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Cartan–Slodkowski spectra, splitting elements and noncommutative spectral mapping theorems

Anar Dosiev

Department of Mathematics, Atilim University, Incek 06836, Ankara, Turkey

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Abstract

In this paper, we prove Slodkowski version of the infinite-dimensional spectral mapping theorem and Cartan–Slodkowski version of the finite-dimensional spectral mapping theorem for nilpotent operator Lie subalgebras with respect to the various noncommutative functional calculi.

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1. Introduction

By spectral mapping theorem we mean inclusions like $f(\sigma(a)) \subseteq \sigma(f(a))$ (forward) or $f(\sigma(a)) \supseteq \sigma(f(a))$ (backward), where $a = (a_1, \ldots, a_n)$ is a n-tuple of bounded linear operators on a complex Banach space, σ is a joint spectrum ($\sigma(a)$ is a subset in $\mathbb{C}^n$), and $f = (f_1, \ldots, f_m)$ is a m-tuple of (noncommutative) functions in n-variables, which by reasonable way acts on the operator family a (in this case $f(a)$ is a m-tuple of operators on the same Banach space) and on $\sigma(a)$ simultaneously. If the equality

$$f(\sigma(a)) = \sigma(f(a)) \tag{1.1}$$

E-mail addresses: dosiev@yahoo.com, adosiev@atilim.edu.tr.

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holds for a certain class of operators a and functions f then we say that σ possesses the full (forward and backward) spectral mapping property on that class. It is well known [12, Section 2.6] that the Taylor spectrum σ_{t} and Slodkowski spectra $\sigma_{\pi,k}$, $\sigma_{\delta,k}$, $k \in \mathbb{Z}_{+}$, possess such property on the class of mutually commuting operators a and holomorphic functions f defined on an open neighborhood of the Taylor spectrum $\sigma_{\mathrm{t}}(a)$. The conventional proof scheme of the equality like (1.1) for some class of operators a and functions f can be divided into the following steps. The first step is to establish the Projection Property

$$\pi_{n-1}(\sigma(a)) = \sigma\left(a'\right),$$

where $a' = (a_1, \ldots, a_{n-1})$ and $\pi_{n-1} : \mathbb{C}^n \to \mathbb{C}^{n-1}$ is the projection map onto the first $n-1$ coordinates. The second step is the inclusion $\sigma(a, f(a)) \subseteq \left\{(\lambda, f(\lambda)) : \lambda \in \mathbb{C}^n\right\}$, which using the Projection Property automatically involves the equality

$$\sigma(a, f(a)) = \{(\lambda, f(\lambda)) : \lambda \in \sigma(a)\}.$$

Finally, using these facts we prove the equality (1.1) itself by the following way

$$\sigma(f(a)) = \pi_m(\sigma(a, f(a))) = \{f(\lambda) : \lambda \in \sigma(a)\} = f(\sigma(a)),$$

where $\pi_m : \mathbb{C}^{n+m} \to \mathbb{C}^m$ is the projection map onto the last m coordinates. The key moment in this framework is the Projection Property that we have applied to the operator family $(a, f(a))$. No doubt on such possibility if a is a commutative family and f is a family of holomorphic functions, in this case $(a, f(a))$ is just another $(n+m)$-tuple of commutative operator family. But this idea cannot be used directly for noncommutative operator families and functions in noncommuting variables. For instance, that fails to be true on the class of operators generating a finite-dimensional nilpotent Lie subalgebra and polynomials generating a finite-dimensional Lie subalgebra in the universal enveloping algebra. Indeed, take an operator family

$$a = (a_1, a_2, a_3), \quad [a_1, a_2] = a_3, \quad [a_i, a_3] = 0, \quad i = 1, 2,$$

generating a Heisenberg algebra, where $[x, y] = xy - yx$ is the Lie multiplication. If $f(a) = a_1 a_2$ is a polynomial then $(a, f(a))$ generates an infinite-dimensional Lie subalgebra (see [6, Example 7.7]).

Nonetheless, in this case the full spectral mapping property is valid for the Taylor spectrum [13] and for Slodkowski spectra [4]. We say in this case that a *finite-dimensional spectral mapping theorem* with respect to polynomials is valid on the class of nilpotent operator Lie subalgebras. Thus in a noncommutative case a (finite) family of functions (even polynomials) f may generate an infinite-dimensional Lie subalgebra despite finite-dimensionality of the nilpotent Lie subalgebra $\mathfrak{g}$ generated by a. This phenomena is crucial to distinguish noncommutative case from commutative one, thereby we are in need of a new method to investigate noncommutative spectral mapping theorems. Some ideas for the particular cases were suggested in [13,9,6]. The finite-dimensional spectral (the Slodkowski version) mapping theorem with respect to the noncommutative holomorphic functional calculus on absolutely convex domains was proved in [9] on the class of nilpotent operator Lie subalgebras. The Taylor version of the full spectral mapping property suggested in [13] was extended to a family of noncommutative rational functions f generating an infinite-dimensional quasinilpotent Banach–Lie (shortly, B–L) subalgebra [6]. It is an infinite-dimensional generalization of the noncommutative spectral mapping theorem that we refer as *infinite-dimensional spectral mapping theorem*. Moreover, the approach suggested in [6] allows to establish some inclusions for the Slodkowski spectra, too. Nevertheless, the full spectral mapping property remained open. Undoubtedly, the proofs of these spectral mapping theorems for each particular case have some common and distinct features. It is reasonable to ask how to prove the spectral mapping theorem with respect to a general noncommutative functional calculi? Our question is also motivated by the latest achievements on the noncommutative functional calculus for the class of nilpotent operator Lie subalgebras (see [8,10,19]).

In this paper, we apply a general framework of spectral mapping properties proposed in [11] to prove the Slodkowski version of the infinite-dimensional spectral mapping theorem from [6] for more general noncommutative functions than rational ones, so called *splitting over Banach* $\mathfrak{g}$*-module elements* (see Section 4.2). Roughly speaking, splitting over Banach $\mathfrak{g}$-module elements of a dominating (function) algebra are those functions in noncommuting variables having trivial actions on cohomologies of the Koszul complex of the Banach $\mathfrak{g}$-module. It is a new phenomena (previously did not observe in papers [9,6]) which allows to extend spectral mapping properties to noncommutative functions f which are not rational ones. For example, all holomorphic functions defined on an open neighborhood of the Taylor spectrum $\sigma_{\mathrm{t}}(a)$ of a mutually commuting operator family a are splitting elements. The latter presented in Taylor's investigations implicitly (see [23, Corollary 4.7, 12, Proposition 2.5.9]). Further, one proves (see Section 6) that many algebras of noncommutative functions previously considered in [9,6] consists of splitting elements. We also generalize the finite-dimensional spectral mapping theorem provided the operator family $f(a)$ generates a finite-dimensional solvable Lie subalgebra and f consists of splitting elements. Herein the key moment plays the Cartan–Slodkowski spectra of operator solvable Lie subalgebras to surpass the obstacle caused by the problem when we do not know whether or not a finite-dimensional Lie subalgebra of the (closed) associative envelope of $\mathfrak{g}$ is automatically nilpotent one, meanwhile it is known (see below Lemmas 3, 16) that it is always a solvable Lie subalgebra.

It is worth to note that our approach generalizes the (classical) scheme mentioned above replacing $\sigma(a, f(a))$ with the spectrum of a certain parametrized Banach space bicomplex connecting a and $f(a)$. We also suggest a modification of the method proposed in [6], which is motivated by the spectral mapping framework [11]. Such modification allows us to overcome the obstacle for the Slodkowski version of the full infinite-dimensional spectral mapping property investigated in [6]. For convenience, we repeat necessary definitions and arguments from [9,6] in Section 3.

Finally, we apply the suggested scheme to various noncommutative functional calculi and obtain relevant spectral mapping theorems in Section 6. In particular, one follows the main results on spectral mapping from [13,9,6].

2. Preliminaries

All linear spaces considered are complex. The linearly ordered set $\mathbb{Z}$ of integers adjoined with the greatest (resp., least) element $\{\infty\}$ (resp., $\{-\infty\}$) is denoted by $\overline{\mathbb{Z}}$ (resp., $\underline{\mathbb{Z}}$), $\mathbb{N}$ is the set of all positive integers and $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$. Let $\mathcal{B}(X, Y)$ be a (semi)normed space of all bounded linear operators between (semi)normed spaces X and Y furnished with the operator (semi)norm and let $\mathcal{B}(X) = \mathcal{B}(X, X)$. If $\mathcal{A}$ is a normed associative algebra then the spectrum (resp., spectral radius) in $\mathcal{A}$ of an element $a \in \mathcal{A}$ is denoted by $\mathrm{sp}_{\mathcal{A}}(a)$ (resp., $\rho(a)$). For $\mathcal{A} = \mathcal{B}(X)$ and $T \in \mathcal{B}(X)$, we write $\mathrm{sp}(T)$ instead of $\mathrm{sp}_{\mathcal{A}}(T)$. If $T \in \mathcal{B}(X, Y)$ then $T^* \in \mathcal{B}(Y^*, X^*)$ denotes the dual operator between (norm) dual spaces Y^* and X^*. The unit ball of a (semi)normed space X is denoted by $X_{(1)}$. Let **BS** be the category of all Banach spaces and bounded morphisms. Its subcategory comprising all left Banach modules over a Banach algebra $\mathcal{A}$ is denoted by $\mathcal{A}$-*mod*. We use the conventional denotation $X \widehat{\otimes} Y$ for the projective tensor product of $X, Y \in$ **BS** and we write $X^{\widehat{\otimes} n}$ instead of n-fold projective tensor product of X on itself. The direct sum $X \oplus Y$ is endowed with the sum-norm. One defines the functors $\mathcal{B}(Y, ?)$, $\mathcal{B}(?, Y)$ and $? \widehat{\otimes} Y$ on **BS** with values in itself. Let S be an infinite set and let $\mathfrak{U}$ be a nontrivial (that is, $\bigcap_{M \in \mathfrak{U}} M = \emptyset$) ultrafilter in S. The ultrafilter $\mathfrak{U}$ is said to be countably incomplete [16] if there exists a countable partition $\{S_n : n \in \mathbb{N}\}$ of S such that $S_n \notin \mathfrak{U}$, $n \in \mathbb{N}$. In the sequel, by an ultrafilter we mean a nontrivial countably incomplete ultrafilter. The ultrapower of a Banach space X (resp., Banach space operator T) with respect to an ultrafilter $\mathfrak{U}$ is denoted by $X_{\mathfrak{U}}$ (resp., $T_{\mathfrak{U}}$). Note that the space X is embedded into $X_{\mathfrak{U}}$ as a closed subspace and $T_{\mathfrak{U}}$ extends T preserving its norm, that is, $\|T_{\mathfrak{U}}\| = \|T\|$. The assignment $X \mapsto X_{\mathfrak{U}}$, $T \mapsto T_{\mathfrak{U}}$ defines a functor $?_{\mathfrak{U}} : \mathbf{BS} \to \mathbf{BS}$.

Now let $\mathcal{A}$ be a Banach algebra, $S \subseteq \mathcal{A}$, $\|S\| = \sup\{\|a\| : a \in S\}$ and let S^n be a subset in $\mathcal{A}$ of all n-times products $a_1 \cdots a_n$, $a_i \in S$. If S is bounded then the number $\rho(S) = \lim_n \|S^n\|^{1/n}$ is called the joint spectral radius of S [21]. Note that the limit exists and it equals to $\inf\left\{\|S^n\|^{1/n} : n \in \mathbb{N}\right\}$. In particular, $\rho(a) = \rho(\{a\})$ for each $a \in \mathcal{A}$.

The Banach algebra of all continuous complex functions on a compact space K furnished with the sup-norm is denoted by $C(K)$.

2.1. Banach space (bi)complexes

A chain Banach space complex is defined as a pair $(\mathfrak{X}, \mathfrak{d})$, where $\mathfrak{X} = \{X_n : n \in \mathbb{Z}\}$ are objects and $\mathfrak{d} = \{d_n : n \in \mathbb{Z}\}$ are morphisms of **BS**, such that $d_{n-1} d_n = 0$ for all n. We also write $(\mathfrak{X}, \mathfrak{d})$ as a sequence:

$$\cdots \longleftarrow X_{n-1} \stackrel{d_{n-1}}{\longleftarrow} X_n \stackrel{d_n}{\longleftarrow} X_{n+1} \longleftarrow \cdots .$$

A cochain Banach space complex is defined as a sequence $\cdots \to X^{n-1} \overset{d^{n-1}}{\to} X^n \overset{d^n}{\to} X^{n+1} \to \cdots$ of objects and morphisms of $\mathbf{BS}$ such that $d^n d^{n-1} = 0$ for all n. The category of all (co)chain complexes in $\mathbf{BS}$ is denoted by $\underline{\mathbf{BS}}$ (resp., $\overline{\mathbf{BS}}$). In particular, $\underline{\mathcal{A}\text{-}mod}$ (resp., $\overline{\mathcal{A}\text{-}mod}$) denotes a subcategory in $\underline{\mathbf{BS}}$ (resp., $\overline{\mathbf{BS}}$) of all chain (resp., cochain) left Banach $\mathcal{A}$-module complexes. The quotient (seminormed) spaces $H_n(\mathfrak{X}, \mathfrak{d}) = \ker(d_{n-1}) / \operatorname{im}(d_n)$ (resp., $H^n(\mathfrak{X}, \mathfrak{d}) = \ker(d^n) / \operatorname{im}(d^{n-1})$), $n \in \mathbb{Z}$, are called groups of (co)homologies of the complex $(\mathfrak{X}, \mathfrak{d})$. The complex $(\mathfrak{X}, \mathfrak{d})$ is said to be exact if all (co)homologies are trivial. If $X_n = \{0\}$ (resp., $X^n = \{0\}$) for all n, $n < 0$, then we say that $(\mathfrak{X}, \mathfrak{d})$ is a nonnegative complex. Note that each chain complex $(\mathfrak{X}, \mathfrak{d})$ makes into a cochain one $\left(\overline{\mathfrak{X}}, \overline{\mathfrak{d}}\right)$ by setting $\overline{X}^n = X_{-n}$ and $\overline{d}^n = d_{-n}$, $n \in \mathbb{Z}$. Similarly, a cochain complex $(\mathfrak{X}, \mathfrak{d})$ makes into a chain one $\left(\underline{\mathfrak{X}}, \underline{\mathfrak{d}}\right)$. That defines a functor $\underline{\mathbf{BS}} \longrightarrow \overline{\mathbf{BS}}$ (resp., $\overline{\mathbf{BS}} \to \underline{\mathbf{BS}}$) called the conjugate functor. Let $(\mathfrak{X}, \mathfrak{d}) \in \underline{\mathbf{BS}}$ and let $\left(\mathfrak{X}^*, \mathfrak{d}^*\right)$ be its dual complex:

$$\cdots \longrightarrow X^*_{n-1} \overset{d^*_{n-1}}{\longrightarrow} X^*_n \overset{d^*_n}{\longrightarrow} X^*_{n+1} \longrightarrow \cdots.$$

By its very definition, $\left(\mathfrak{X}^*, \mathfrak{d}^*\right) = \mathcal{B}((\mathfrak{X}, \mathfrak{d}), \mathbb{C}) \in \overline{\mathbf{BS}}$. A well known [18, item 7.6.13] the Sequence Prime Principle asserts that $(\mathfrak{X}, \mathfrak{d})$ is exact iff so is its dual complex $\left(\mathfrak{X}^*, \mathfrak{d}^*\right)$.

