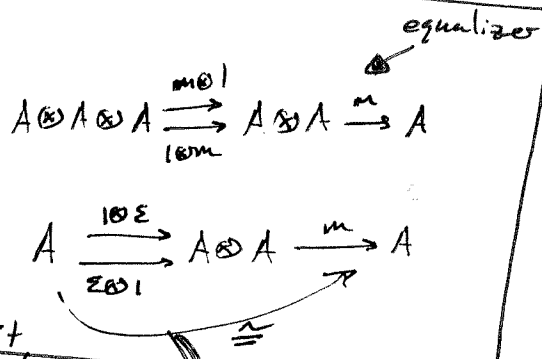


(Goal: Apply quantum groups to knots)

Reference for this lecture: Shahn Majid A Quantum Groups Primer

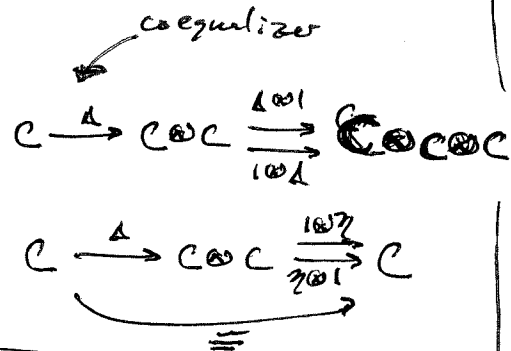
Idea: Quantum groups are group-object of noncommutative geometry

Def: A unital alg  $A$  over a field  $k$  is  
 v.s. over  $k$   
 w/ mult. map  $m: A \otimes A \rightarrow A$   
 unit map  $\varepsilon: k \rightarrow A$



Remark: My personal notation for unit/commit is reverse of Öğür's and Majid's.

Def: A coalgebra  $C$  over  $k$  is  
 v.s. over  $k$   
 w/ coproduct map  $\Delta: C \rightarrow C \otimes C$   
 counit map  $\eta: C \rightarrow k$



Coalg Ex: Ring of regular functions on alg group

$\eta$  = evaluate function at 1

$\Delta$  = map on functions induced by  $G \times G \rightarrow G$   
 (looks like matrix mult.)

Notation:  $m(a \otimes b) = ab$   
 $\varepsilon(1) = 1$

$\Delta c = \sum_i c_{i1} \otimes c_{i2} \stackrel{\text{alt}}{=} \sum_i c_{c(i)} \otimes c_{c(2)} \stackrel{\text{alt}}{=} c_{(1)} \otimes c_{(2)}$

Einstein summation convention

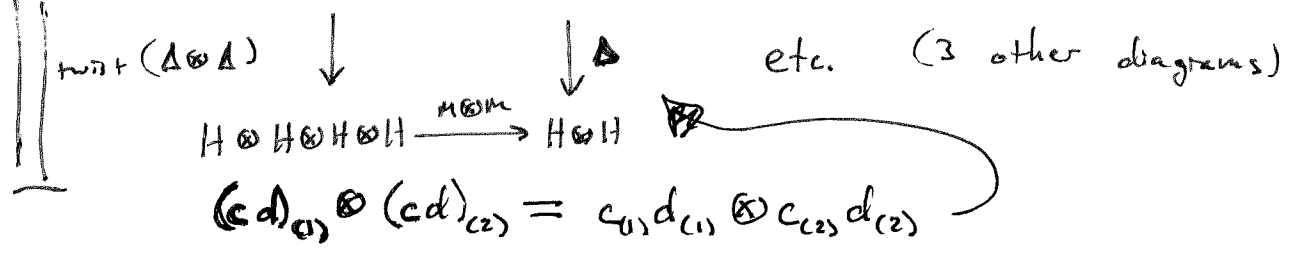
Ex  $C \xrightarrow{\Delta} C \otimes C \xrightarrow[1 \otimes \Delta]{\Delta \otimes 1} C \otimes C \otimes C \rightsquigarrow c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}$

Ex  $C \xrightarrow{\Delta} C \otimes C \xrightarrow[\eta \otimes 1]{1 \otimes \eta} C \rightsquigarrow \begin{cases} \eta(c_{(1)}) \otimes c_{(2)} = c \\ c_{(1)} \otimes \eta(c_{(2)}) = c \end{cases}$

Def: A bialgebra  $H$  is

- 1) An algebra  $(H, m, \varepsilon)$
- 2) A coalgebra  $(H, \Delta, \eta)$
- 3) Compatibility  $\Delta, \eta$  are algebra morphisms  
 $\updownarrow$   
 $(m, \varepsilon$  are coalgebra morphisms)

