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Goal: Computations in $u_q(sl(2))$

q-binomials (moral: everything has "q-analogue")

Let $q \in k^*$. $\left[{n \atop m}\right]_q = \dfrac{[n]_q!}{[m]_q!\,[n-m]_q!}$ w/ $\begin{cases} [m]_q = \dfrac{1-q^m}{1-q} \\ \quad = 1+\cdots+q^{m-1} \\ [m]_q! = [m]_q \cdot [m-1]_q \cdots \end{cases}$

Lemma (q-Binomial formula):

If A, B are algebra elements w/ $BA = qAB$

then $$(A+B)^n = \sum_{m=0}^{n} \left[{n \atop m}\right]_q A^m B^{n-m}$$

Proof (induction)

$n=1$: $(A+B) = \sum_{m=0}^{1} \left[{1 \atop m}\right]_q A^m B^{n-m} = \left[{1 \atop 0}\right]_q A + \left[{1 \atop 1}\right]_q B$

$= A + B. \quad ([0]_q = 1)$

induction: $(A+B)^n = (A+B)^{n-1}(A+B)$

$$= \left(\sum_{m=0}^{n-1} \left[{n-1 \atop m}\right]_q A^m B^{n-1-m}\right)(A+B)$$

$$= \sum_{m=1}^{n} \left[{n-1 \atop m-1}\right]_q q^{n-m} A^m B^{n-m}$$ shift in index

$$+ \sum_{m=0}^{n-1} \left[{n-1 \atop m}\right] A^m B^{n-m}$$

$$= A^n + B^n + \sum_{m=1}^{n-1} \left(q^{n-m} \left[{n-1 \atop m-1}\right]_q + \left[{n-1 \atop m}\right]_q\right) A^m B^{n-m}$$

→ **Claim** $q^{n-m}\left[{n-1 \atop m-1}\right]_q + \left[{n-1 \atop m}\right]_q = \left[{n \atop m}\right]_q$

((Follows from $q^{n-m}[m]_q + [n-m]_q = [n]_q$

$q^{n-m}(1+q+\cdots+q^{m-1}) + (1+q+\cdots+q^{n-m-1}) = 1+q+\cdots+q^{n-1}$))

$$q^{n-m}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ m \end{bmatrix}_q = q^{n-m}\frac{[n-1]_q!}{[m-1]_q!\,[n-m]_q!} + \frac{[n-1]_q!}{[m]_q!\,[n-m-1]_q!}$$

$$= \frac{[n-1]_q!}{[m]_q!\,[n-m]_q!}\underbrace{\left(q^{n-m}[m]_q + [n-m]_q\right)}_{[n]_q}$$

$$= \begin{bmatrix} n \\ m \end{bmatrix}_q$$

q-Exponentials

Def: $e_q^A = \sum_{m=0}^{\infty}{}^{'} \frac{A^m}{[m]_q!}$ (Similar to power series $e^x = \sum{}^{'} \frac{x^n}{n!}$)

↳ In case of convergence worries, say "it is a formal power series"

Lemma: If $BA = qAB$
then $$e_q^{A+B} = e_q^A \cdot e_q^B$$

← Note order of sum: $e_q^{B+A} = e_{q^{-1}}^{B} \cdot e_{q^{-1}}^{A}$

Proof

$$e_q^{A+B} = \sum{}^{'} \frac{(A+B)^n}{[n]_q!} = \sum{}^{'}\sum{}^{'} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{A^m B^{n-m}}{[n]_q!}$$

$$= \sum{}^{'}\sum{}^{'} \frac{A^m}{[m]_q!} \cdot \frac{B^{n-m}}{[n-m]_q!}$$

$(k = n-m)$
$$= \sum{}^{'}\sum{}^{'} \frac{A^m}{[m]_q!} \cdot \frac{B^k}{[k]_q!} = e_q^A \cdot e_q^B$$ ∎

((Now we will prove two helper lemmas to be used later...))