The following simple assertion will be used later.

Lemma 1. *Let $(\mathfrak{X}, \mathfrak{d})$ be an exact cochain complex of seminormed spaces and let $\delta = \{\delta_n : n \in \mathbb{Z}\}$ be a morphism of complexes acting from $(\mathfrak{X}, \mathfrak{d})$ into itself. If the operators $\delta_{n-1} \in \mathcal{B}\left(X^{n-1}\right)$ and $\delta_{n+1} \in \mathcal{B}\left(X^{n+1}\right)$ are nilpotent for some n, then so is the operator $\delta_n \in \mathcal{B}(X^n)$.*

Proof. Assume that $(\delta_{n-1})^s = 0$ and $(\delta_{n+1})^t = 0$. Take $x \in X^n$. Then $d^n (\delta_n)^t x = (\delta_{n+1})^t d^n x = 0$. But $H^n(\mathfrak{X}, \mathfrak{d}) = \{0\}$, therefore $(\delta_n)^t x = d^{n-1} y$ for some $y \in X^{n-1}$. It follows that $(\delta_n)^{s+t} x = (\delta_n)^s d^{n-1} y = d^{n-1} (\delta_{n-1})^s x = 0$, that is, $(\delta_n)^{s+t} = 0$. $\square$

Let $X \in \mathbf{BS}$ and let $\wedge^n X$ be its nth exterior power (see [6]). We set $C^n(X, Y) = \mathcal{B}(\wedge^n X, Y)$, $Y \in \mathbf{BS}$. The latter is a Banach space of all continuous skew-symmetric n-linear forms on X with values in Y. Let $Y \in \mathbf{BS}$ and $(\mathfrak{X}, \mathfrak{d}) \in \underline{\mathbf{BS}}$. The functor $\mathcal{B}(Y, ?) : \mathbf{BS} \to \mathbf{BS}$ transforms the complex $(\mathfrak{X}, \mathfrak{d})$ into a new complex $\mathcal{B}(Y, (\mathfrak{X}, \mathfrak{d}))$:

$$\cdots \leftarrow \mathcal{B}(Y, X_{n-1}) \overset{\beta_{n-1}}{\longleftarrow} \mathcal{B}(Y, X_n) \overset{\beta_n}{\longleftarrow} \mathcal{B}(Y, X_{n+1}) \leftarrow \cdots,$$

where $\beta_n(T) = d_n \cdot T$, $T \in \mathcal{B}(Y, X_n)$. A Banach space cochain complex $\mathcal{B}((\mathfrak{X}, \mathfrak{d}), Y)$ is analogously defined. A Banach space Y is said to be *projective* (resp., *injective*) if the complex $\mathcal{B}(Y, (\mathfrak{X}, \mathfrak{d}))$ (resp., $\mathcal{B}((\mathfrak{X}, \mathfrak{d}), Y)$) is exact for each exact Banach space complex $(\mathfrak{X}, \mathfrak{d})$. A Banach space Y is said to be *flat* if its dual space Y^* is injective.

The class of all projective (resp., flat) Banach spaces is denoted by $Proj$ (resp., Flat). The following assertion was proved in [11].

Lemma 2. *Let* $Y \in \mathbf{BS}$, $n \in \mathbb{N}$. *Then* $\wedge^n Y \in Proj$ (*resp.*, $\wedge^n Y \in \text{Flat}$) *whenever* $Y \in Proj$ (*resp.*, $Y \in \text{Flat}$).

A Banach space bicomplex is a triple $(\mathfrak{X}, \mathfrak{d}', \mathfrak{d}'')$ with $\mathfrak{X} = \{X^{n,m} : n, m \in \mathbb{Z}\}$ being underline Banach spaces, $\mathfrak{d}' = \left\{d_{\prime}^{n,m} \in \mathcal{B}\left(X^{n,m}, X^{n,m+1}\right)\right\}$ the column differentials, and the row differentials $\mathfrak{d}'' = \left\{d_{\prime\prime}^{n,m} \in \mathcal{B}\left(X^{n,m}, X^{n+1,m}\right)\right\}$, such that the following diagram

$$
\begin{array}{ccccccc}
 & & \uparrow & & & & \\
\cdots & \longrightarrow & X^{n+1,m} & \longrightarrow & \cdots & & \\
 & & \uparrow d_{\prime}^{n,m} & & \uparrow & & \\
\cdots & \longrightarrow & X^{n,m} & \overset{d_{\prime\prime}^{n,m}}{\longrightarrow} & X^{n,m+1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}
$$

is commutative, and all columns $\left(\mathfrak{X}^{\bullet,m}, \mathfrak{d}_{\prime}^{\bullet,m}\right)$ and all rows $\left(\mathfrak{X}^{n,\bullet}, \mathfrak{d}_{\prime\prime}^{n,\bullet}\right)$ are Banach space complexes, where $\mathfrak{X}^{\bullet,m} = \left\{X^{k,m}\right\}$, $\mathfrak{X}^{n,\bullet} = \left\{X^{n,k}\right\}$ and $\mathfrak{d}_{\prime}^{\bullet,m} = \left\{d_{\prime}^{k,m}\right\}$, $\mathfrak{d}_{\prime\prime}^{n,\bullet} = \left\{d_{\prime\prime}^{n,k}\right\}$. By analogy, it is defined other versions of bicomplexes. For us, of interest will be the "double-cochain" (proposed above) and also "double-chain" versions of a bicomplex, and we briefly say chain (resp., cochain) bicomplex instead of "double-chain" (resp., "double-cochain"). One can easily check that $d_{\prime}^{n,m}\left(\ker\left(d_{\prime\prime}^{n,m}\right)\right) \subseteq \ker\left(d_{\prime\prime}^{n+1,m}\right)$ and $d_{\prime}^{n,m}\left(\operatorname{im}\left(d_{\prime\prime}^{n,m-1}\right)\right) \subseteq \operatorname{im}\left(d_{\prime\prime}^{n+1,m-1}\right)$, therefore the quotient operator

$$
D_{\prime}^{n,m} : H^m\left(\mathfrak{X}^{n,\bullet}, \mathfrak{d}_{\prime\prime}^{n,\bullet}\right) \to H^m\left(\mathfrak{X}^{n+1,\bullet}, \mathfrak{d}_{\prime\prime}^{n+1,\bullet}\right), \quad D_{\prime}^{n,m}\left(x^{\sim}\right) = d_{\prime}^{n,m}(x)^{\sim},
$$

is well defined. Moreover, the sequence

$$
\cdots \longrightarrow H^m\left(\mathfrak{X}^{n,\bullet}, \mathfrak{d}_{\prime\prime}^{n,\bullet}\right) \overset{D_{\prime}^{n,m}}{\longrightarrow} H^m\left(\mathfrak{X}^{n+1,\bullet}, \mathfrak{d}_{\prime\prime}^{n+1,\bullet}\right) \longrightarrow \cdots
$$

is a complex called *mth vertical cohomology complex of the bicomplex*. By analogy, one defines *n*th horizontal cohomology complex

$$
\cdots \longrightarrow H^n\left(\mathfrak{X}^{\bullet,m}, \mathfrak{d}_{\prime}^{\bullet,m}\right) \overset{D_{\prime\prime}^{n,m}}{\longrightarrow} H^n\left(\mathfrak{X}^{\bullet,m+1}, \mathfrak{d}_{\prime}^{\bullet,m+1}\right) \longrightarrow \cdots
$$

of the bicomplex. We say that a bicomplex $(\mathfrak{X}, \mathfrak{d}', \mathfrak{d}'')$ is bounded below if one can find $N \in \mathbb{Z}$ such that $X^{n,m} = \{0\}$ whenever $n < N$ or $m < N$. The space $X^{N,N}$ is called the *initial space* of the bicomplex. If $N = 0$ then we say that $(\mathfrak{X}, \mathfrak{d}', \mathfrak{d}'')$ is a nonnegative bicomplex with the initial space $X^{0,0}$.

Let $(\mathfrak{X}, \mathfrak{d}', \mathfrak{d}'')$ be a bounded below Banach space bicomplex, $X^n = \bigoplus_{k+s=n} X^{k,s} \in$ **BS** a sum of (bounded) diagonals of the bicomplex. One defines a Banach space complex $\cdots \longrightarrow X^n \stackrel{\delta^n}{\longrightarrow} X^{n+1} \longrightarrow \cdots$, where $\delta^n(x) = d''^{k,s}(x) + (-1)^s d'^{k,s}(x)$ whenever $x \in X^{k,s}$, $k+s=n$, $n \in \mathbb{Z}$. The latter is called *the total complex of* $(\mathfrak{X}, \mathfrak{d}', \mathfrak{d}'')$ and it is denoted by $\operatorname{Tot}(\mathfrak{X}, \mathfrak{d}', \mathfrak{d}'')$.

2.2. Slodkowski spectra

Let Ω be a topological space and let $\mathfrak{X} = \{X_n : n \in \mathbb{Z}\}$ be a family of Banach spaces. Assume that there exists a family of continuous maps $\mathfrak{d} = \{d_n : n \in \mathbb{Z}\}$, $d_n : \Omega \to \mathcal{B}(X_{n+1}, X_n)$, such that $(\mathfrak{X}, \mathfrak{d}(\lambda))$ is a chain Banach space complex $\cdots \longleftarrow X_{n-1} \stackrel{d_{n-1}(\lambda)}{\longleftarrow} X_n \stackrel{d_n(\lambda)}{\longleftarrow} X_{n+1} \longleftarrow \cdots$, for each $\lambda \in \Omega$, where $\mathfrak{d}(\lambda) = \{d_n(\lambda)\}$. The collection of Banach space complexes $(\mathfrak{X}, \mathfrak{d}(\lambda))$, $\lambda \in \Omega$, is called (see [11]) a *parametrized chain Banach space complex* or *chain Ω-Banach complex* and it is denoted by $(\mathfrak{X}, \mathfrak{d})$. If $(\mathfrak{X}, \mathfrak{d}(\lambda))$ is a cochain complex for each $\lambda \in \Omega$, then $(\mathfrak{X}, \mathfrak{d})$ is said to be a cochain Ω-Banach complex. By reasonable way it is defined a morphism of Ω-Banach complexes. Using the functors $\mathcal{B}(Y, ?)$, $\mathcal{B}(?, Y)$, $?\widehat{\otimes} Y$, and $?_{\mathfrak{U}}$, one may associate new Ω-Banach complexes from the original Ω-Banach complex $(\mathfrak{X}, \mathfrak{d})$. In particular, $\mathcal{B}((\mathfrak{X}, \mathfrak{d}), \mathbb{C}) = (\mathfrak{X}^*, \mathfrak{d}^*) = \{(\mathfrak{X}^*, \mathfrak{d}(\lambda)^*) : \lambda \in \Omega\}$ is the dual parametrized complex.

A parametrized (co)chain Banach space bicomplex is defined as a certain bicomplex $\left(\mathfrak{X}, \mathfrak{d}', \mathfrak{d}''\right)$ such that all its rows $\left(\mathfrak{X}_{n,\bullet}, \mathfrak{d}''_{n,\bullet}\right)$ are Ω-Banach complexes, columns $\left(\mathfrak{X}_{\bullet,m}, \mathfrak{d}'_{\bullet,m}\right)$ are Λ-Banach complexes, and $\left(\mathfrak{X}, \mathfrak{d}'(\lambda), \mathfrak{d}''(\mu)\right)$ is a Banach space bicomplex for all $\lambda \in \Omega$ and $\mu \in \Lambda$. In this case we say that $\left(\mathfrak{X}, \mathfrak{d}', \mathfrak{d}''\right)$ is a $\Omega \times \Lambda$*-Banach bicomplex*.

Now let $(\mathfrak{X}, \mathfrak{d})$ be a (co)chain parametrized Banach space complex,

$$\Sigma_n(\mathfrak{X}, \mathfrak{d}) = \{\lambda \in \Omega : H_n(\mathfrak{X}, \mathfrak{d}(\lambda)) \neq \{0\}\},$$

and $\Sigma^n(\mathfrak{X}, \mathfrak{d}) = \{\lambda \in \Omega : H^n(\mathfrak{X}, \mathfrak{d}(\lambda)) \neq \{0\}\}$ if $(\mathfrak{X}, \mathfrak{d})$ is a cochain complex, $n \in \mathbb{Z}$. Further, let

$$\sigma_{\delta,n}(\mathfrak{X}, \mathfrak{d}) = \bigcup_{k \leqslant n} \Sigma_k(\mathfrak{X}, \mathfrak{d})$$

and let $\sigma_{\pi,n}(\mathfrak{X}, \mathfrak{d})$ be the set of those $\lambda \in \Omega$ such that $\lambda \in \bigcup_{k \geqslant n} \Sigma_k(\mathfrak{X}, \mathfrak{d})$ or the image of the operator $d_{n-1}(\lambda)$ is not closed. By analogy, we set

$$\sigma^{\delta,n}(\mathfrak{X}, \mathfrak{d}) = \bigcup_{k \geqslant n} \Sigma^k(\mathfrak{X}, \mathfrak{d})$$

and $\sigma^{\pi,n}(\mathfrak{X},\mathfrak{d})$ is a set of those $\lambda \in \Omega$ such that $\lambda \in \bigcup_{k\leqslant n}\Sigma^k(\mathfrak{X},\mathfrak{d})$ or the image of the operator $d^n(\lambda)$ is not closed, whenever $(\mathfrak{X},\mathfrak{d})$ is a cochain complex. Note that $\sigma^{\delta,n}(\mathfrak{X},\mathfrak{d}) = \sigma_{\delta,n}\left(\underline{\mathfrak{X}},\underline{\mathfrak{d}}\right)$ and $\sigma_{\pi,n}(\mathfrak{X},\mathfrak{d}) = \sigma^{\pi,n}\left(\overline{\mathfrak{X}},\overline{\mathfrak{d}}\right)$, $n \in \mathbb{Z}$.

