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Quantum Groups I

(Goal: Apply quantum groups to knots)

Reference for this lecture: Shahn Majid *A Quantum Groups Primer*

Idea: Quantum groups are group-object of noncommutative geometry

Def: A unital alg A over a field k is
v.s. over k
w/ mult. map $m: A\otimes A \to A$
unit map $\varepsilon: k \to A$

equalizer

$$A\otimes A\otimes A \underset{1\otimes m}{\overset{m\otimes 1}{\rightrightarrows}} A\otimes A \xrightarrow{m} A$$

$$\begin{tikzcd} A \arrow[r, shift left, "1\otimes\varepsilon"] \arrow[r, shift right, "\varepsilon\otimes 1"'] \arrow[rr, bend right, "="'] & A\otimes A \arrow[r, "m"] & A \end{tikzcd}$$

Remark: My personal notation for unit/counit is reverse of Özgür's and Majid's.

Def: A coalgebra C over k is
v.s. over k
w/ coproduct map $\Delta: C \to C\otimes C$
counit map $\eta: C \to k$

coequalizer

$$C \xrightarrow{\Delta} C\otimes C \underset{1\otimes\Delta}{\overset{\Delta\otimes 1}{\rightrightarrows}} C\otimes C\otimes C$$

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[rr, bend right, "="'] & C\otimes C \arrow[r, shift left, "1\otimes\eta"] \arrow[r, shift right, "\eta\otimes 1"'] & C \end{tikzcd}$$

Coalg Ex: Ring of regular functions on alg group
η = evaluate function at 1
Δ = map on functions induced by $G\times G \to G$
(looks like matrix mult.)

Notation: $m(a\otimes b) = ab$
$\varepsilon(1) = 1$

$$\Delta c = \sum_i c_{i1}\otimes c_{i2} \overset{\text{alt}}{=} \sum c_{(1)}\otimes c_{(2)} \overset{\text{alt}}{=} c_{(1)}\otimes c_{(2)}$$

Einstein summation convention.

EX $C \xrightarrow{\Delta} C\otimes C \underset{1\otimes\Delta}{\overset{\Delta\otimes 1}{\rightrightarrows}} C\otimes C\otimes C \iff c_{(1)(1)}\otimes c_{(1)(2)}\otimes c_{(2)} = c_{(1)}\otimes c_{(2)(1)}\otimes c_{(2)(2)}$

EX $\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[rr, bend right, "="'] & C\otimes C \arrow[r, shift left, "1\otimes\eta"] \arrow[r, shift right, "\eta\otimes 1"'] & C \end{tikzcd} \iff \begin{cases} \eta(c_{(1)})\otimes c_{(2)} = c \\ c_{(1)}\otimes\eta(c_{(2)}) = c \end{cases}$

Def: A bialgebra H is
1) An <u>algebra</u> (H, m, ε)
2) A <u>coalgebra</u> (H, Δ, η)
3) <u>Compatibility</u> Δ, η are algebra morphisms
$\Updownarrow$
(m, ε are coalgebra morphisms)

<u>Compatibility</u>:

$$\begin{tikzcd} H\otimes H \arrow[r, "m"] \arrow[d, "\mathrm{twist}(\Delta\otimes\Delta)"'] & H \arrow[d, "\Delta"] \\ H\otimes H\otimes H\otimes H \arrow[r, "m\otimes m"] & H\otimes H \end{tikzcd}$$

etc. (3 other diagrams)

$$(cd)_{(1)} \otimes (cd)_{(2)} = c_{(1)}d_{(1)} \otimes c_{(2)}d_{(2)}$$

Bialgebra	$\longleftrightarrow$	Quantum Semigroup
Hopf algebra	$\longleftrightarrow$	Quantum Group

Intuition

Def: A Hopf algebra H is
1) a <u>bialgebra</u> $(H, m, \Delta, \varepsilon, \eta)$
2) w/ <u>antipode</u> map $S : H \to H$ w/

$$(Sh_{(1)})\, h_{(2)} = \eta(h) = h_{(1)} (Sh_{(2)})$$

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[rrr, bend right, "\eta"'] & H\otimes H \arrow[r, shift left, "S\otimes 1"] \arrow[r, shift right, "1\otimes S"'] & H\otimes H \arrow[r, "m"] & H \end{tikzcd}$$

alt:

$$\begin{tikzcd} H\otimes H \arrow[rr, "1\otimes S"] & & H\otimes H \arrow[d, "m"] \\ H \arrow[u, "\Delta"] \arrow[d, "\Delta"'] \arrow[r, "\eta"] & k \arrow[r, "\varepsilon"] & H \\ H\otimes H \arrow[rr, "S\otimes 1"'] & & H\otimes H \arrow[u, "m"'] \end{tikzcd}$$

