<u>Recall</u>: $M$ is a geometric object
$A$ an algebra of functions on $M$
} often, $M$ can be recovered from $A$.

<u>Idea</u>: Replace comm. alg. $A$ by something noncomm.
→ Make sense of what "$M$" is...

Suppose $M = G$ is an affine algebraic group / alg. closed field $k$

→ Let $A = \mathcal{O}(G)$ be ring of regular functions on $G$

→ Classical $G = \mathrm{Spec}(A)$.

↳ group structure? (does not get encoded in alg str. of $A$)

- Encoded in <u>coalgebra</u> structure of $A$.

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<u>Definitions of algebra & coalgebra</u>:

$(A, m, \varepsilon)$ is an <u>algebra</u> if $m: A \otimes A \to A$ (multiplication)
$\varepsilon: k \to A$ (unit)

<u>with</u>
- $m$ associative

$$A \otimes A \otimes A \overset{m \otimes 1}{\underset{1 \otimes m}{\rightrightarrows}} A \otimes A \xrightarrow{m} A$$

- $\varepsilon$ is unit for $m$ ($m$ is compatible w/ $\varepsilon$)

$$\begin{tikzcd} k \otimes A \arrow[r, "\varepsilon \otimes 1"] \arrow[rr, bend right, "\cong"'] & A \otimes A \arrow[r, "m"] & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \otimes k \arrow[r, "1 \otimes \varepsilon"] \arrow[rr, bend right, "\cong"'] & A \otimes A \arrow[r, "m"] & A \end{tikzcd}$$

$(C, \Delta, \eta)$ is a <u>(counital) coalgebra</u> if $\Delta: C \to C \otimes C$ (com…
$\eta: C \to k$ (coun…

<u>with</u>
- $\Delta$ coassociative

$$C \longrightarrow C \otimes C \rightrightarrows C \otimes C \otimes C$$

- $\Delta$ compatible w/ $\eta$

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[rr, bend right, "\cong"'] & C \otimes C \arrow[r, "\eta \otimes 1"] & k \otimes C \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C \arrow[r, "\Delta"] \arrow[rr, bend right, "\cong"'] & C \otimes C \arrow[r, "1 \otimes \eta"] & C \otimes k \end{tikzcd}$$

Morphisms of algebras & coalgebras are defined as expected...
(all $m, \varepsilon$ / $\Delta, \eta$ structure maps commute w/ morphism)

**Def:** A bialgebra has both structures, and they are compatible:

$(A, m, \varepsilon, \Delta, \eta)$ w/
- $(A, m, \varepsilon)$ algebra
- $(A, \Delta, \eta)$ coalgebra
- $\begin{cases} \Delta, \eta & \text{algebra morphisms} \\ m, \varepsilon & \text{coalgebra morphisms} \end{cases}$

(Note: $\Delta$ an algebra morphism means

$$\begin{tikzcd} A \otimes A \arrow[r, "\Delta_\otimes"] \arrow[d, "m_A"'] & (A \otimes A) \otimes (A \otimes A) \arrow[d, "m_\otimes"] \\ A \arrow[r, "\Delta_A"] & A \otimes A \end{tikzcd}$$
)

$\Rightarrow$ Back to our example: $A = \mathcal{O}(G)$, $G$ an affine group.
$A$ an algebra.

Then $A$ has coproduct $\Delta : A \to A \otimes A = \mathcal{O}(G) \otimes \mathcal{O}(G) \cong \mathcal{O}(G \times G)$

(by $\Delta(f)(x, y) = f(xy)$)

(coassoc b/c
$(\mathrm{id} \otimes \Delta)(\Delta f)(x, y, z) = f(x(yz))$
$(\Delta \otimes \mathrm{id})(\Delta f)(x, y, z) = f((xy)z)$ )

The inverse of $G$ is encoded in map
$$S : A \to A$$
w/ $S(f)(x) = f(x^{-1})$

$\Rightarrow$ $A$ is a Hopf algebra.

**Def:** Let $A$ be a bialgebra. $S : A \to A$ is an antipode if

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[d, "\Delta"'] & k \arrow[r, "\varepsilon"] & A \\ A \otimes A \arrow[rr, "S \otimes 1"] & & A \otimes A \arrow[u, "m"'] \end{tikzcd}$$

**Def:** A Hopf algebra is a bialgebra w/ anti-pode.

⟹ Example 1: $A = \mathcal{O}(G)$ as already discussed.

⟹ Example 2: $A = \mathcal{U}(\mathfrak{g})$ univ. envelop. alg. of Lie alg. $\mathfrak{g}$

Recall: $T\mathfrak{g} = k \oplus \mathfrak{g} \oplus \mathfrak{g}\otimes\mathfrak{g} \oplus \cdots$ tensor alg

$$\mathcal{U}\mathfrak{g} = T\mathfrak{g} \big/ (x\otimes y - y\otimes x - [x,y])$$

coalg. structure from
- $\Delta x = x\otimes 1 + 1\otimes x$ for $x \in \mathfrak{g}$ extended to $T\mathfrak{g}$ by alg hom
- $\eta x = 0$ for $x \in \mathfrak{g}^{\otimes > 0}$
- $Sx = -x$ for $x \in \mathfrak{g}$

$\mathcal{O}(G)$ vs. $\mathcal{U}(\mathfrak{g})$ vs. Quantum Groups

| | |
|---|---|
| $\mathcal{O}(G)$: commutative, **not** co-commutative | $\mathcal{U}(\mathfrak{g})$: **not** commutative, co-commutative |

Quantum Groups: **not** commutative, **not** co-commutative