Definition 1. The set-valued functions $\sigma_{\delta,n}$, $\sigma_{\pi,n}$ (resp., $\sigma^{\delta,n}$, $\sigma^{\pi,n}$), $n \in \mathbb{Z}$, defined on the class of all parametrized (co)chain Banach space complexes are called the Slodkowski spectra. The set

$$\sigma_{\mathrm{t}}(\mathfrak{X},\mathfrak{d}) = \sigma_{\delta,\infty}(\mathfrak{X},\mathfrak{d}) = \sigma_{\pi,-\infty}(\mathfrak{X},\mathfrak{d}) = \bigcup_{n\in\mathbb{Z}}\Sigma_n(\mathfrak{X},\mathfrak{d})$$

(resp., $\sigma_{\mathrm{t}}(\mathfrak{X},\mathfrak{d}) = \sigma^{\pi,\infty}(\mathfrak{X},\mathfrak{d}) = \sigma^{\delta,-\infty}(\mathfrak{X},\mathfrak{d}) = \bigcup_{n\in\mathbb{Z}}\Sigma^n(\mathfrak{X},\mathfrak{d})$ for the cochain complex $(\mathfrak{X},\mathfrak{d})$) is called the Taylor spectrum of $(\mathfrak{X},\mathfrak{d})$. We set $\mathfrak{S} = \mathfrak{S}_\delta \cup \mathfrak{S}_\pi$ (resp., $\mathfrak{S} = \mathfrak{S}^\delta \cup \mathfrak{S}^\pi$), where $\mathfrak{S}_\delta = \left\{\sigma_{\delta,n} : n \in \overline{\mathbb{Z}}\right\}$, $\mathfrak{S}_\pi = \left\{\sigma_{\pi,n} : n \in \underline{\mathbb{Z}}\right\}$ $\Big($resp., $\mathfrak{S}^\delta = \left\{\sigma^{\delta,n} : n \in \underline{\mathbb{Z}}\right\}$, $\mathfrak{S}^\pi = \left\{\sigma^{\pi,n} : n \in \overline{\mathbb{Z}}\right\}\Big)$.

Further, $\sigma \in \mathfrak{S}$ denotes one of the Slodkowski spectra if the latter will not specially be indicated. Let $(\mathfrak{X},\mathfrak{d})$ be a Ω-Banach complex. Using the Sequence Prime Principle, we obtain (see [22]) that

$$\sigma^{\pi,n}\left(\mathfrak{X}^*,\mathfrak{d}^*\right) = \sigma_{\delta,n}(\mathfrak{X},\mathfrak{d}), \quad \sigma_{\delta,n}\left(\mathfrak{X}^*,\mathfrak{d}^*\right) = \sigma^{\pi,n}(\mathfrak{X},\mathfrak{d}). \tag{2.1}$$

Moreover, if $(\mathfrak{X},\mathfrak{d})$ is a chain Ω-Banach complex then $\sigma^{\delta,n}\left(\mathfrak{X}^*,\mathfrak{d}^*\right) = \sigma_{\pi,n}(\mathfrak{X},\mathfrak{d})$, and $\sigma_{\pi,n}\left(\mathfrak{X}^*,\mathfrak{d}^*\right) = \sigma^{\delta,n}(\mathfrak{X},\mathfrak{d})$ whenever $(\mathfrak{X},\mathfrak{d})$ is a cochain complex (see [11]).

Theorem 1 (*Dosiev [11]*). *Let $(\mathfrak{X},\mathfrak{d})$ be a (co)chain Ω-Banach complex and let $Y \in$* **BS**.

(a) *Then $\sigma(\mathfrak{X},\mathfrak{d}) \subseteq \sigma(\mathcal{B}(Y,(\mathfrak{X},\mathfrak{d})))$ and $\sigma(\mathfrak{X},\mathfrak{d}) \subseteq \sigma\left((\mathfrak{X},\mathfrak{d})\widehat{\otimes}Y\right)$ for all $\sigma \in \mathfrak{S}$. Moreover, these inclusion become equalities whenever $Y \in Proj$ and $Y \in$* Flat, *respectively.*

(b) *If $\mathfrak{U}$ is an ultrafilter then $\sigma_{\pi,n}(\mathfrak{X}_{\mathfrak{U}},\mathfrak{d}_{\mathfrak{U}}) = \bigcup_{k\geqslant n}\Sigma_k(\mathfrak{X}_{\mathfrak{U}},\mathfrak{d}_{\mathfrak{U}})$ for all $\sigma_{\pi,n} \in \mathfrak{S}_\pi$. Moreover, $\sigma(\mathfrak{X},\mathfrak{d}) = \sigma(\mathfrak{X}_{\mathfrak{U}},\mathfrak{d}_{\mathfrak{U}})$ for all $\sigma \in \mathfrak{S}$.*

Now let us remind a cochain version of the spectral mapping properties for π-type Slodkowski spectra (see Definition 1) proposed in [11].

Let $(\mathfrak{X},\mathfrak{d})$ and $\left(\mathfrak{Y},\overline{\mathfrak{d}}\right)$ be a nonnegative cochain parametrized Banach space complexes such that both complexes have the same first term $X = X^0 = Y^0$ and let Ω and Λ be their space of parameters, respectively. We say that these complexes are π-*spectrally connected* if there exists a nonnegative cochain $\Omega \times \Lambda$-Banach bicomplex $\left(\mathfrak{Z},\mathfrak{d}',\mathfrak{d}''\right)$ such $\left(\mathfrak{Z}^{0,\bullet},\mathfrak{d}_{''}^{0,\bullet}\right) = (\mathfrak{X},\mathfrak{d})$,$\left(\mathfrak{Z}^{\bullet,0},\mathfrak{d}_{'}^{\bullet,0}\right) = \left(\mathfrak{Y},\overline{\mathfrak{d}}\right)$ and $\sigma\left(\mathfrak{Z}^{s,\bullet},\mathfrak{d}_{''}^{s,\bullet}\right) \subseteq \sigma(\mathfrak{X},\mathfrak{d})$,

$\sigma\left(\mathfrak{Z}^{\bullet,m}, \mathfrak{d}_{\prime}^{\bullet,m}\right) \subseteq \sigma\left(\mathfrak{Y}, \overline{\mathfrak{d}}\right)$ for all $\sigma \in \mathfrak{S}^{\pi}$, and $s, m \in \mathbb{N}$. Thus $(\mathfrak{Z}, \mathfrak{d}_{\prime}(\lambda), \mathfrak{d}_{\prime\prime}(\mu))$ is a nonnegative Banach space bicomplex with the initial space X for each $(\lambda, \mu) \in \Omega \times \Lambda$. Their total complexes $\operatorname{Tot}(\mathfrak{Z}, \mathfrak{d}_{\prime}(\lambda), \mathfrak{d}_{\prime\prime}(\mu))$, $(\lambda, \mu) \in \Omega \times \Lambda$, define $\Omega \times \Lambda$-Banach complex $\operatorname{Tot}(\mathfrak{Z}, \mathfrak{d}_{\prime}, \mathfrak{d}_{\prime\prime})$ and let $\sigma(\mathfrak{Z}, \mathfrak{d}_{\prime}, \mathfrak{d}_{\prime\prime})$ denotes a Slodkowski spectrum of the latter complex.

Definition 2. Let $(\mathfrak{X}, \mathfrak{d})$ and $\left(\mathfrak{Y}, \overline{\mathfrak{d}}\right)$ be π-spectrally connected complexes parametrized on a topological spaces Ω and Λ, respectively, and let $(\mathfrak{Z}, \mathfrak{d}_{\prime}, \mathfrak{d}_{\prime\prime})$ be a $\Omega \times \Lambda$-bicomplex connecting these complexes. By spectral mapping with respect to $(\mathfrak{Z}, \mathfrak{d}_{\prime}, \mathfrak{d}_{\prime\prime})$ we mean a continuous map $f : \Omega \to \Lambda$ such that

(a) all vertical cohomology complexes

$$0 \to H^m(\mathfrak{X}, \mathfrak{d}(\lambda)) \to \cdots \to H^m\left(\mathfrak{Z}^{n,\bullet}, \mathfrak{d}_{\prime\prime}^{n,\bullet}(\lambda)\right) \stackrel{D_{\prime}^{n,m}(\mu)}{\longrightarrow} H^m\left(\mathfrak{Z}^{n+1,\bullet}, \mathfrak{d}_{\prime\prime}^{n+1,\bullet}(\lambda)\right) \to \cdots$$

of the bicomplex $(\mathfrak{Z}, \mathfrak{d}_{\prime}(\lambda), \mathfrak{d}_{\prime\prime}(\mu))$ are exact whenever $\mu \neq f(\lambda)$;

(b) $D_{\prime}^{0,m}(f(\lambda)) = 0$ whenever the cohomology space $H^m(\mathfrak{X}, \mathfrak{d}(\lambda))$ is Hausdorff.

If only the second condition (b) is satisfied then we say that f is a prespectral mapping.

Note that the first condition (a) of Definition 2, means that all terms ${}'E_2^{mn}(\lambda, \mu)$, $m, n \in \mathbb{Z}_+$, of the first spectral sequence associated by the bicomplex $(\mathfrak{Z}, \mathfrak{d}_{\prime}(\lambda), \mathfrak{d}_{\prime\prime}(\mu))$ (see [15, Ch. 1, item 4.8]) are vanishing whenever $\mu \neq f(\lambda)$.

The following forward and backward spectral mapping properties of spectrally connected complexes were proved in [11].

Theorem 2. *Let* $(\mathfrak{X}, \mathfrak{d})$ *and* $\left(\mathfrak{Y}, \overline{\mathfrak{d}}\right)$ *be a cochain complexes parametrized on* Ω *and* Λ, *respectively, and let* $\mathfrak{U}$ *be an ultrafilter. If* $(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}})$ *and* $\left(\mathfrak{Y}, \overline{\mathfrak{d}}\right)$ *are* π*-spectrally connected and* $f : \Omega \to \Lambda$ *is a prespectral mapping then*

$$f(\sigma(\mathfrak{X}, \mathfrak{d})) \subseteq \sigma\left(\mathfrak{Y}, \overline{\mathfrak{d}}\right)$$

for all $\sigma \in \mathfrak{S}^{\pi}$.

Theorem 3. *Let* $(\mathfrak{X}, \mathfrak{d})$ *and* $\left(\mathfrak{Y}, \overline{\mathfrak{d}}\right)$ *be* π*-spectrally connected Banach space complexes parametrized on* Ω *and* Λ, *respectively,* $f : \Omega \to \Lambda$ *a spectral mapping with respect to a* $\Omega \times \Lambda$*-bicomplex* $(\mathfrak{Z}, \mathfrak{d}_{\prime}, \mathfrak{d}_{\prime\prime})$ *connecting* $(\mathfrak{X}, \mathfrak{d})$ *and* $\left(\mathfrak{Y}, \overline{\mathfrak{d}}\right)$, *and let* $\sigma \in \mathfrak{S}^{\pi}$. *If* $\sigma\left(\mathfrak{Y}, \overline{\mathfrak{d}}\right) = \Pi_{\Lambda}(\sigma(\mathfrak{Z}, \mathfrak{d}_{\prime}, \mathfrak{d}_{\prime\prime}))$ *then*

$$\sigma\left(\mathfrak{Y}, \overline{\mathfrak{d}}\right) \subseteq f(\sigma(\mathfrak{X}, \mathfrak{d})),$$

where $\Pi_{\Lambda} : \Omega \times \Lambda \to \Lambda$ *is the canonical projection.*

2.3. Full subalgebras

Let $\mathcal{A}$ be a unital associative algebra. A subalgebra $\mathcal{B} \subseteq \mathcal{A}$ is said to be a full (or inverse closed) subalgebra [2, Ch.1, Section 1.4], if any invertible in $\mathcal{A}$ element of $\mathcal{B}$ is invertible in $\mathcal{B}$. Thus $\mathrm{sp}_{\mathcal{A}}(b) = \mathrm{sp}_{\mathcal{B}}(b)$ for all $b \in \mathcal{B}$. One easily check that the full subalgebras are stable under taking arbitrary intersections, so it makes sense to define the full subalgebra $\mathcal{R}(M)$ in $\mathcal{A}$ generated by a subset $M \subseteq \mathcal{A}$. The elements of the subalgebra $\mathcal{R}(M)$ can be interpreted as a set of values of all formal "rational functions" with the set M of variables in the algebra $\mathcal{A}$ (see [26, 5, Section 2]). Namely, let S be a set with a mapping $\pi : S \to \mathcal{A}$ into the algebra $\mathcal{A}$ and let $M = \mathrm{im}(\pi)$. One can define the "rational functions" with the set S of variables and their actions in $\mathcal{A}$ as a collection of formal expressions from $\mathcal{R}_{S,\pi} = \bigcup_{n\in\mathbb{Z}_+} \mathcal{R}^n_{S,\pi}$ with the canonical mapping $\widehat{\pi} : \mathcal{R}_{S,\pi} \to \mathcal{A}$, $\widehat{\pi}(f(S)) = f(M)$, extending π, which is inductively defined by the following way. Let $\mathcal{R}^0_{S,\pi}$ be the free algebra (of all polynomials) generated by the set S, and let $\mathcal{R}^0(M) = \left\{ f(M) : f(S) \in \mathcal{R}^0_{S,\pi} \right\}$ be the values of all polynomials in the algebra $\mathcal{A}$ following π. If the collection $\mathcal{R}^{n-1}_{S,\pi}$ and their images $\mathcal{R}^{n-1}(M)$ have been defined, then the expressions from $\mathcal{R}^n_{S,\pi}$ is defined as the free algebra (of all polynomials) generated by $\mathcal{R}^{n-1}_{S,\pi}$ and all formal expressions $f^{-1}(S)$, $f(S) \in \mathcal{R}^{n-1}_{S,\pi}$, for which $f(M)$ is invertible in $\mathcal{A}$. We set $\widehat{\pi}\left(f^{-1}(S)\right) = f^{-1}(M) = f(M)^{-1}$. Thus $\mathcal{R}(M) = \bigcup_{n\in\mathbb{Z}_+} \mathcal{R}^n(M)$. If $f(S) \in \mathcal{R}^n_{S,\pi}$ then we say that $f(S)$ has an order n. Note also that if $\varepsilon : \mathcal{A} \to \mathcal{B}$ is a unital algebra homomorphism and E is the image of the family M by means of ε, that is, $E = \varepsilon(M)$, then $\mathcal{R}_{S,\pi} \subseteq \mathcal{R}_{S,\varepsilon\cdot\pi}$, and $\varepsilon(f(M)) = f(E)$ for $f(S) \in \mathcal{R}_{S,\pi}$. The following useful lemma was proved by Turovskii [25].

Lemma 3. *Let $\mathcal{B}$ be a Banach algebra and let $\mathfrak{g}$ be its finite-dimensional nilpotent Lie subalgebra such that the full subalgebra $\mathcal{R}(\mathfrak{g}) \subseteq \mathcal{B}$ generated by $\mathfrak{g}$ is dense in $\mathcal{B}$. Then $\mathcal{B}$ is commutative modulo its Jacobson radical* $\mathrm{Rad}\,\mathcal{B}$.

The following simple assertions will be used in Sections 3, 4.

