Automorphism Quantum Groups

Motivation: Recall $M_q(2)$ coacting on quantum plane

$$\mathbb{A}_q^2 \longrightarrow \mathbb{A}_q^2 \otimes M_q(2)$$

$\mathbb{A}_q^2 = k\langle x,y\rangle / (yx = qxy)$

(This can be thought of as the "automorphism group" of $\mathbb{A}_q^2$ → "coautomorphism"?

→ Generalize: starting w/ algebra A form an "automorphism quantum group" $M(A)$

Let A be an algebra.

Def: A comeasuring of A is (B, β)

- B = algebra
- $\beta : A \longrightarrow A \otimes B$ algebra map

- A morphism between comeasurings $(B,\beta) \longrightarrow (B',\beta')$ is $\phi : B \longrightarrow B'$ algebra map compatible w/ β, β'

Compatibility:

$$\begin{array}{ccc} A & = & A \\ \beta \downarrow & \circlearrowright & \downarrow \beta' \\ A\otimes B & \xrightarrow{1\otimes\phi} & A\otimes B' \end{array}$$

Let $(M(A), \beta_u)$ be the initial object in category of comeasurings (if it exists)

We will show that for A finite dimensional this will always exist.

Remark: If $(M(A), \beta_u)$ exists, then

- $M(A)$ is a bialgebra
- β_u makes A an $M(A)$ comodule algebra) (β_u gives coaction)
- $M(A)$ is the universal bialgebra w/ coaction

$\rightsquigarrow$ This is the direct analog of automorphism group.

① $M(A)$ is a bialgebra b/c $(M(A)\otimes M(A), \beta_u \otimes \beta_u)$ is comeas

universality $\Rightarrow \exists \; \Delta : M(A) \longrightarrow M(A) \otimes M(A)$

w/

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\beta_u"'] & A \arrow[d, "\beta_u \otimes \beta_u"] \\ A\otimes M(A) \arrow[r, "1\otimes\Delta"'] & A\otimes(M(A)\otimes M(A)) \end{tikzcd}$$

Actually $(\beta_u \otimes 1)\circ\beta_u$

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unit comes from comeas. $(k, \, id\otimes 1)$

$\Rightarrow \exists \; \eta : M(A) \longrightarrow k$

w/

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\beta_u"'] & A \arrow[d] \\ A\otimes M(A) \arrow[r, "1\otimes\eta"'] & A\otimes k \end{tikzcd}$$

HW: These satisfy bialgebra axioms

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③ If $\beta : A \to A\otimes B$ is another bialg w/ coaction

universality $\Rightarrow$

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\beta_u"'] & A \arrow[d, "\beta"] \\ A\otimes M(A) \arrow[r, "1\otimes\pi"'] & A\otimes B \end{tikzcd}$$

$\pi : M(A) \longrightarrow B$

HW: compatible w/ bialgebra coactions

Existence of $M(A)$:

→ Let A be finite dimensional.

Plan: Construct $M(A)$ in the case where A not necess. unital
– deal w/ unit later. (formally adjoin unit)
(In this case comeasuring means only $A \xrightarrow{\beta} A \otimes B$ multiplicative)

Fix a basis $\{e_i\}_0^{\dim(A)-1}$ of A.

- define $M_1(A) = k\langle t_j^i \rangle / \sim$ where $i, j \in \{0, \dots, \dim(A)-1\}$

  where $c_{ij}^a t_a^k \sim c_{ab}^k t_i^a t_j^b$

  c_{ij}^k are structure constants of $\{e_i\}$

  $e_i e_j = c_{ij}^k e_k$

coalg. $\begin{cases} \Delta(t_j^i) = t_a^i \otimes t_j^a & \text{(Matrix multiplication.)} \\ \eta(t_j^i) = \delta_j^i \end{cases}$

comeas. $\left(\beta_u(e_i) = e_a \otimes t_i^a \right.$

→ Check that β_u is multiplicative:

$$\begin{aligned} \beta_u(e_i e_j) = \beta_u(c_{ij}^k e_k) &= c_{ij}^k \beta_u(e_k) \\ &= c_{ij}^a (e_k \otimes t_a^k) \\ &= c_{ab}^k e_k \otimes t_i^a t_j^b \\ &= e_a e_b \otimes t_i^a t_j^b \\ &= \beta_u(e_i) \cdot \beta_u(e_j) \end{aligned}$$

swap notation $a \leftrightarrow k$.

⇝ check universality:

If (B, β) is comeas.

say $\beta(e_i) = e_a \otimes \overline{t}^a_i$

β mult $\Longrightarrow$

$$c_{ij}^{a} \, \overline{t}^{k}_{a} = c_{ab}^{k} \, \overline{t}^{a}_{i} \, \overline{t}^{b}_{j}$$

Define: $\pi : M_1(A) \longrightarrow B$ by

$$t^i_j \longmapsto \overline{t}^i_j$$