Quasitriangular structures

Idea: want things which are ~~both~~ commutative & not cocommutative BUT are still close to being cocommutative.

Def: A quasitriangular bialgebra/Hopf algebra is
$$(H, R)$$
where (1) H is a bialgebra/Hopf algebra

(2) $R \in H \otimes H$ is invertible and obeys
$$\tau \circ \Delta h = R(\Delta h)R^{-1}$$

(3) $(\Delta \otimes id)R = R_{13}R_{23} \in H \otimes H \otimes H$

$(id \otimes \Delta)R = R_{13}R_{12} \in H \otimes H \otimes H$

If $R = a_i \otimes b_i \in H \otimes H$

$R_{13} = a_i \otimes 1 \otimes b_i \in H \otimes H \otimes H$

$R_{23} = 1 \otimes a_i \otimes b_i \in H \otimes H \otimes H$

$R_{12} = a_i \otimes b_i \otimes 1 \in H \otimes H \otimes H$

Lemma: Suppose (H, R) is quasi-triangular bialgebra. Then:

(1) $(\eta \otimes id)R = (id \otimes \eta)R = 1$

(2) (H, R_{21}^{-1}) is also quasi-triangular $\quad (R_{21} = b_i \otimes a_i)$

(3) $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ (Yang-Baxter Equation)

Proof:

(1) $(\eta \otimes id \otimes id)(\Delta \otimes id)R = R_{23} \in k \otimes H \otimes H \quad$ (b/c $(\eta \otimes id)\Delta = id$)

$(\eta \otimes id \otimes id) R_{13}R_{23} =$

so $(\eta \otimes id \otimes id)R_{13} = 1_{H \otimes H \otimes H}$ (b/c R_{23} is invertible)

$\Rightarrow (\eta \otimes id) R = 1$ ...

(2) $\tau \circ \Delta h = R(\Delta h)R^{-1}$

$$\Delta h = \tau \circ (R(\Delta h)R^{-1})$$
$$= R_{21}(\tau\circ\Delta h)R_{21}^{-1}$$
$$\Longrightarrow R_{21}^{-1}(\Delta h)R_{21} = \tau\circ\Delta h$$

$$(\Delta\otimes id)R_{21}^{-1} = ((\Delta\otimes id)R_{21})^{-1}$$
$$= (R_{31}R_{32})^{-1}$$
$$= R_{32}^{-1}R_{31}^{-1}$$

...

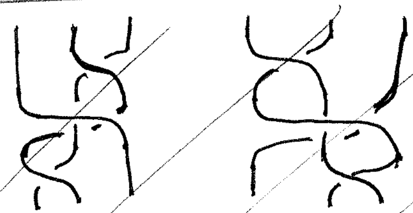
(3) $(id\otimes(\tau\circ\Delta))R = (id\otimes\tau)(id\otimes\Delta)R$

$$= (id\otimes\tau)R_{13}R_{12}$$
$$= R_{12}R_{13}$$

$$(id\otimes(\tau\circ\Delta))R = R_{23}((id\otimes\Delta)R)R_{23}^{-1}$$
$$= R_{23}(R_{13}R_{12})R_{23}^{-1}$$

So ... $R_{12}R_{13} = R_{23}R_{13}R_{12}R_{23}^{-1}$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$



Lemma: Let (H,R) be a quasitriangular Hopf algebra. Then

$$(S\otimes id)R = R^{-1}$$
$$(id\otimes S)R^{-1} = R$$
$$(S\otimes S)R = R$$

Proof:

Notation: $R = a\otimes b$ (drop the i in $a_i\otimes b_i$ for convenience)

$\Delta a = a_{(1)}\otimes a_{(2)}$ (subscript will be coproduct parts)

~~Claim: $a_{(1)}\otimes Sa_{(2)}\otimes b = 1$~~

Claim: $a_{(1)}Sa_{(2)}\otimes b = 1$

part 1: Show $a_{(1)}Sa_{(2)} = \eta(a)\cdot 1$ via $m(id\otimes S)\Delta = \eta$

part 2: Show $\eta(a)\otimes b = 1$ via $(\eta\otimes id)\circ\Delta = 1$

So $(m\otimes id)(id\otimes S\otimes id)(\Delta\otimes id)R = 1$

$\parallel$

$(m\otimes id)(id\otimes S\otimes id) R \; R$

$$(m\otimes id)(id\otimes S\otimes id)(\Delta\otimes id)R = 1$$

$$\|$$

$$(m\otimes id)(id\otimes S\otimes id)\,(a\otimes 1\otimes b)\cdot(1\otimes a\otimes b)$$

$$\cancel{(m\otimes id)\,(a\otimes Sa\otimes b^2)} \qquad (m\otimes id)(a\otimes 1\otimes b)(id\otimes S\otimes id)(1\otimes a\otimes b)$$

$$\cancel{a\,Sa\otimes b^2} \;=\!=\; R(S\otimes id)R$$

So $R(S\otimes id)R = 1$

$$\Longrightarrow (S\otimes id)R = R^{-1}$$

others are similar. $\square$