Lemma 4. *Let S be a set and let $\pi : S \to \mathcal{A}$, $\varsigma : S \to \mathcal{B}$ be mappings into algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. If $\mathcal{R}_{S,\pi} \subseteq \mathcal{R}_{S,\varsigma}$ and $\mathrm{sp}_{\mathcal{A}}(f(\pi(S))) = \mathrm{sp}_{\mathcal{B}}(f(\varsigma(S)))$ for all $f(S) \in \mathcal{R}_{S,\pi}$, then $\mathcal{R}_{S,\pi} = \mathcal{R}_{S,\varsigma}$.*

Proof. We proceed by induction on the order of rational functions taken from $\mathcal{R}_{S,\varsigma}$. It is beyond a doubt $\mathcal{R}^0_{S,\varsigma} \subseteq \mathcal{R}_{S,\pi}$. Take $f(S) \in \mathcal{R}^n_{S,\varsigma}$. By definition, $f(S) = p(\Phi)$ is a (free) polynomial taken by a set $\Phi = \left\{ g_\iota(S), g_\kappa^{-1}(S) : g_\iota(S), g_\kappa(S) \in \mathcal{R}^{n-1}_{S,\varsigma} \right\}$. By induction hypothesis, $\bigcup_{k=0}^{n-1} \mathcal{R}^k_{S,\varsigma} \subseteq \mathcal{R}_{S,\pi}$. Therefore, one suffices to assume that $f(S) = g^{-1}(S)$ for some $g(S) \in \mathcal{R}^{n-1}_{S,\varsigma}$. Then $g(S) \in \mathcal{R}_{S,\pi}$ and $g(\varsigma(S))$ is invertible in $\mathcal{B}$. With $\mathrm{sp}_{\mathcal{A}}(g(\pi(S))) = \mathrm{sp}_{\mathcal{B}}(g(\varsigma(S)))$ in mind, infer that $g(\pi(S))$ is invertible in $\mathcal{A}$, too. The latter in turn implies that $g^{-1}(S) \in \mathcal{R}_{S,\pi}$, that is, $f(S) \in \mathcal{R}_{S,\pi}$. $\square$

Now let S and W be sets with a surjective map $\tau : S \to W$ and let $\varsigma : W \to \mathcal{A}$ be a mapping into an algebra $\mathcal{A}$. One define maps $\widehat{\varsigma} : \mathcal{R}_{W,\varsigma} \to \mathcal{A}$ and $\widehat{\pi} : \mathcal{R}_{S,\pi} \to \mathcal{A}$ extending ς and π, respectively, where $\pi = \varsigma \cdot \tau$.

Lemma 5. *There exists a unique mapping* $\widetilde{\tau} : \mathcal{R}_{S,\pi} \to \mathcal{R}_{W,\varsigma}$ *extending* τ *such that* $\widehat{\varsigma} \cdot \widetilde{\tau} = \widehat{\pi}$.

Proof. We proceed by induction on the order of rational functions. Note that τ extend uniquely up to an algebra homomorphism $\tau^0 : \mathcal{R}^0_{S,\pi} \to \mathcal{R}^0_{W,\varsigma}$, $\tau^0 (f (S)) = f (W)$. Evidently, $\widehat{\varsigma} \cdot \tau^0 = \widehat{\pi}$.

By induction hypothesis, one uniquely defines a map $\tau^{n-1} : \mathcal{R}^{n-1}_{S,\pi} \to \mathcal{R}^{n-1}_{W,\varsigma}$ such that $\widehat{\varsigma} \cdot \tau^{n-1} = \widehat{\pi}$. Take $f^{-1} (S) \in \mathcal{R}_{S,\pi}$ such that $f (S) \in \mathcal{R}^{n-1}_{S,\pi}$. By definition, $f (\pi (S))$ is invertible in $\mathcal{A}$. But, $f (\pi (S)) = \widehat{\pi} (f (S)) = \widehat{\varsigma}\tau^{n-1} (f (S))$, whence $g^{-1} (W) \in \mathcal{R}^n_{W,\varsigma}$, where $g (W) = \tau^{n-1} (f (S))$. We set $\tau^n \left(f^{-1} (S)\right) = g^{-1} (W)$. Then $\widehat{\varsigma}\left(\tau^n \left(f^{-1} (S)\right)\right) = \widehat{\varsigma}\left(g^{-1} (W)\right) = \widehat{\varsigma}(g (W))^{-1} = \widehat{\pi}(f (S))^{-1} = \widehat{\pi}\left(f^{-1} (S)\right)$. One defines a mapping $\tau^n : \mathcal{R}^{n-1}_{S,\pi} \cup \overline{\mathcal{R}}^{n-1}_{S,\pi} \to \mathcal{R}^n_{W,\varsigma}$, where $\overline{\mathcal{R}}^{n-1}_{S,\pi} = \left\{f^{-1} (S) : f (S) \in \mathcal{R}^{n-1}_{S,\pi}\right\}$. The latter is uniquely extended up to an algebra homomorphism $\tau^n : \mathcal{R}^n_{S,\pi} \to \mathcal{R}^n_{W,\varsigma}$. Obviously, $\widehat{\varsigma} \cdot \tau^n = \widehat{\pi}$. □

3. The complexes induced by a B–L algebra representation

To apply the spectral mapping framework (Section 2.2) to a spectral theory of B–L (Banach–Lie) algebra representations, we consider parametrized Banach space complexes induced by a B–L algebra representation and some technical machinery to operate with them, that is the aim of this section.

3.1. Banach modules over B–L algebras

A normed Lie algebra (resp., B–L algebra) E is a normed (resp., Banach) space and a Lie algebra with its jointly continuous Lie brackets $[\cdot, \cdot] : E \times E \to E$, $(a, b) \mapsto [a, b]$. We say that a B–L algebra E is a *quasinilpotent B–L algebra generated by* $S \subseteq E$ if the Lie subalgebra generated by S is dense in E and all operators $ad (a) \in \mathcal{B} (E)$, $ad (a)\, b = [a, b]$ $(a \in E)$, of the adjoint representation are quasinilpotent (see [6]).

A Banach module over a B–L algebra E (shortly, a Banach E-module) is a Banach space X with a bounded Lie representation $\alpha : E \to \mathcal{B} (X)$. To indicate the Lie representation, we briefly say that the pair (X, α) is a E-module. A functional $\lambda \in E^*$ is said to be a character of E, if $\lambda ([E, E]) = 0$. The space of all characters (furnished with the $*$-weak topology) of a B–L algebra E is denoted by $\Delta (E)$ $(\subseteq E^*)$. The module dual to X is defined as the pair (X^*, α^*), where $\alpha^* : E^{op} \to \mathcal{B} (X^*)$, $\alpha^* (a) = \alpha (a)^*$ is the dual Lie representation, E^{op} is the opposite Lie algebra. A Banach E-module (X, α) generates the following chain Banach space complex:

$$C_\bullet (\alpha) : 0 \leftarrow X \xleftarrow{d_0} X\widehat{\otimes}E \xleftarrow{d_1} \cdots \xleftarrow{d_{n-1}} X\widehat{\otimes} \wedge^n E \xleftarrow{d_n} \cdots$$

with the differential

$$d_n x \otimes \underline{a} = \sum_{i=1}^{n+1} (-1)^{i+1} \alpha(a_i)\, x \otimes \underline{a}_i + \sum_{i<j} (-1)^{i+j-1}\, x \otimes \left[a_i, a_j\right] \wedge \underline{a}_{i,j},$$

where $\underline{a} = a_1 \wedge \cdots \wedge a_{n+1} \in \wedge^{n+1} E$, $\underline{a}_i$ (resp., $\underline{a}_{i,j}$) is obtained from $\underline{a}$ throwing out of ith vector a_i (resp., a_i and a_j). If $\dim(E) < \infty$ then the latter complex is known as the Koszul complex of E-module X. The module (X, α) generates also the cochain complex

$$C^\bullet(\alpha) : 0 \to X \xrightarrow{d^0} C^1(E, X) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} C^n(E, X) \xrightarrow{d^n} \cdots$$

with the differential

$$d^n \omega\left(\underline{a}\right) = \sum_{i=1}^{n+1} (-1)^{i+1} \alpha(a_i)\, \omega\left(\underline{a}_i\right) + \sum_{i<j} (-1)^{i+j}\, \omega\left(\left[a_i, a_j\right] \wedge \underline{a}_{i,j}\right),$$

where $\omega \in C^n(E, X) = \mathcal{B}(\wedge^n E, X)$. The parametrized on $\Delta(E)$ (co)chain Banach space complex $C_\bullet(\alpha - \lambda)$ (resp., $C^\bullet(\alpha - \lambda)$), $\lambda \in \Delta(E)$, is denoted by $\mathcal{C}_\bullet(\alpha)$ (resp., $\mathcal{C}^\bullet(\alpha)$). We write $\mathcal{C}(\alpha)$ to indicate one of these complexes (chain or cochain). Their spectra $\sigma(\mathcal{C}(\alpha))$, $\sigma \in \mathfrak{S}$, are called the Slodkowski spectra (resp., Taylor spectrum) of the E-module X or the Lie representation α and we denote them by $\sigma(\alpha)$, $\sigma \in \mathfrak{S}$. Since $C_\bullet(\alpha - \lambda)^* = C^\bullet(\alpha^* - \lambda)$ to within an isomorphism in $\overline{\mathbf{BS}}$, it follows using (2.1) that

$$\sigma^{\delta,n}\left(\alpha^*\right) = \sigma_{\pi,n}(\alpha), \quad \sigma^{\pi,n}\left(\alpha^*\right) = \sigma_{\delta,n}(\alpha), \quad n \in \mathbb{Z}_+. \tag{3.1}$$

Now, let $\mathfrak{U}$ be an ultrafilter, $X \in \mathbf{BS}$ and let $X_{\mathfrak{U}}$ be the ultrapower of X. The Lie representation $\alpha_{\mathfrak{U}} : E \to \mathcal{B}(X_{\mathfrak{U}})$, $\alpha_{\mathfrak{U}}(a) = \alpha(a)_{\mathfrak{U}}$, is called an ultrapower of α. Thus $(X_{\mathfrak{U}}, \alpha_{\mathfrak{U}})$ is a Banach E-module. The following notion was introduced in [6].

Definition 3. Let (X, α) be a Banach E-module and $\sigma \in \mathfrak{S}$. We define ultraspectrum $\sigma_u(\alpha)$ (resp., $\sigma^u(\alpha)$) of the module (X, α), or the Lie representation α, as the union of spectra $\sigma(\alpha_{\mathfrak{U}})$ taken over all countably incomplete ultrafilters $\mathfrak{U}$.

Note that (see [6, Lemma 5.4]) if $\dim(E) < \infty$ then

$$\mathcal{C}(\alpha)_{\mathfrak{U}} = \mathcal{C}(\alpha_{\mathfrak{U}}) \quad \text{and} \quad \sigma_u(\alpha) = \sigma(\alpha) \quad (\text{resp., } \sigma^u(\alpha) = \sigma(\alpha)) \tag{3.2}$$

for all $\sigma \in \mathfrak{S}$.

3.2. The Lie representation θ

Now let E be a B–L algebra and let I be its closed ideal. Then its exterior power $\wedge^n I$ makes into a Banach E-module due to the representation

$$T_{n,I} : E \to \mathcal{B}\left(\wedge^n I\right), \quad T_{n,I}(a)\left(\underline{u}\right) = \sum_{i=1}^{n} (-1)^{i-1} \left(ad\,(a)\, u_i\right) \wedge \underline{u}_i,$$

where $\underline{u} = u_1 \wedge \cdots \wedge u_n$. The latter extends the adjoint representation of E. If (X, α) is a Banach E-module, then the space $C^n(I, X)$ furnishes a Banach E-module structure by the Lie representation

$$\theta_{n,I} : E \to \mathcal{B}\left(C^n(I, X)\right), \quad \theta_{n,I}(a) = L_{\alpha(a)} - R_{T_{n,I}(a)},$$

where $L_{\alpha(a)}$ (resp., $R_{T_{n,I}(a)}$) is the left (resp., right) multiplication operator. We set $T_n = T_{n,E}$ and $\theta_n = \theta_{n,E}$. Note that (see [3, Ch. 1]),

$$d^n \theta_n(a) = \theta_{n+1}(a)\, d^n, \tag{3.3}$$

$$d^{n-1} i_n(a) + i_{n+1}(a)\, d^n = \theta_n(a), \tag{3.4}$$

$$\theta_{n-1}(a)\, i_n(b) - i_n(b)\, \theta_n(a) = i_n([a, b]), \tag{3.5}$$

where d^n is the differential of the complex $C^\bullet(\alpha)$ and $i_n(a) : C^n(E, X) \to C^{n-1}(E, X)$, $(i_n(a)\,\omega)\, b = \omega(a \wedge b)$, is so called *a homotopy operator*. Respectively, $X \widehat{\otimes} \wedge^n E$ is a E-module by the representation $\vartheta_n : E \to \mathcal{B}\left(X \widehat{\otimes} \wedge^n E\right)$, $\vartheta_n(a) = \alpha(a) \otimes 1 + 1 \otimes T_n(a)$, and formulae similar (3.3), (3.4) and (3.5) are also valid for the chain complex $C_\bullet(\alpha)$.

Proposition 1. *Let $\imath_n : \left(X \widehat{\otimes} \wedge^n E\right)^* \to \mathcal{B}(\wedge^n E, X^*)$ be the canonical isomorphism in* **BS** *given by the rule $\imath_n(f)\left(\underline{u}\right)(x) = f\left(x \otimes \underline{u}\right)$, $x \in X$, $\underline{u} \in \wedge^n E$. Then $\theta_n'(a)\, \imath_n = \imath_n \vartheta_n(a)^*$, where $\theta_n' : E^{op} \to \mathcal{B}(C^n(E, X^*))$, $\theta_n'(a) = L_{\alpha^*(a)} - R_{T_n^{op}(a)}$, is the Lie representation induced by the dual representation $\alpha^* : E^{op} \to \mathcal{B}(X^*)$.*

Proof. Let $T_n^{op} : E^{op} \to \mathcal{B}(\wedge^n E)$ be the extension of the adjoint representation of E^{op}. One easily check that $T_n^{op}(a) = -T_n(a)$, $a \in E$. Then

$$\begin{aligned} R_{T_n^{op}(a)} \imath_n(f)\left(\underline{u}\right)(x) &= -\imath_n(f)\left(T_n(a)\,\underline{u}\right)(x) = -f\left(x \otimes T_n(a)\,\underline{u}\right) \\ &= -\left(1 \otimes T_n(a)\right)^*(f)\left(x \otimes \underline{u}\right) \\ &= -\imath_n\left(\left(1 \otimes T_n(a)\right)^*(f)\right)\left(\underline{u}\right)(x), \end{aligned}$$

that is, $-R_{T_n^{op}(a)}\iota_n = \iota_n\,(1\otimes T_n\,(a))^*$. Further,

$$L_{\alpha^*(a)}\iota_n\,(f)\,(\underline{u})\,(x) = \alpha^*\,(a)\,(\iota_n\,(f)\,(\underline{u}))\,(x) = \iota_n\,(f)\,(\underline{u})\,(\alpha\,(a)\,x) = f\,(\alpha\,(a)\,x\otimes\underline{u})$$
$$= (\alpha\,(a)\otimes 1)^*\,(f)\,(x\otimes\underline{u}) = \iota_n\,((\alpha\,(a)\otimes 1)^*\,(f))\,(\underline{u})\,(x)\,,$$

that is, $L_{\alpha^*(a)}\iota_n\,(f) = \iota_n\,((\alpha\,(a)\otimes 1)^*\,(f))$. It follows that

$$\theta_n'\,(a)\,\iota_n = \left(L_{\alpha^*(a)} - R_{T_n^{op}(a)}\right)\iota_n = \iota_n\left((\alpha\,(a)\otimes 1)^* + (1\otimes T_n\,(a))^*\right) = \iota_n\vartheta_n\,(a)^*\,,$$

that is, $\theta_n'\,(a)\,\iota_n = \iota_n\vartheta_n\,(a)^*$. □

Note that all ι_n are isomorphisms and $\left(X\widehat{\otimes}\wedge^n E\right)^* = X^*\widehat{\otimes}\,(\wedge^n E)^*$ whenever $\dim(E) < \infty$, and in this case, the dual representation $\vartheta^* : E^{op}\to\mathcal{B}\left(\left(X\widehat{\otimes}\wedge E\right)^*\right)$, $\vartheta^*\,(a) = \vartheta\,(a)^*$ (here $\vartheta\,(a) = \sum_n \vartheta_n\,(a)$), is reduced (to within an isomorphism) to the Lie representation $\theta' : E^{op}\to\mathcal{B}\,(\mathcal{B}\,(\wedge E, X^*))$, $\theta'\,(a) = \sum_n \theta_n'\,(a)$, by Proposition 1.