Lemma If $\left\{\begin{array}{l} BA = qAB \\ AD = qDA \\ [C, A] = B - D \end{array}\right\}$ then $[C, A^n] = [n]_q (A^{n-1}B - DA^{n-1})$

Proof (induction):

$\underline{n=1}$ $\quad [C, A^1] = [1]_q (B - D) = B - D \quad$ (assumption)

$\underline{\text{induction}}$
$$\begin{aligned}
[C, A^n] &= CA^n - A^n C \\
&= CA^n - ACA^{n-1} + ACA^{n-1} - A^n C \\
&= [C, A] A^{n-1} + A [C, A^{n-1}] \\
&= (B - D) A^{n-1} + A [n-1]_q (A^{n-2}B - DA^{n-2}) \\
&= (\underbrace{q^{n-1} + [n-1]_q}_{q^{n-1}[1]_q + [n-1]_q}) A^{n-1} B - (\underbrace{1 + q [n-1]_q}_{[1]_q + q^1 [n-1]_q}) DA^{n-1} \\
&= [n]_q A^{n-1} B - [n]_q DA^{n-1}
\end{aligned}$$
$\blacksquare$

Lemma: If $\left\{\begin{array}{l} BA = qAB \\ AD = qDA \\ [C, A] = B - D \end{array}\right\}$ then $[C, e_q^A] = e_q^A B - D e_q^A$

Proof:
$$\begin{aligned}
[C, e_q^A] &= \sum_{n=0}^{\infty}{}' \frac{[C, A^n]}{[n]_q!} \\
&= \sum_{n=0}^{\infty}{}' \frac{[n]_q (A^{n-1}B - DA^{n-1})}{[n]_q!} \\
&= \sum_{1}^{\infty}{}' \frac{A^{n-1}}{[n-1]_q!} B - D \sum_{1}^{\infty}{}' \frac{A^{n-1}}{[n-1]_q!} = e_q^A B - D e_q^A
\end{aligned}$$

(note $n=0$ term: $[C, A^0] = C - C = 0$)

$\blacksquare$

(Step 1 $\mathcal{U}_q(sl_2)$

Step 2 $u_q(sl_2)$)

- $\mathcal{U}_q(sl_2)$

Recall $sl_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ w/ } tr = 0 \right\}$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad X_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad X_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$\rightsquigarrow$ H, $X_\pm$ generate sl_2 as a vector space & $\mathcal{U}(sl_2)$ as algebra

relation: $[H, X_\pm] = \pm 2 X_\pm$

$[X_+, X_-] = H$

"Quantize" $\mathcal{U}(sl_2)$:

Pick $q \in \mathbb{C}^*$.

Def: $\mathcal{U}_q(sl_2)$ is $\mathbb{C}\langle E, F, g, g^{-1} \rangle$ / relations

relations: $gEg^{-1} = q^2 E$

$gFg^{-1} = q^{-2} F$

$[E, F] = \dfrac{g - g^{-1}}{q - q^{-1}}$

connectio

$g = q^H = e$

$E = X_+ \cancel{g}$

$F = g^{-H/2} X_-$

$g^{H/2} X_+$

$g^{-H/2} X_-$

$$gEg^{-1} = q^2 E \longleftrightarrow (I + hH + \cdots)(X_+ q^{H/2})(I - hH + \cdots)$$

$$\| $$

$$(I + hH + \cdots)(X_+)(I - hH + \cdots) q^{H/2}$$

$$\| $$

$$(X_+ + h[H, X_+] + \cdots) q^{H/2}$$

$$\| $$

$$e^{2h}(X_+ q^{H/2})$$

Coalgebra structure:

$\Delta g = g \otimes g$ $\quad$ $\Delta E = E \otimes g + 1 \otimes E$ $\quad$ $\left(\begin{array}{l} \eta(E) = \eta(F) = 0 \\ S(E) = -Eg^{-1} \end{array} \right)$

$(\eta(g) = 1$ $\quad$ $\Delta F = F \otimes 1 + g^{-1} \otimes F$