Prop: (1) The antipode map of Hopf algebra is unique map w/ its properties

(2) $S(hg) = S(g)S(h)$, $S(1) = 1$ } S is an anti-algebra map

(3) $S\otimes S\,(h_{(1)}\otimes h_{(2)}) = (Sh)_{(2)} \otimes (Sh)_{(1)}$, $\eta Sh = \eta h$ } S is an anti-coalg. map

Proof:

(1) Suppose S_1 & S_2 are both antipode maps.

$$S_1 h = S_1(h_{(1)}\eta h_{(2)})$$

$$= S_1(h_{(1)})\,\eta h_{(2)}$$

$$= S_1(h_{(1)})\,h_{(2)(1)}\,S_2(h_{(2)(2)}) \quad (S_1\otimes id\otimes S_2)(\underbrace{h_{(1)}\otimes h_{(2)(1)}\otimes h_{(2)(2)}}_{=})$$

$$= S_1(h_{(1)(1)})\,h_{(1)(2)}\,S_2(h_{(2)}) \quad (S_1\otimes id\otimes S_2)(h_{(1)(1)}\otimes h_{(1)(2)}\otimes h_{(2)})$$

$$= \eta h_{(1)}\; S_2(h_{(2)})$$

$$= S_2(\eta h_{(1)}\, h_{(2)})$$

$$= S_2 h$$

This is analogous to proof that inverse is unique for groups: $g_{(2)}^{-1} = g_{(2)}^{-1} g\, g_{(1)}^{-1} = g_{(1)}^{-1}$

(2) $$(S(h_{(1)(1)}g_{(1)(1)}))\,h_{(1)(2)}g_{(1)(2)} \otimes g_{(2)} \otimes h_{(2)} = \underbrace{(S((h_{(1)}g_{(1)})_{(1)}))(h_{(1)}g_{(1)})_{(2)}}_{} \otimes g_{(2)} \otimes h_{(2)}$$

$$= \eta(h_{(1)}g_{(1)}) \otimes g_{(2)} \otimes h_{(2)}$$

$$= \eta(h_{(1)})\eta(g_{(1)}) \otimes g_{(2)} \otimes h_{(2)}$$

$$= 1 \otimes \eta(g_{(1)})g_{(2)} \otimes \eta(h_{(1)})h_{(2)}$$

$$= 1 \otimes g \otimes h$$

$$(S(h_{(1)(1)}g_{(1)(1)}))\,h_{(1)(2)}\,g_{(1)(2)}\,S(g_{(2)}) \otimes h_{(2)} = S(g)\otimes h$$

$$S(h_{(1)(1)}g_{(1)})\,h_{(1)(2)}\,g_{(2)(1)}\,S(g_{(2)(2)}) \otimes h_{(2)} =$$

$$S(h_{(1)(1)}g_{(1)})\,h_{(1)(2)}\,\eta(g_{(2)}) \otimes h_{(2)} =$$

$$S(h_{(1)(1)}\, g_{(1)}\, \eta(g_{(2)}))\; h_{(1)(2)} \otimes h_{(2)} =$$

$$S(h_{(1)(1)}\, g)\; h_{(1)(2)} \otimes h_{(2)} =$$

So
$$S(h_{(1)(1)}\, g)\, h_{(1)(2)} \otimes h_{(2)} = S(g) \otimes h$$

$$S(h_{(1)(1)}\, g)\, h_{(1)(2)} \cdot S(h_{(2)}) = S(g) \cdot S(h)$$

$$S(h_{(1)}\, g)\; h_{(2)(1)}\; S(h_{(2)(2)}) =$$

$$S(h_{(1)}\, g)\; \eta(h_{(2)}) =$$

$$S(h_{(1)}\, \eta(h_{(2)})\, g) =$$

$$S(h\, g) = \qquad \blacksquare$$