Corollary 1. *Let E be a finite-dimensional nilpotent Lie algebra and let (X,α) be a Banach E-module. The dual representation $\theta^* : E^{op}\to\mathcal{B}\left(\mathcal{B}\,(\wedge E, X)^*\right)$, $\theta^*\,(a) = \theta\,(a)^*$, is reduced (to within an isomorphism) to the representation $\theta' : E^{op}\to\mathcal{B}\,(\mathcal{B}\,(\wedge E, X^*))$, $\theta'\,(a) = L_{\alpha^*(a)} - R_{T^{op}(a)}$.*

Proof. Let $n = \dim(E)$. At first, note that $\wedge^k E^* = \left(\wedge^k E\right)^*$ and the map $\gamma_k : X\otimes\wedge^k E^*\to\mathcal{B}\left(\wedge^k E, X\right)$, $\gamma_k\,(x\otimes f)\,(\underline{u}) = f\,(\underline{u})\,x$, $f\in\wedge^k E^*$, $\underline{u}\in\wedge^k E$, is an isomorphism in **BS**. Moreover, $\theta_k\,(a)\,\gamma_k = \gamma_k\left(\alpha\,(a)\otimes 1 - 1\otimes T_k\,(a)^*\right)$. Now let $\tau_{\underline{w}}^{(k)} : \wedge^k E^*\to\wedge^{n-k}E$ be an isomorphism depending on the choice of some fixed $\underline{w}\in\wedge^n E$ (see [1, Ch.1, Section 11]). Taking into account that E is a nilpotent Lie algebra, we conclude $\tau_{\underline{w}}^{(k)} T_k\,(a)^* = -T_{n-k}\,(a)\,\tau_{\underline{w}}^{(k)}$ by virtue of Corollary 1 from [1, Ch. 1, Section 11]. It follows that

$$\left(1_X\otimes\tau_{\underline{w}}^{(k)}\right)\left(\alpha\,(a)\otimes 1 - 1\otimes T_k\,(a)^*\right) = (\alpha\,(a)\otimes 1 + 1\otimes T_{n-k}\,(a))\left(1_X\otimes\tau_{\underline{w}}^{(k)}\right).$$

Thus the linear map $\varepsilon = \sum_k\left(1_X\otimes\tau_{\underline{w}}^{(k)}\right)\gamma_k^{-1}$ implements a topological isomorphism

$$\varepsilon : \mathcal{B}\,(\wedge E, X)\to X\otimes\wedge E$$

such that $\varepsilon\theta\,(a) = \vartheta\,(a)\,\varepsilon$ for all $a\in E$, that is, $\theta = \vartheta$ to within an isomorphism. Using Proposition 1, we infer that $\theta^* = \vartheta^* = \theta'$ to within an isomorphism. □

Now let E be a finite-dimensional Lie algebra, G a subspace in E such that $[E, G] = \{0\}$ (in particular, G is a Lie ideal), W a complemented to G subspace in E and let

$P \in \mathcal{B}(E)$ be a projection onto G along W. For brevity, we write (up to the end of this section) G_s and E_k instead of $\wedge^s G$ and $\wedge^k E$, respectively. Fix $n \in \mathbb{N}$ and let us introduce the following subspaces $G_s \wedge E_k$ (resp., $G_s \wedge W \wedge E_{k-1}$) in E_n, $s+k=n$, generated by vectors $\underline{u} \wedge \underline{v}$ (resp., $\underline{u} \wedge w \wedge \underline{v}$), where $\underline{u} = u_1 \wedge \cdots \wedge u_s \in G_s$, $\underline{v} = v_1 \wedge \cdots \wedge v_k \in E_k$ (resp., $w \in W$, $\underline{v} \in E_{k-1}$). Note that all subspaces $G_s \wedge E_k$ are E-submodules in E_n, that is, they are invariant under all operators $T_n(a)$, $a \in E$. Indeed, $T_n(a)\left(\underline{u} \wedge \underline{v}\right) = \underline{u} \wedge T_k(a)\,\underline{v}$, thus, $T_n(a)\,|G_s \wedge E_k = 1_{G_s} \wedge T_k(a)$. Moreover, one can easily verify that

$$G_s \wedge E_k = G_{s+1} \wedge E_{k-1} \oplus G_s \wedge W \wedge E_{k-1},$$

therefore the operator $T_n(a)\,|G_s \wedge E_k$ has a triangular operator matrix with respect to this decomposition. To find that, we introduce the following operators $A_a \in \mathcal{B}(W, G)$, $A_a = P\,\mathrm{ad}(a)\,|_W$, and $B_a \in \mathcal{B}(W)$, $B_a = (1-P)\,\mathrm{ad}(a)\,|_W$, where $a \in E$. We also set $\overline{A}_a = 1_{G_s} \wedge A_a \wedge 1_{E_{k-1}}$ and $\overline{B}_a = 1_{G_s} \wedge B_a \wedge 1_{E_{k-1}}$. Then

$$T_n(a)\left(\underline{u} \wedge w \wedge \underline{v}\right) = \underline{u} \wedge A_a w \wedge \underline{v} + \underline{u} \wedge B_a w \wedge \underline{v} + \underline{u} \wedge w \wedge T_{k-1}(a)\,\underline{v}$$

whenever $\underline{u} \wedge w \wedge \underline{v} \in G_s \wedge W \wedge E_{k-1}$. It follows that

$$T_n(a)\,|G_s \wedge E_k = \begin{pmatrix} 1_{G_{s+1}} \wedge T_{k-1}(a) & \overline{A}_a \\ 0 & 1_{G_s \wedge W} \wedge T_{k-1}(a) + \overline{B}_a \end{pmatrix}.$$

Now let $F = E/G$ be the quotient Lie algebra and let $\tau : E \to F$ be the quotient map. Bearing in mind that the exterior power $\wedge F$ of the space F is a (grading) F-module by the Lie representation $T : F \to \mathcal{B}(\wedge F)$ extending the adjoint representation, we conclude that it makes into a E-module via pull back along the Lie homomorphism τ. The exterior power $\wedge\tau : \wedge E \to \wedge F$ of τ is a (grading) E-module morphism, because of $\wedge\tau \cdot T_n(a) = T_n(\tau(a)) \cdot \wedge\tau$ for all $a \in E$. In particular, all spaces $G_s \otimes F_k$ ($F_k = \wedge^k F$) are turning into E-modules by means of the Lie representations $1 \otimes T_k\tau : E \to \mathcal{B}(G_s \otimes F_k)$, $(1 \otimes T_k\tau)\,a = 1_{G_s} \otimes T_k(\tau(a))$.

Lemma 6. *The sequence* $0 \leftarrow G_s \otimes F_k \xleftarrow{\tau_{s,k}} G_s \wedge E_k \xleftarrow{\iota} G_{s+1} \wedge E_{k-1} \leftarrow 0$ *is an exact sequence of E-modules, where* ι *is the embedding and* $\tau_{s,k}\left(\underline{u} \wedge \underline{v}\right) = \underline{u} \otimes \wedge^k\tau\left(\underline{v}\right)$.

Proof. One needs (see [13, Proposition 2.5]) only to prove that $\tau_{s,k}$ is a E-module morphism. For the latter, one suffices to prove that $\left(1_{G_s} \otimes T_k\tau(a)\right) \cdot \tau_{s,k} = \tau_{s,k} \cdot \left(1_{G_s} \wedge T_k(a)\right)$ for all $a \in E$, which verifies immediately. □

Now let $X \in \mathbf{BS}$ and let $\alpha : E \to \mathcal{B}(X)$ be a Lie representation of a finite-dimensional Lie algebra E on X. As we noted above the Banach space $\mathcal{B}(\wedge E, X)$ makes into a (grading) E-module by means of the representation θ. Undoubtedly, all spaces $\mathcal{B}(G_s \wedge E_k, X)$ are E-submodules in $\mathcal{B}(E_n, X)$, where G is defined as above and $n = s + k$. Moreover,

$$\mathcal{B}(G_s \wedge E_k, X) = \mathcal{B}(G_{s+1} \wedge E_{k-1}, X) \oplus \mathcal{B}(G_s \wedge W \wedge E_{k-1}, X) \tag{3.6}$$

and the operator $\theta(a)$ $(a \in E)$ has the following matrix form with respect to this decomposition

$$\theta_n(a)\,|\mathcal{B}(G_s \wedge E_k, X) = \begin{pmatrix} 1_{G_{s+1}} \wedge \theta_{k-1}(a) & 0 \\ -R_{\overline{A}_a} & 1_{G_s \wedge W} \wedge \theta_{k-1}(a) - R_{\overline{B}_a} \end{pmatrix}, \tag{3.7}$$

where $1_{G_{s+1}} \wedge \theta_{k-1}(a) = L_{\alpha(a)} - R_{1_{G_{s+1}} \wedge T_{k-1}(a)}$ and $1_{G_s \wedge W} \wedge \theta_{k-1}(a) = L_{\alpha(a)} - R_{1_{G_s \wedge W} \wedge T_{k-1}(a)}$. Let us introduce the following operators

$$D_{s,k}(a) = \begin{pmatrix} 1_{G_{s+1}} \wedge \theta_{k-1}(a) & 0 \\ 0 & 1_{G_s \wedge W} \wedge \theta_{k-1}(a) - R_{\overline{B}_a} \end{pmatrix}, N_{s,k}(a) = \begin{pmatrix} 0 & 0 \\ R_{\overline{A}_a} & 0 \end{pmatrix}.$$

It is clear that $\theta_n(a)\,|\mathcal{B}(G_s \wedge E_k, X) = D_{s,k}(a) - N_{s,k}(a)$, $a \in E$. Moreover, the correspondence $D_{s,k} : E \to \mathcal{B}(\mathcal{B}(G_s \wedge E_k, X))$, $a \mapsto D_{s,k}(a)$, is a Lie representation. Indeed, taking into account that $\left[1_{G_s \wedge W} \wedge \theta_{k-1}(b), R_{\overline{B}_a}\right] = 0$, $a, b \in E$, one suffices to prove that $[B_a, B_b] = B_{[a,b]}$ for all $a, b \in E$. The latter verifies immediately:

$$\begin{aligned} [B_a, B_b](w) &= (1-P)([a,(1-P)[b,w]] - [b,(1-P)[a,w]]) \\ &= (1-P)([a,[b,w]] - [b,[a,w]]) = (1-P)[[a,b],w] \\ &= B_{[a,b]}(w) \end{aligned}$$

(we used here that $[E, G] = \{0\}$). Note also that $N_{s,k}(a)\,N_{s,k}(b) = 0$, $a, b \in E$.

Lemma 7. *Let $\mathcal{E}_{s,k}$ be a Lie subalgebra $\mathcal{B}(G_s \wedge E_k, X)$ generated by operators $N_{s,k}(a)$, $D_{s,k}(b)$, $a, b \in E$. If E is a nilpotent Lie algebra then so is $\mathcal{E}_{s,k}$.*

Proof. Let us assume that $E^{(t+1)} = \{0\}$, where $E^{(k)}$ is the kth member of the lower central series of E, $t \in \mathbb{N}$. Note that $(1-P)([a,b]) \in E^{(2)} + G$ for all $a, b \in E$. Then $[(1-P)([a,b]), c] \in \left[E^{(2)} + G, E\right] \subseteq \left[E^{(2)}, E\right] = E^{(3)}$, $c \in E$. It follows that $\operatorname{im}\left(A_a B_{b_1} \cdots B_{b_m}\right) \subseteq P\left(E^{(m+2)}\right)$, whence $A_a B_{b_1} \cdots B_{b_{t-1}} = 0$ for all $a, b_1, \ldots, b_{t-1} \in E$. But, the direct computations show that

$$\operatorname{ad}\left(D_{s,k}(b_{t-1})\right) \cdots \operatorname{ad}\left(D_{s,k}(b_1)\right)\left(N_{s,k}(a)\right) = \begin{pmatrix} 0 & 0 \\ R_{C(a,b_1,\ldots,b_{t-1})} & 0 \end{pmatrix} = 0,$$

where $C(a, b_1, \ldots, b_{t-1}) = (-1)^{t-1}\, 1_{G_s} \wedge A_a B_{b_1} \cdots B_{b_{t-1}} \wedge 1_{E_{k-1}}$. It remains to note that $D_{s,k}$ is a Lie representation and operators $N_{s,k}(a)$ generate a commutative Lie ideal. $\square$

Proposition 2. *Let E be a finite-dimensional nilpotent Lie algebra, $G \subseteq E$ a subspace such that $[E, G] = \{0\}$, (X, α) a Banach E-module and let $\mathcal{A}_\theta$ be a closed full subalgebra in $\mathcal{B}(\mathcal{B}(\wedge E, X))$ generated by the Lie subalgebra $\theta(E)$. Then all $\mathcal{B}(G_s \wedge E_k, X)$ are $\mathcal{A}_\theta$-invariant subspaces in $\mathcal{B}(\wedge E, X)$.*

