Def: Two Hopf algebras are dually paired if there is a pairing
$$\langle -,- \rangle : H \otimes H' \longrightarrow k$$
respecting algebra/coalgebra structure
$$\left.\begin{aligned} \langle \phi\psi, h \rangle &= \langle \phi \otimes \psi, \Delta h \rangle \\ \langle \phi, hg \rangle &= \langle \Delta\phi, h \otimes g \rangle \end{aligned}\right\} \text{product \& coproduct are adjoint}$$
$$\langle S\phi, h \rangle = \langle \phi, Sh \rangle$$
... unit/counit structure also.

EX: If H is finite dimensional then there is a dual pairing w/ H^* via usual dual pairing map
$$\langle -,- \rangle : H \otimes H^* \longrightarrow k$$
by evaluation
$$\langle h, \phi \rangle = \phi(h)$$

(**Recall:** $\Delta : H \to H \otimes H$ induces product $\Delta^* : H^* \otimes H^* \longrightarrow H^*$ (for finite dim'l H) etc).

Remark: "Dually paired" $\not\Longrightarrow$ perfect pairing (in general)

EX: G a finite group, $\underbrace{k(G)^*}_{\text{functions on } G} \cong \underbrace{kG}_{\text{group Hopf algebra}}$ are dually paired

EX: G affine algebraic group
(take $G \subset M_n(\mathbb{C})$)

$\mathcal{U}(\mathfrak{g})$ and $\mathbb{C}[G]$ are dually paired as follows.

$\mathfrak{g}$: Lie algebra of G

Let $\rho: \mathfrak{g} \to M_n(\mathbb{C})$ be the defining rep'n of $\mathfrak{g}$

Define
$$\langle \xi, x^i_j \rangle = \underbrace{\rho(\xi)^i_j}_{\text{is entry of matrix representing } \xi}$$

(Since the Hopf structure extends ~~freely~~ from generators, this is enough to define pairing.)

An *action* of a bialgebra (or Hopf algebra) is an action of underlying algebra structure

→ Coalgebra structure allows further tensors of representations:

V, W are H-modules then $V \otimes W$ gets H-module st.

$$h \triangleright (v \otimes w) = (h_{(1)} \triangleright v) \otimes (h_{(2)} \triangleright w)$$
$$= (\Delta h) \triangleright (v \otimes w)$$

"acting on"

Remark: So H acts on monoidal category of H-modules w/ $\otimes$ via Δ (correct notation?)

Def: A bialgebra H acts on an algebra A if

(1) H acts on A as a v.s.

(2) $m_A: A \otimes A \to A$ commutes w/ action of H

(3) $1_A: k \to A$ commutes w/ action of H.

→ $h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b)$

EX: (1) For kG:

$$g \triangleright (ab) = (g \triangleright a)(g \triangleright b)$$

Recall: $\Delta g = g \otimes g$

(2) For $\mathcal{U}(\mathfrak{g})$:

$$\xi \triangleright (ab) = (\xi \triangleright a) b + a (\xi \triangleright b)$$

(Derivation)

Recall: $\Delta \xi = \xi \otimes 1 + 1 \otimes \xi$

(3) For $k(G)$:

If A is G-graded algebra,

$\deg : A \to G$

(!) $f \triangleright a = f(\deg a) \cdot a$ if a is homogeneous

Prop: (adjoint action) Every Hopf algebra H acts on itself (viewed as an algebra) by

$$Ad_h(g) = h_{(1)} g \cdot S h_{(2)}$$

(Remark: This generalizes adjoint action on groups
if $H = kG$ then $Ad_g(h) = g h g^{-1}$ $\left\{ \text{b/c } \begin{array}{l} \Delta g = g \otimes g \\ S g = g^{-1} \end{array} \right.$)

Proof:

- $h \triangleright (g \triangleright a) = h \triangleright (g_{(1)} a \, S g_{(2)})$
$= h_{(1)} g_{(1)} a \, S g_{(2)} \, S h_{(2)}$
$= (hg)_{(1)} a \, S(hg)_{(2)}$
- $1 \triangleright a = 1 \, a \, S(1) = a$

This is an algebra action. (part (1) of def)

- $h \triangleright (ab) = h_{(1)} ab \, S h_{(2)}$
$= h_{(1)} a \, S h_{(2)} h_{(3)} b \, S h_{(4)}$
$= (h_{(1)} \triangleright a) \cdot (h_{(2)} \triangleright b)$

Recall: coassoc. says $\Delta \Delta \Delta h = h_{(1)} \otimes h_{(2)} \otimes h_{(3)} \otimes h_{(4)}$ well defined w/o more precise notation

action respects alg (coalg duality (part (2) def

- $h \triangleright 1 = h_{(1)} \cdot 1 \cdot S h_{(2)}$

Ex $H = \mathcal{U}(\mathfrak{g})$

$$Ad_\xi(h) = \xi h\, S(1) + 1\, h\, S(\xi)$$
$$= \xi h - h \xi$$
$$= [\xi, h]$$

(Note: $\Delta\xi = \xi \otimes 1 + 1 \otimes \xi$
$S\xi = -\xi$
$S1 = 1$)

Next: Coactions & examples

$\hookrightarrow SL_q(n)$ coacts on "quantum n-space"