX_+ - X_+H = 2X_+$$

$$[H, X_-] = 2X_-$$

$$[X_+, X_-] = H$$

Recall: $U(\mathfrak{sl}_2)$ = universal enveloping algebra of $\mathfrak{sl}_2$

= freely generated by H, X_+, X_- modulo

$$HX_+ - X_+H = 2X_+$$
$$HX_- - X_-H = 2X_-$$
$$X_+X_- - X_-X_+ = H$$

Hopf algebra w/

$$\begin{cases} \Delta(X_+) = X_+ \otimes 1 + 1 \otimes X_+ \\ \Delta(X_-) = X_- \otimes 1 + 1 \otimes X_- \\ \Delta(H) = H \otimes 1 + 1 \otimes H \end{cases}$$

(noncommutative but co-commutative Hopf algebra)

$$\eta(X_+) = \eta(X_-) = \eta(H) = 0$$

$$\begin{cases} S(X_+) = -X_+ \\ S(X_-) = -X_- \end{cases}$$

$U_q(sl_2)$ $\quad (q \in \mathbb{C}^*)$

As an algebra $U_q(sl_2)$ is freely generated by E, F, g, g^{-1} modulo some relations

$$U_q(sl_2) = \mathbb{C}\langle E, F, g, g^{-1}\rangle \big/ \sim$$

$$\begin{cases} gEg^{-1} \sim q^2 E \\ gFg^{-1} \sim q^{-2} F \\ [E,F] \sim \dfrac{g - g^{-1}}{q - q^{-1}} & (q \neq \pm 1) \end{cases}$$

Hopf algebra w/

$$\begin{cases} \Delta(E) = E \otimes g + 1 \otimes E \\ \Delta(F) = F \otimes 1 + g^{-1} \otimes F \\ \Delta(g) = g \otimes g \end{cases}$$

$$\eta(g) = 1 \quad \text{and} \quad \eta(E) = \eta(F) = 0$$

$$\begin{cases} S(E) = -Eg^{-1} \\ S(F) = -gF \\ S(g) = g^{-1} \end{cases}$$

Relationship to $U(sl_2)$:

$$\left(\text{Intermediate notation: } U_{q,n}(sl_2) = U_q(sl_2) \big/ \sim \quad \begin{cases} g^n \sim 1 \\ E^n \sim 0 \\ F^n \sim 0 \end{cases} \text{ w/ } q^n = 1 \right)$$

$$\left[\begin{array}{l} q = e^h \\ g = q^H = e^{hH} = I + hH + h^2 \dfrac{H^2}{2!} + h^3 \dfrac{H^3}{3!} + \cdots \end{array} \right]$$

<u>Fact</u> ~~All~~ finite dimensional irred representations of $sl_2(\mathbb{C})$ are indexed by natural #'s.

The n^{th} one is

$$H \longleftrightarrow \begin{bmatrix} n-1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 3 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & -3 & \\ 0 & & & & & & \ddots \\ & & & & & & -n+1 \end{bmatrix} \quad (n \text{ even})$$

$$\begin{bmatrix} n-1 & & & & & 0 \\ & \ddots & & & & \\ & & 2 & & & \\ & & & 0 & & \\ & & & & -2 & \\ 0 & & & & & \ddots \\ & & & & & & -n+1 \end{bmatrix} \quad (n \text{ odd})$$

$$X_+ \longleftrightarrow \begin{bmatrix} 0 & 1 & & & 0 \\ & \ddots & 2 & & \\ & & \ddots & \ddots & \\ & & & & n-1 \\ 0 & & & & 0 \end{bmatrix}$$

$$X_- \longleftrightarrow \begin{bmatrix} 0 & & & & 0 \\ n-1 & \ddots & & & \\ & \ddots & \ddots & & \\ & & 2 & & \\ 0 & & & 1 & 0 \end{bmatrix}$$

<u>Note:</u> $(X_+)^n = (X_-)^n = 0$

$U_{q,n}(sl_2) \sim$ irred rep'n indexed by n.

in this rep'n: $g = q^H = \begin{bmatrix} q^{n-1} & & & 0 \\ & q^{n-3} & & \\ 0 & & \ddots & \\ & & & q^{-n+1} \end{bmatrix}$

$(E)^n = 0$ $\qquad E = X_+ q^{H/2} = \begin{bmatrix} 0 & q^{\frac{n-3}{2}} & & & 0 \\ & \ddots & 2q^{\frac{n-5}{2}} & & \\ & & \ddots & \ddots & \\ 0 & & & 0 & (n-1)q^{\frac{-n+1}{2}} \end{bmatrix}$

$(F)^n = 0$ $\qquad F = q^{-H/2} X_- = \begin{bmatrix} 0 & & & 0 \\ (n-1)q^{\frac{-n+1}{2}} & \ddots & & \\ & \ddots & \ddots & \end{bmatrix}$