Proof. We proceed by induction on s. Obviously, the assertion is true for $s = 0$. Assume that $\mathcal{B}(G_s \wedge E_k, X)$ is invariant under $\mathcal{A}_\theta$ and let $B_{s,\theta} = \mathcal{A}_\theta | \mathcal{B}(G_s \wedge E_k, X)$. Let us prove that so is the subspace $\mathcal{B}(G_{s+1} \wedge E_{k-1}, X)$. The Lie subalgebra $\mathcal{E}_{s,k} \subseteq \mathcal{B}(\mathcal{B}(G_s \wedge E_k, X))$ generated by operators $N_{s,k}(a)$, $D_{s,k}(b)$, $a, b \in E$, is nilpotent by virtue of Lemma 7. By Lemma 3, the closure B_s of the full subalgebra $\mathcal{R}\left(\mathcal{E}_{s,k}\right) \subseteq \mathcal{B}(\mathcal{B}(G_s \wedge E_k, X))$ generated by $\mathcal{E}_{s,k}$ is commutative modulo its Jacobson radical $Rad B_s$. Then $N_{s,k}(a) \in Rad B_s$, $a \in E$. Since $\theta(E) | \mathcal{B}(G_s \wedge E_k, X) \subseteq \mathcal{E}_{s,k}$, it follows that $B_{s,\theta} \subseteq B_s$. Moreover, $\mathcal{R}_{E,D_{s,k}} \subseteq \mathcal{R}_{E,\theta}$ (see Section 2.3) by virtue of (3.7) (a triangular operator matrix with invertible diagonal entries is invertible itself), thereupon $r(\theta(E)) - r\left(D_{s,k}(E)\right) \in Rad B_s$ for all $r(E) \in \mathcal{R}_{E,D_{s,k}}$. Taking into account that B_s is a full subalgebra, we conclude that $\mathrm{sp}\left(r\left(D_{s,k}(E)\right)\right) = \mathrm{sp}_{B_s}\left(r\left(D_{s,k}(E)\right)\right) = \mathrm{sp}_{B_s}(r(\theta(E))) = \mathrm{sp}(r(\theta(E)))$, $r(E) \in \mathcal{R}_{E,D_{s,k}}$. Using Lemma 4, we infer that $\mathcal{R}_{E,D_{s,k}} = \mathcal{R}_{E,\theta}$ and $r(\theta(E))$ has a lower triangular operator matrix with respect to the decomposition (3.6) for all rational functions $r(E) \in \mathcal{R}_{E,\theta}$ due to (3.7). Thus $B_{s,\theta}(\mathcal{B}(G_{s+1} \wedge E_{k-1}, X)) \subseteq \mathcal{B}(G_{s+1} \wedge E_{k-1}, X)$, that is, $\mathcal{B}(G_{s+1} \wedge E_{k-1}, X)$ is invariant under the subalgebra $\mathcal{A}_\theta$. $\square$

Now let again $F = E/G$, $[E, G] = \{0\}$, X a Banach F-module with a Lie representation $\beta : F \to \mathcal{B}(X)$ and let $\alpha : E \to \mathcal{B}(X)$, $\alpha = \beta \cdot \tau$, where $\tau : E \to F$ is the quotient map. The spaces $\mathcal{B}(\wedge F, X)$ and $\mathcal{B}(\wedge E, X)$ are turning into modules over F and E, respectively by the θ type representations. To distinct their denotations we write θ_β and θ_α for them, respectively. Consider the cochain complex $C^\bullet(\alpha)$ generated by α. One can easily verify that $d^k \mathcal{B}(G_s \wedge E_k, X) \subseteq \mathcal{B}(G_s \wedge E_{k+1}, X)$ ($G \subseteq \ker(\alpha)$), where d^k is the differential of the complex $C^\bullet(\alpha)$, whence all $C_s^\bullet(\alpha) = \left\{\mathcal{B}(G_s \wedge E_k, X), d^k, k \in \mathbb{Z}_+\right\}$ are subcomplexes of the cochain complex $C^\bullet(\alpha)$. Moreover, as follows from the above reasoning and (3.3), $C_s^\bullet(\alpha)$ is a complex of E-modules by the representation θ_α. Let us also introduce a cochain complex $\mathcal{B}(G_s, C^\bullet(\beta))$ as the result of action of the functor $\mathcal{B}(G_s, ?)$ subjected to the cochain complex $C^\bullet(\beta)$ generated by β. The latter is a complex of F-modules by the left regular representation taken by θ_β. Therefore $\mathcal{B}(G_s, C^\bullet(\beta))$ is a complex of E-modules along the Lie homomorphism τ.

Lemma 8. *The following sequences*

$$0 \to \mathcal{B}(G_s, \mathcal{B}(F_k, X)) \xrightarrow{\tau_{s,k}^X} \mathcal{B}(G_s \wedge E_k, X) \xrightarrow{\iota^X} \mathcal{B}(G_{s+1} \wedge E_{k-1}, X) \to 0$$

are exact sequences of E-modules, which for each $s \in \mathbb{Z}_+$ associate an exact sequence

$$0 \to \mathcal{B}\left(G_s, C^\bullet(\beta)\right) \xrightarrow{\tau_s^X} C_s^\bullet(\alpha) \xrightarrow{\iota^X} C_{s+1}^\bullet(\alpha) \to 0$$

of E-module complexes, where $\tau_{s,k}^X \omega = \omega \cdot \tau_{s,k}$, $\iota^X \omega = \omega \cdot \iota$.

Proof. The exactness of all these complexes follows from Lemma 6. It is beyond a doubt, ι^X is a morphism of E-modules, even $\mathcal{A}_{\theta_\alpha}$-modules (see Proposition 2). Demonstrate that $\tau_{s,k}^X$ is a E-module morphism. Take $\omega \in \mathcal{B}(G_s, \mathcal{B}(F_k, X))$, $a \in E$, $\underline{u} \wedge \underline{v} \in G_s \wedge E_k$. Then

$$\begin{aligned}
\tau_{s,k}^X \left(L_{\theta_\beta \tau(a)} \omega\right) \underline{u} \wedge \underline{v} &= L_{\theta_\beta \tau(a)} \omega \left(\tau_{s,k} \left(\underline{u} \wedge \underline{v}\right)\right) = L_{\theta_\beta \tau(a)} \omega \left(\underline{u} \otimes \wedge^k \tau\left(\underline{v}\right)\right) \\
&= \left(\theta_\beta \left(\tau\left(a\right)\right) \omega\left(\underline{u}\right)\right) \left(\wedge^k \tau\left(\underline{v}\right)\right) \\
&= \alpha\left(a\right) \omega\left(\underline{u}\right) \left(\wedge^k \tau\left(\underline{v}\right)\right) - \omega\left(\underline{u}\right) \left(T_k\left(\tau\left(a\right)\right) \wedge^k \tau\left(\underline{v}\right)\right) \\
&= \alpha\left(a\right) \omega\left(\underline{u}\right) \left(\wedge^k \tau\left(\underline{v}\right)\right) - \omega\left(\underline{u}\right) \left(\wedge^k \tau T_k\left(a\right)\left(\underline{v}\right)\right) \\
&= \alpha\left(a\right) \omega\left(\tau_{s,k}\left(\underline{u} \wedge \underline{v}\right)\right) - \omega\left(\tau_{s,k}\left(\underline{u} \wedge T_k\left(a\right) \underline{v}\right)\right) \\
&= \theta_\alpha\left(a\right) \tau_{s,k}^X\left(\omega\right) \underline{u} \wedge \underline{v},
\end{aligned}$$

that is, $\tau_{s,k}^X L_{\theta_\beta \tau(a)} = \theta_\alpha(a)\, \tau_{s,k}^X$. The rest is clear. □

Remark 1. Note that the grading powers of the complex morphisms τ_s^X and ι^X from Lemma 8 are equal 0 and -1, respectively.

Now let $H_\beta^i = H^i\left(C^\bullet(\beta)\right)$, $H_\alpha^i = H^i\left(C^\bullet(\alpha)\right)$, $H_s^i = H^i\left(C_s^\bullet(\alpha)\right)$, $i, s \in \mathbb{Z}_+$, be the cohomologies.

Lemma 9. *Let* $n = \dim(E)$, $m = \dim(F)$. *Then* $H_\alpha^i = \{0\}$ *for all* i, $n - k \leqq i \leqq n$, *iff* $H_\beta^i = \{0\}$ *for all* i, $m - k \leqq i \leqq m$, *where* $k \in \mathbb{Z}_+$.

Proof. By Lemma 8, we have exact sequence

$$0 \to \mathcal{B}\left(G_s, C^\bullet(\beta)\right) \xrightarrow{\tau_s^X} C_s^\bullet(\alpha) \xrightarrow{\iota^X} C_{s+1}^\bullet(\alpha) \to 0$$

of complexes, for each $s \in \mathbb{Z}_+$. Taking into account Remark 1, let us write the induced long exact sequence of cohomologies:

$$\begin{aligned}
0 &\to \mathcal{B}\left(G_s, H_\beta^0\right) \to H_s^0 \to 0 \to \cdots \to H_{s+1}^{i-2} \to \mathcal{B}\left(G_s, H_\beta^i\right) \to H_s^i \to H_{s+1}^{i-1} \to \cdots \\
&\to \mathcal{B}\left(G_s, H_\beta^m\right) \to H_s^m \to H_{s+1}^{m-1} \to 0 \to H_s^{m+1} \to H_{s+1}^m \to 0 \to \cdots. \qquad (3.8)
\end{aligned}$$

It follows that $H_s^{m+i} = H_{s+1}^{m+i-1}$, $0 \leqslant s \leqslant n-m$, $i \in \mathbb{N}$. The complex (3.8) for $s = n-m$ has the following form:

$$0 \to H_\beta^0 \to H_{n-m}^0 \to 0 \to \cdots \to H_{n-m+1}^{i-2} \to H_\beta^i \to H_{n-m}^i \to H_{n-m+1}^{i-1} \to \cdots$$
$$\to H_\beta^m \to H_{n-m}^m \to H_{n-m+1}^{m-1} \to 0 \to H_{n-m}^{m+1} \to H_{n-m+1}^m \to 0 \to \cdots.$$

With $G_{n-m+1} = \{0\}$ in mind, infer that $H_{n-m+1}^i = 0$ and $H_\beta^i = H_{n-m}^i$, $i \in \mathbb{Z}_+$. By definition, $H_\alpha^i = H_0^i$ and from (3.8) for $s = 0$, we obtain that the following exact complex

$$0 \to H_\beta^0 \to H_\alpha^0 \to 0 \to \cdots \to H_1^{i-2} \to H_\beta^i \to H_\alpha^i \to H_1^{i-1} \to \cdots$$
$$\to H_\beta^m \to H_\alpha^m \to H_1^{m-1} \to 0 \to H_\alpha^{m+1} \to H_1^m \to 0 \to \cdots.$$

Now assume that $H_\beta^i = \{0\}$ for all i, $m-k \leqslant i \leqslant m$. For fixed i, $n-k \leqslant i \leqslant n$, from the latter sequence we deduce that $H_\alpha^i = H_1^{i-1}$. By the same reasoning, from (3.8) for $s = 1$, we deduce that $H_1^{i-1} = H_2^{i-2}$ and so on. Then $H_\alpha^i = H_{n-m}^{i-n+m} = H_\beta^{i-n+m} = \{0\}$ $(m-k \leqslant i-n+m)$.

Conversely, assume that $H_0^i = H_\alpha^i = \{0\}$, $n-k \leqslant i \leqslant n$. We have to prove that $H_\beta^{m-i} = \{0\}$, $0 \leqslant i \leqslant k$. We proceed by induction on k. If $k = 0$ then $H_\beta^m = H_{n-m}^m = H_{n-m-1}^{m+1} = \cdots = H_0^n = \{0\}$, whence $H_\beta^m = \{0\}$.

Now let $k > 0$. By induction hypothesis, $H_\beta^{m-i} = \{0\}$, $0 \leqslant i \leqslant k-1$. Using the exactness (3.8) again, we infer that $H_s^m = H_{s+1}^{m-1}$, $H_s^{m-1} = H_{s+1}^{m-2}, \ldots, H_s^{m-k+1} = H_{s+1}^{m-k}$ for all s. Fix t, $1 \leqslant t \leqslant n-m$. For $s = t-1$, we deduce that $H_t^{m-k} = H_{t-1}^{m-k+1}$, for $s = t-2$, $H_{t-1}^{m-k+1} = H_{t-2}^{m-k+2}$ etc. We conclude that $H_t^{m-k} = H_{t-k}^m$. In particular, $H_\beta^{m-k} = H_{n-m}^{m-k} = H_{n-m-k}^m$. But $H_{n-m-k}^m = H_{n-m-k-1}^{m+1} = \cdots = H_0^{n-k} = H_\alpha^{n-k} = \{0\}$, that is, $H_\beta^{m-k} = \{0\}$. □

Proposition 3. *Let E be a finite-dimensional nilpotent Lie algebra, $\tau : E \to F$ a Lie epimorphism, and let (X, β) be a F-module. Then $\sigma(\beta \cdot \tau) = \sigma(\beta) \cdot \tau$ for all $\sigma \in \mathfrak{S}$.*

Proof. At first, note that spectra $\sigma_{\delta,k}$ and $\sigma^{\delta,k}$ (resp., $\sigma_{\pi,k}$ and $\sigma^{\pi,k}$) coincide on the class of nilpotent Lie algebra representations due to [13, Proposition 3.1]. Moreover, one suffices to prove the assertion only for spectra $\sigma = \sigma^{\delta,k} \in \mathfrak{S}^\delta$. Indeed, if the assertion has been proved for all $\sigma \in \mathfrak{S}^\delta$ then, by using (3.1), we conclude that

$$\sigma^{\pi,k}(\beta \cdot \tau) = \sigma_{\pi,k}(\beta \cdot \tau) = \sigma^{\delta,k}\left((\beta \cdot \tau)^*\right) = \sigma^{\delta,k}\left(\beta^* \cdot \tau\right) = \sigma^{\delta,k}\left(\beta^*\right) \cdot \tau = \sigma_{\pi,k}(\beta) \cdot \tau$$
$$= \sigma^{\pi,k}(\beta) \cdot \tau.$$

Now assume that $\sigma = \sigma^{\delta,k}$. Take $\mu \in \sigma(\beta \cdot \tau)$ and $u \in E$ such that $\tau(u) = 0$. By the Projection Property [13] (see also below Proposition 9), $\mu(u) \in \mathrm{sp}(\beta(\tau(u))) = \{0\}$, whence the functional $\lambda \in \Delta(F)$, $\lambda \cdot \tau = \mu$, is defined correctly. Moreover, $\beta \cdot \tau - \mu = (\beta - \lambda) \cdot \tau$. If $[G, E] = \{0\}$, where $G = \ker(\tau)$, then $\lambda \in \sigma(\beta)$ and vice-verse, by virtue of Lemma 9.

Now assume that $[G, E] \neq \{0\}$. We use the argument from the proof of [13, Proposition 2.6]. Let us introduce Lie ideals $G_0 = G$, $G_k = [E, G_{k-1}]$, $k \geqslant 1$. Since E is a nilpotent Lie algebra, it follows that $G_{s+1} = \{0\}$ for some s. Moreover, the epimorphism τ splits into the product $\tau_1 \tau_2 \cdots \tau_{s+1}$, where $\tau_i : E/G_i \to E/G_{i-1}$ is the quotient map ($E/G_0 = F$). But $\ker(\tau_i) = G_{i-1}/G_i$ and $[E/G_i, \ker(\tau_i)] = \{0\}$ for all i. Then $\sigma(\beta)\tau = \sigma(\beta)\tau_1\tau_2\cdots\tau_{s+1} = \sigma(\beta\tau_1)\tau_2\cdots\tau_{s+1} = \cdots = \sigma(\beta\tau_1\tau_2\cdots\tau_{s+1}) = \sigma(\beta\tau)$, that is, $\sigma(\beta)\tau = \sigma(\beta\tau)$. □

4. The algebras dominating over a $\mathfrak{g}$-module

In this section, we introduce a formal model of a noncommutative functional calculus for a nilpotent Lie algebra $\mathfrak{g}$ and prove the relevant spectral mapping theorem. The central role plays splitting over a Banach $\mathfrak{g}$-module X elements motivating holomorphic functions in noncommuting variables from $\mathfrak{g}$ acting on X.