Compatibility:  $H \otimes H \xrightarrow{m} H$



$$(cd)_{c1} \otimes (cd)_{c2} = c_{c1} d_{c1} \otimes c_{c2} d_{c2}$$

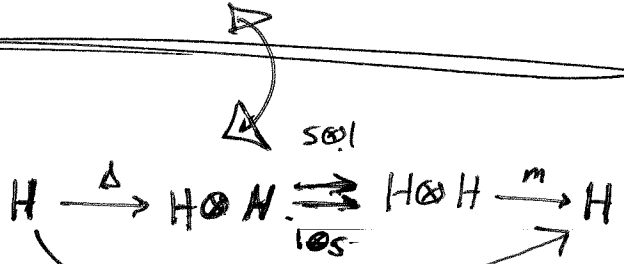
Bialgebra  $\leftrightarrow$  Quantum Semigroup  
 Hopf algebra  $\leftrightarrow$  Quantum Group

Intuition

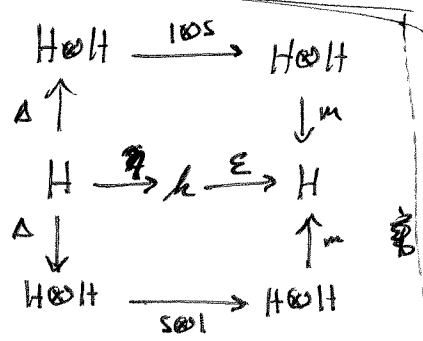
Def: A Hopf algebra  $H$  is

- 1) a bialgebra  $(H, m, \Delta, \varepsilon, \eta)$
- 2) w/ antipode map  $S: H \rightarrow H$  w/

$$(S h_{c1}) h_{c2} = \eta(h) = h_{c1} (S h_{c2})$$



alt:



Prop: (1) The antipode map of Hopf algebra is unique map w/ its properties

$$(2) \left. \begin{aligned} S(hg) &= S(g)S(h) \\ S(1) &= 1 \end{aligned} \right\} S \text{ is an anti-algebra map}$$

$$(3) \left. \begin{aligned} S \otimes S (h_{(1)} \otimes h_{(2)}) &= (S h)_{(2)} \otimes (S h)_{(1)} \\ \eta S h &= \eta h \end{aligned} \right\} S \text{ is an anti-coalg. map}$$

Proof:

(1) Suppose  $S_1$  &  $S_2$  are both antipode maps.  
 $S_1 h = S_1 (h_{(1)} \eta h_{(2)})$

This is analogous to proof that inverse is unique for groups:  
 $g^{-1} = g^{-1} g g^{-1} = g^{-1}$

$$\begin{aligned} &= S_1 (h_{(1)}) \eta h_{(2)} \\ &= S_1 (h_{(1)}) \underbrace{h_{(2)(1)} S_2 (h_{(2)(2)})}_{(S_1 \otimes id \otimes S_2) (h_{(1)} \otimes h_{(2)(1)} \otimes h_{(2)(2)})} \\ &= S_1 (h_{(1)}) \underbrace{h_{(2)(2)} S_2 (h_{(2)})}_{(S_1 \otimes id \otimes S_2) (h_{(1)} \otimes h_{(2)(2)} \otimes h_{(2)})} \\ &= \eta h_{(1)} S_2 (h_{(2)}) \\ &= S_2 (\eta h_{(1)} h_{(2)}) \\ &= S_2 h \end{aligned}$$

$$\begin{aligned} (2) (S (h_{(1)} g_{(1)})) h_{(2)} g_{(2)} \otimes g_{(3)} \otimes h_{(2)} &= (S ((h_{(1)} g_{(1)})_{(1)})) (h_{(1)} g_{(1)})_{(2)} \otimes g_{(2)} \otimes h_{(2)} \\ &= \eta (h_{(1)} g_{(1)}) \otimes g_{(2)} \otimes h_{(2)} \\ &= \eta (h_{(1)}) \eta (g_{(1)}) \otimes g_{(2)} \otimes h_{(2)} \\ &= 1 \otimes \eta (g_{(1)}) g_{(2)} \otimes \eta (h_{(1)}) h_{(2)} \\ &= 1 \otimes g \otimes h \end{aligned}$$

$$(S (h_{(1)} g_{(1)})) h_{(2)} g_{(2)} S (g_{(2)}) \otimes h_{(2)} = S(g) \otimes h$$

$$S (h_{(1)} g_{(1)}) h_{(2)} g_{(2)} S (g_{(2)}) \otimes h_{(2)} =$$

$$S (h_{(1)} g_{(1)}) h_{(2)} \eta (g_{(2)}) \otimes h_{(2)} =$$

$$S(h_{(1)} g_{(1)} \eta(g_{(2)})) h_{(1)(2)} \otimes h_{(2)} =$$

$$S(h_{(1)} g) h_{(1)(2)} \otimes h_{(2)} =$$

So  $S(h_{(1)} g) h_{(1)(2)} \otimes h_{(2)} = S(g) \otimes h$

$$S(h_{(1)} g) h_{(1)(2)} \cdot S(h_{(2)}) = S(g) \cdot S(h)$$

$$S(h_{(1)} g) h_{(1)(2)} S(h_{(2)}) =$$

$$S(h_{(1)} g) \eta(h_{(2)}) =$$

$$S(h_{(1)} \eta(h_{(2)}) g) =$$

$$S(h g) =$$