Everywhere below $\mathfrak{g}$ denotes a finite-dimensional nilpotent Lie algebra, $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, and (X, α) is a Banach $\mathfrak{g}$-module. Let $\mathcal{A}$ be a topological algebra. By a normed Lie subalgebra in $\mathcal{A}$ we mean a Lie subalgebra $\mathfrak{F} \subseteq \mathcal{A}$ furnished with a certain norm $\|\cdot\|$ such that $(\mathfrak{F}, \|\cdot\|)$ is a normed Lie algebra and the identity embedding $\mathfrak{F} \hookrightarrow \mathcal{A}$ is continuous. For instance, if $\mathcal{A}$ is a Banach algebra then a normed Lie algebra $(\mathfrak{F}, \|\cdot\|)$ is a normed Lie subalgebra in $\mathcal{A}$ whenever $\mathfrak{F} \subseteq \mathcal{A}$ and $\|\cdot\| \geqslant \|\cdot\|_{\mathcal{A}}$. The space of all continuous characters on $\mathcal{A}$ furnished with the $*$-weak topology is denoted by $\mathrm{Spec}(\mathcal{A})$.

4.1. Properties of the dominating algebras

The following definition generalizes the dominated Banach algebras proposed in [6, Section 7].

Definition 4. Let $\mathcal{A}_{\mathfrak{g}}$ be a Hausdorff locally convex algebra with a fixed Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathcal{A}_{\mathfrak{g}}$. We say that $\mathcal{A}_{\mathfrak{g}}$ dominates over the module (X, α) and write $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha)$, if there exists a continuous algebra homomorphism $\widehat{\theta} : \mathcal{A}_{\mathfrak{g}} \to \mathcal{B}(\mathcal{B}(\wedge\mathfrak{g}, X))$ such that $\widehat{\theta} \cdot \pi = \theta$ and $\widehat{\theta}(\mathrm{im}(\widehat{\pi}))$ is dense in $\widehat{\theta}(\mathcal{A}_{\mathfrak{g}})$, where $\widehat{\pi} : \mathcal{R}_{\mathfrak{g},\pi} \to \mathcal{A}_{\mathfrak{g}}$ is the extension of the map π (see Section 2.3). The elements from the subalgebra $\mathrm{im}(\widehat{\pi}) (= \mathcal{R}(\mathrm{im}(\pi)))$ are called rational functions in $\mathcal{A}_{\mathfrak{g}}$ acting on X.

Example 1. If $\mathcal{A}_{\mathfrak{g}}$ is the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ furnished with the finest locally convex topology and π is the canonical embedding $\mathfrak{g} \to \mathcal{U}(\mathfrak{g})$, then $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha)$ for each $\mathfrak{g}$-module X.

Example 2. With respect to each $\mathfrak{g}$-module (X, α) one defines a dominating over that module Banach algebra $\mathcal{A}_\theta$ as the closure of the full subalgebra $\mathcal{R}(\theta(\mathfrak{g})) \subseteq \mathcal{B}(\mathcal{B}(\wedge\mathfrak{g}, X))$ generated by the Lie subalgebra $\theta(\mathfrak{g})$, and the representation θ stands itself instead of a Lie homomorphism $\pi : \mathfrak{g} \to \mathcal{A}_\theta$. Undoubtedly, $\mathcal{A}_\theta \succ (X, \alpha)$.

Other examples will be considered later in Section 6 (see also [6]).

Lemma 10. *If $\mathcal{A}_\mathfrak{g} \succ (X, \alpha)$ then $\mathcal{A}_\mathfrak{g}^{op} \succ (X^*, \alpha^*)$, where $\mathcal{A}_\mathfrak{g}^{op}$ is the opposite to $\mathcal{A}_\mathfrak{g}$ algebra. Moreover, $\mathcal{A}_\mathfrak{g} \succ (X_\mathfrak{U}, \alpha_\mathfrak{U})$ for an ultrafilter $\mathfrak{U}$.*

Proof. By Corollary 1, the dual (to θ) representation $\theta^* : \mathfrak{g}^{op} \to \mathcal{B}\left(\mathcal{B}(\wedge\mathfrak{g}, X)^*\right)$ is reduced (to within an isomorphism) to the representation $\theta' : \mathfrak{g}^{op} \to \mathcal{B}(\mathcal{B}(\wedge\mathfrak{g}, X^*))$, $\theta'(a) = L_{\alpha^*(a)} - R_{T^{op}(a)}$, extended the dual representation α^*. Then $\widehat{\theta}^*\pi(a) = \theta'(a)$ to within an isomorphism for all $a \in \mathfrak{g}$, where $\widehat{\theta}^* : \mathcal{A}_\mathfrak{g}^{op} \to \mathcal{B}\left(\mathcal{B}(\wedge\mathfrak{g}, X)^*\right)$, $\widehat{\theta}^*(a) = \widehat{\theta}(a)^*$, is the dual (to $\widehat{\theta}$) representation. The latter means that $\mathcal{A}_\mathfrak{g}^{op} \succ (X^*, \alpha^*)$ by Definition 4. The relation $\mathcal{A}_\mathfrak{g} \succ (X_\mathfrak{U}, \alpha_\mathfrak{U})$ can be proved on the same matter as in [6, Lemma 7.2]. $\square$

Now let $\mathcal{A}_\mathfrak{g} \succ (X, \alpha)$. It is clear that $C^k(\mathfrak{g}, X)$ is a complemented $\mathcal{A}_\mathfrak{g}$-invariant subspace in $\mathcal{B}(\wedge\mathfrak{g}, X)$ for each k. We set $\widehat{\theta}_k(a) = \widehat{\theta}(a)|_{C^k(\mathfrak{g},X)}$, $a \in \mathcal{A}_\mathfrak{g}$. In particular, $X \in \mathcal{A}_\mathfrak{g}$-*mod* $(X = C^0(\mathfrak{g}, X))$. We denote the relevant bounded representation $\mathcal{A}_\mathfrak{g} \to \mathcal{B}(X)$ by $\alpha|^{\mathcal{A}_\mathfrak{g}}$, thus $\alpha|^{\mathcal{A}_\mathfrak{g}} \cdot \pi = \alpha$ and $\alpha|^{\mathcal{A}_\mathfrak{g}}(\mathcal{R}(\mathrm{im}(\pi)))$ is dense in $\alpha|^{\mathcal{A}_\mathfrak{g}}\left(\mathcal{A}_\mathfrak{g}\right)$.

Let I be a Lie ideal in $\mathfrak{g}$. Then $C^k(I, X)$ is a $\mathfrak{g}$-module by the representation

$$\theta_{k,I} : \mathfrak{g} \to \mathcal{B}\left(C^k(I, X)\right), \quad \theta_{k,I}(u) = L_{\alpha(u)} - R_{T_{k,I}(u)}$$

(see Section 3.2) and the restriction map $C^k(\mathfrak{g}, X) \to C^k(I, X)$, $\omega \mapsto \omega|_I$ $(\omega|_I = \omega|_{\wedge^k I})$, is a $\mathfrak{g}$-module homomorphism.

Proposition 4. *Let $\mathcal{A}_\mathfrak{g} \succ (X, \alpha)$. Then $C^k(I, X)$ makes into a Banach $\mathcal{A}_\mathfrak{g}$-module extending its $\mathfrak{g}$-module structure such that the restriction map $C^k(\mathfrak{g}, X) \to C^k(I, X)$ is a morphism in $\mathcal{A}_\mathfrak{g}$-mod.*

Proof. Since $\mathfrak{g}$ is a nilpotent Lie algebra, the ideal I can be included into a Jordan–Holder series of ideals having one-dimensional gaps by virtue of Engel theorem [3, Ch. 1, Section 4]. Therefore, one suffices to prove the assertion for an ideal I of codimension 1. Take such an ideal I and let $e \notin I$. Note that the map

$$C^k(\mathfrak{g}, X) \to C^k(I, X) \oplus C^{k-1}(I, X), \quad \omega \mapsto (\omega|_I, (i_k(e)\,\omega)|_I),$$

implements a topological isomorphism in **BS** due to [6, Lemma 6.3]. If we identify $C^k(\mathfrak{g}, X)$ with the direct sum $C^k(I, X) \oplus C^{k-1}(I, X)$ by means of the isomorphism then the restriction map $C^k(\mathfrak{g}, X) \to C^k(I, X)$ would be the projection onto first subspace. Fix $a \in \mathfrak{g}$. The operator $\theta_k(a)$ has the following matrix form

$$\begin{pmatrix} \theta_{k,I}(a) & 0 \\ G_k([e,a]) & \theta_{k-1,I}(a) \end{pmatrix} \tag{4.1}$$

with respect to the decomposition, where

$$G_k(b) : C^k(I, X) \to C^{k-1}(I, X), \quad G_k(b)(\omega|_I) = (i_k(b)\,\omega)|_I,$$

$\omega \in C^k(\mathfrak{g}, X)$, $b \in I$. Indeed, if A is the matrix (4.1) then using (3.5), we deduce

$$\begin{aligned} A\omega &= A(\omega|_I, (i_k(e)\,\omega)|_I) = \left(\theta_{k,I}(a)(\omega|_I), G_k([e,a])(\omega|_I) + \theta_{k-1,I}(a)(i_k(e)\,\omega)|_I\right) \\ &= ((\theta_k(a)\,\omega)|_I, (i_k([e,a])\,\omega + \theta_{k-1}(a)\,i_k(e)\,\omega)|_I) \\ &= ((\theta_k(a)\,\omega)|_I, (i_k(e)\,\theta_k(a)\,\omega)|_I) \\ &= \theta_k(a)\,\omega. \end{aligned}$$

Now let us introduce the following operators

$$D_k(a) = \begin{pmatrix} \theta_{k,I}(a) & 0 \\ 0 & \theta_{k-1,I}(a) \end{pmatrix}, \quad N_k(b) = \begin{pmatrix} 0 & 0 \\ G_k(b) & 0 \end{pmatrix},$$

where $a \in \mathfrak{g}$, $b \in I$. Using (3.5) again, we infer that $[D_k(a), N_k(b)] = N_k([a,b])$. Moreover, $[D_k(a_1), D_k(a_2)] = D_k([a_1,a_2])$ and $N_k(b_1)\,N_k(b_2) = 0$ for all $a_i \in \mathfrak{g}$ and $b_i \in I$, $i = 1,2$. It follows that the Lie subalgebra $E \subseteq \mathcal{B}\left(C^k(I,X)\right)$ generated by these operators is a finite-dimensional nilpotent Lie algebra. By Lemma 3, the closure B of the full subalgebra $\mathcal{R}(E) \subseteq \mathcal{B}\left(C^k(I,X)\right)$ generated by E is commutative modulo its radical $Rad B$. Then $N_k(b) \in Rad B$, $b \in I$, and $\mathcal{R}_{\mathfrak{g},D_k} \subseteq \mathcal{R}_{\mathfrak{g},\theta}$. Moreover, $\widehat{\theta}_k(r(\overline{\mathfrak{g}})) - r(D_k(\mathfrak{g})) \in Rad B$, $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},D_k}$, where $\overline{\mathfrak{g}} = \operatorname{im}(\pi)$. Taking into account that B is a full subalgebra, we deduce that

$$\operatorname{sp}(r(D_k(\mathfrak{g}))) = \operatorname{sp}_B(r(D_k(\mathfrak{g}))) = \operatorname{sp}_B\left(\widehat{\theta}_k(r(\overline{\mathfrak{g}}))\right) = \operatorname{sp}\left(\widehat{\theta}_k(r(\overline{\mathfrak{g}}))\right)$$

for all $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},D_k}$. By Lemma 4, $\mathcal{R}_{\mathfrak{g},D_k} = \mathcal{R}_{\mathfrak{g},\theta}$ and

$$\widehat{\theta}_k(r(\overline{\mathfrak{g}})) = \begin{pmatrix} r\left(\theta_{k,I}(\mathfrak{g})\right) & 0 \\ * & r\left(\theta_{k-1,I}(\mathfrak{g})\right) \end{pmatrix}$$

for all $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\theta}$. It follows that $C^k(I,X)$ is invariant under $\widehat{\theta}_k(r(\overline{\mathfrak{g}}))$ and

$$\left(\widehat{\theta}_k(r(\overline{\mathfrak{g}}))\,\omega\right)|_I = r\left(\theta_{k,I}(\mathfrak{g})\right)(\omega|_I),$$

$\omega \in C^k(\mathfrak{g},X)$. But, $\widehat{\theta}_k(\mathrm{im}(\widehat{\pi}))$ is dense in $\widehat{\theta}_k\left(\mathcal{A}_{\mathfrak{g}}\right)$ (see Definition 4), therefore $C^k(I,X)$ is a $\mathcal{A}_{\mathfrak{g}}$-submodule and the restriction map $C^k(\mathfrak{g},X) \to C^k(I,X)$ is a $\mathcal{A}_{\mathfrak{g}}$-module homomorphism. □

Let $\widehat{\theta}_{k,I} : \mathcal{A}_{\mathfrak{g}} \to \mathcal{B}\left(C^k(I,X)\right)$ be a bounded representation defining $\mathcal{A}_{\mathfrak{g}}$-module structure on $C^k(I,X)$ suggested in Proposition 4. Then $\widehat{\theta}_{k,I}\cdot\pi = \theta_{k,I}$ and $\mathcal{B}(\wedge I, X)$ makes into a Banach $\mathcal{A}_{\mathfrak{g}}$-module by the representation $\widehat{\theta}_I = \oplus_{k\in\mathbb{Z}_+}\widehat{\theta}_{k,I}$.

Corollary 2. *Let $\mathcal{A}_{\mathfrak{g}} \succ (X,\alpha)$ and let $\mathfrak{F}$ be a normed Lie subalgebra in $\mathcal{A}_{\mathfrak{g}}$. Then $\sigma\left(\widehat{\theta}_k|_{\mathfrak{F}}\right) \subseteq \sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$, $\sigma \in \mathfrak{S}$, $k \in \mathbb{N}$. In particular, $\sigma\left(\widehat{\theta}|_{\mathfrak{F}}\right) = \sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$ for all $\sigma \in \mathfrak{S}$.*

Proof. One suffices to prove that $\sigma\left(\widehat{\theta}_{k,I}|_{\mathfrak{F}}\right) \subseteq \sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$ for all ideals $I \subseteq \mathfrak{g}$. As in the proof of Lemma 8.5 from [6], we proceed by induction on the pair $(k, \dim(I))$. Take an ideal $J \subset I$ such that $\dim(I/J) = 1$ and $\left[\mathfrak{g}, I\right] \subseteq J$. We have an admissible ($\mathbb{C}$-split) sequence $0 \to C^{k-1}(J,X) \longrightarrow C^k(I,X) \longrightarrow C^k(J,X) \to 0$ of Banach $\mathcal{A}_{\mathfrak{g}}$-modules by virtue of Proposition 4, and this in turn associates an exact sequence of $\Delta(\mathfrak{F})$-Banach complexes $0 \to \mathcal{C}\left(\widehat{\theta}_{k-1,J}|_{\mathfrak{F}}\right) \longrightarrow \mathcal{C}\left(\widehat{\theta}_{k,I}|_{\mathfrak{F}}\right) \longrightarrow \mathcal{C}\left(\widehat{\theta}_{k,J}|_{\mathfrak{F}}\right) \to 0$. It follows that $\sigma\left(\widehat{\theta}_{k,I}|_{\mathfrak{F}}\right) \subseteq \sigma\left(\widehat{\theta}_{k-1,J}|_{\mathfrak{F}}\right) \cup \sigma\left(\widehat{\theta}_{k,J}|_{\mathfrak{F}}\right)$ by [6, Corollary 3.5]. By induction hypothesis, $\sigma\left(\widehat{\theta}_{k-1,J}|_{\mathfrak{F}}\right) \cup \sigma\left(\widehat{\theta}_{k,J}|_{\mathfrak{F}}\right) \subseteq \sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$, therefore $\sigma\left(\widehat{\theta}_{k,I}|_{\mathfrak{F}}\right) \subseteq \sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$. Finally, $\sigma\left(\widehat{\theta}|_{\mathfrak{F}}\right) = \sigma\left(\oplus_{k\in\mathbb{Z}_+}\widehat{\theta}_k|_{\mathfrak{F}}\right) = \bigcup_{k\in\mathbb{Z}_+}\sigma\left(\widehat{\theta}_k|_{\mathfrak{F}}\right) = \sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$. □

Corollary 3. *Let $\mathcal{A}_{\mathfrak{g}} \succ (X,\alpha)$. Then*

$$d(\lambda)\,\widehat{\theta}(a) = \widehat{\theta}(a)\,d(\lambda), \quad \mathrm{sp}\left(\widehat{\theta}(a)\right) = \mathrm{sp}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}(a)\right)$$

for all $a \in \mathcal{A}_{\mathfrak{g}}$, where $d(\lambda)$ is the differential of the complex $C^\bullet(\alpha-\lambda)$, $\lambda \in \Delta(\mathfrak{g})$. In particular, $C^\bullet(\alpha-\lambda) \in \overline{\mathcal{A}_{\mathfrak{g}}\text{-}mod}$.

Proof. By Definition 4, $\mathrm{im}\left(\widehat{\theta}\right) \subseteq \mathcal{A}_\theta$, where $\mathcal{A}_\theta = \overline{\mathcal{R}(\theta(\mathfrak{g}))}$ (see Example 2). Moreover, using (3.3), we conclude that $d(\lambda)\,\theta_\lambda(a) = \theta_\lambda(a)\,d(\lambda)$ for all $a \in \mathfrak{g}$, where $\theta_\lambda(a) = L_{(\alpha-\lambda)(a)} - R_{T(a)}$. Note that $\theta_\lambda(a) = \theta(a) - \lambda(a)$, whence $d(\lambda)\,\theta(a) = \theta(a)\,d(\lambda)$. The latter obviously implies that $d(\lambda)\,T = T\,d(\lambda)$ for all $T \in \mathcal{A}_\theta$.

To prove the equality $\mathrm{sp}\left(\widehat{\theta}(a)\right) = \mathrm{sp}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}(a)\right)$, one suffices to set $\mathfrak{F} = \mathbb{C}a$ and $\sigma = \sigma_{\mathrm{t}}$ in Corollary 2. $\square$

Corollary 4. *Let (X, α) be a Banach $\mathfrak{g}$-module. Then $\mathcal{R}_{\mathfrak{g},\theta} = \mathcal{R}_{\mathfrak{g},\alpha}$.*

Proof. By definition, $\theta = \oplus_{k\in\mathbb{Z}_+}\theta_k$ and $\theta_0 = \alpha$. It follows that $\mathcal{R}_{\mathfrak{g},\theta} \subseteq \mathcal{R}_{\mathfrak{g},\alpha}$. Further, if $\mathcal{A}_\theta$ is the closed full subalgebra in $\mathcal{B}(\mathcal{B}(\wedge\mathfrak{g}, X))$ generated by $\theta(\mathfrak{g})$, then $\mathcal{A}_\theta \succ (X, \alpha)$ (see Example 2). By Corollary 3, $\mathrm{sp}(r(\theta(\mathfrak{g}))) = \mathrm{sp}(r(\alpha(\mathfrak{g})))$ for all $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\theta}$. It remains to use Lemma 4. $\square$

Now let $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha)$, $\mathfrak{F}$ a normed Lie subalgebra in $\mathcal{A}_{\mathfrak{g}}$ and let $\widehat{\mathfrak{F}}$ be the norm-completion of $\mathfrak{F}$. Let us introduce a bicomplex (as in [6]) connecting parametrized Banach space complexes $\mathcal{C}^\bullet(\alpha)$ and $\mathcal{C}^\bullet\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\widehat{\mathfrak{F}}}\right)$. The following diagram

$$\begin{array}{ccccc} & & \vdots & & \\ & & \beta_\mu \uparrow & & \\ \cdots & \stackrel{\delta_\lambda}{\to} & C^s\left(\widehat{\mathfrak{F}}, C^k(\mathfrak{g}, X)\right) & \stackrel{\delta_\lambda}{\to} & \cdots \\ & & \beta_\mu \uparrow & & \\ & & \vdots & & \end{array}$$

is commutative, where $\delta_\lambda(\Phi) = d^k(\lambda)\cdot\Phi$, $\Phi \in C^s\left(\widehat{\mathfrak{F}}, C^k(\mathfrak{g}, X)\right)$ ($d^k(\lambda)$ is the differential of the complex $C^\bullet(\alpha-\lambda)$) and β_μ is the differential of $C^\bullet\left(\widehat{\theta}_k|_{\widehat{\mathfrak{F}}} - \mu\right)$ (see Corollary 3) Thus we deal with a parametrized Banach space bicomplex $\mathcal{B}_{\lambda,\mu}(\mathfrak{g}, \mathfrak{F}, X)$, $\lambda \in \Delta(\mathfrak{g})$, $\mu \in \Delta(\mathfrak{F})$, with rows $\mathcal{B}\left(\wedge^s\widehat{\mathfrak{F}}, C^\bullet(\alpha-\lambda)\right)$, $s \in \mathbb{Z}_+$, and columns $C^\bullet\left(\widehat{\theta}_k|_{\widehat{\mathfrak{F}}} - \mu\right)$, $k \in \mathbb{Z}_+$, for which we use the denotation $\mathcal{B}(\mathfrak{g}, \mathfrak{F}, X)$. The total complex of $\mathcal{B}_{\lambda,\mu}(\mathfrak{g}, \mathfrak{F}, X)$ is denoted by $\mathrm{Tot}_{\lambda,\mu}(\mathfrak{g}, \mathfrak{F}, X)$. Then

$$\mathrm{Tot}(\mathfrak{g}, \mathfrak{F}, X) = \left\{\mathrm{Tot}_{\lambda,\mu}(\mathfrak{g}, \mathfrak{F}, X) : (\lambda, \mu) \in \Delta(\mathfrak{g}) \times \Delta(\mathfrak{F})\right\}$$

is a parametrized Banach space complex and their Slodkowski spectra are denoted by $\sigma(\mathfrak{g}, \mathfrak{F}, X)$, $\sigma \in \mathfrak{S}$.

Proposition 5. *Let $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha)$, $\mathfrak{F}$ a normed Lie subalgebra in $\mathcal{A}_{\mathfrak{g}}$, $\mathfrak{U}$ an ultrafilter and let $\widetilde{\alpha} = \alpha|^{\mathcal{A}_{\mathfrak{g}}}$. If $\widehat{\mathfrak{F}} \in Proj$ then parametrized Banach space complexes $\mathcal{C}^\bullet(\alpha_{\mathfrak{U}})$ and $\mathcal{C}^\bullet(\widetilde{\alpha}_{\mathfrak{U}}|_{\widehat{\mathfrak{F}}})$ are π-spectrally connected by means of the $\Delta(\mathfrak{g}) \times \Delta(\mathfrak{F})$-Banach bicomplex $\mathcal{B}(\mathfrak{g}, \mathfrak{F}, X_{\mathfrak{U}})$.*

Proof. Note that $\mathcal{A}_{\mathfrak{g}} \succ (X_{\mathfrak{U}}, \alpha_{\mathfrak{U}})$ by Lemma 10. Let $\widehat{\theta_{\mathfrak{U}}} : \mathcal{A}_{\mathfrak{g}} \to \mathcal{B}(\mathcal{B}(\wedge\mathfrak{g}, X_{\mathfrak{U}}))$ be the representation extending $\alpha_{\mathfrak{U}}$. Since $C^\bullet(\alpha_{\mathfrak{U}}) \in \overline{\mathcal{A}_{\mathfrak{g}}\text{-}mod}$, it follows that $\mathcal{B}(\mathfrak{g}, \mathfrak{F}, X_{\mathfrak{U}})$ is a $\Delta(\mathfrak{g}) \times \Delta(\mathfrak{F})$-Banach bicomplex. Further,

$$\sigma\left(\mathcal{C}^{\bullet}\left(\widehat{\theta}_{\mathfrak{U}}|_{\mathfrak{F}}\right)\right) = \bigcup_{k \in \mathbb{Z}_+} \sigma\left(\widehat{\theta}_{k\mathfrak{U}}|_{\mathfrak{F}}\right) \subseteq \sigma\left(\widetilde{\alpha}_{\mathfrak{U}}|_{\mathfrak{F}}\right), \quad \sigma \in \mathfrak{S}^{\pi},$$

by virtue of Corollary 2. Moreover,

$$\bigcup_{s \in \mathbb{Z}_+} \sigma\left(\mathcal{B}\left(\wedge^{s}\widehat{\mathfrak{F}}, \mathcal{C}^{\bullet}(\alpha_{\mathfrak{U}})\right)\right) \subseteq \sigma\left(\mathcal{C}^{\bullet}(\alpha_{\mathfrak{U}})\right), \quad \sigma \in \mathfrak{S}^{\pi},$$

by Lemma 2 and Theorem 1(a). It follows that the Banach space complexes $\mathcal{C}^{\bullet}(\alpha_{\mathfrak{U}})$ and $\mathcal{C}^{\bullet}(\widetilde{\alpha}_{\mathfrak{U}}|_{\mathfrak{F}})$ are π-spectrally connected (see Section 2.2) by means of $\mathcal{B}(\mathfrak{g}, \mathfrak{F}, X_{\mathfrak{U}})$. $\square$

Remark 2. One can prove the chain version (using the chain complex $\mathcal{C}_{\bullet}(\alpha_{\mathfrak{U}})$) of this result replacing the requirement $\widehat{\mathfrak{F}} \in Proj$ with $\widehat{\mathfrak{F}} \in$ Flat and using Theorem 1(a).

4.2. The subset of splitting over $\mathfrak{g}$*-module elements*

The splitting elements over a $\mathfrak{g}$-module play a fundamental role on the backward spectral mapping property.

Let (X, α) be a Banach $\mathfrak{g}$-module. As follows from (3.4), $\theta(u) - \lambda(u) = d(\lambda)\, i(u) + i(u)\, d(\lambda)$ for all $u \in \mathfrak{g}$, where $\lambda \in \Delta(\mathfrak{g})$, $d(\lambda) \in \mathcal{B}(\mathcal{B}(\wedge\mathfrak{g}, X))$ is the differential of the complex $C^{\bullet}(\alpha - \lambda)$ and $i(u) \in \mathcal{B}(\mathcal{B}(\wedge\mathfrak{g}, X))$ is the homotopy operator induced by u. The latter relation can be enlarged to all rational functions acting on X by the following way.

Proposition 6. *Let $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha)$, $\mu : \mathcal{A}_{\mathfrak{g}} \to \mathbb{C}$ a character and let a be a rational function in $\mathcal{A}_{\mathfrak{g}}$ acting on X. There exists an operator $i_{\mu}(a) \in \mathcal{B}(\mathcal{B}(\wedge\mathfrak{g}, X))$ such that*

$$\widehat{\theta}(a) - \mu(a) = d(\mu \cdot \pi)\, i_{\mu}(a) + i_{\mu}(a)\, d(\mu \cdot \pi).$$

Moreover, if $\lambda \in \sigma_{\mathrm{t}}(\alpha)$ then assignment $\widetilde{\lambda} : \mathcal{R}(\theta(\mathfrak{g})) \to \mathbb{C}$, $r(\theta(\mathfrak{g})) \mapsto r(\lambda(\mathfrak{g}))$, defines a character $\widetilde{\lambda} \in \operatorname{Spec}(\mathcal{A}_{\theta})$. In particular, $\widehat{\theta}(a) - \lambda|^{\mathcal{A}_{\mathfrak{g}}}(a) = d(\lambda)\, i_{\lambda}(a) + i_{\lambda}(a)\, d(\lambda)$, where $\lambda|^{\mathcal{A}_{\mathfrak{g}}} = \widetilde{\lambda} \cdot \widehat{\theta} \in \operatorname{Spec}\left(\mathcal{A}_{\mathfrak{g}}\right)$ and $i_{\lambda}(a) \in \mathcal{B}(\mathcal{B}(\wedge\mathfrak{g}, X))$.

Proof. To prove the first equality, one suffices to proceed by induction on the order of rational function a and use (3.4) and (3.3). Take $\lambda \in \sigma_{\mathrm{t}}(\alpha)$. On the same ground as in [6, Lemma 8.3], one can prove that $r(\lambda(\mathfrak{g})) \in \operatorname{sp}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}\overline{r}(\mathfrak{g})\right)$ for all $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\theta}$, where $\overline{r}(\mathfrak{g}) = \widehat{\pi}(r(\mathfrak{g}))$. Thus $\mathcal{R}_{\mathfrak{g},\theta} \subseteq \mathcal{R}_{\mathfrak{g},\lambda}$. If $r_1(\theta(\mathfrak{g})) = r_2(\theta(\mathfrak{g}))$ for some $r_1(\mathfrak{g}), r_2(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\theta}$, then $(r_1 - r_2)(\lambda(\mathfrak{g})) \in \operatorname{sp}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}(\overline{r}_1(\mathfrak{g}) - \overline{r}_2(\mathfrak{g}))\right) = \operatorname{sp}((r_1 - r_2)(\theta(\mathfrak{g}))) = \{0\}$ by virtue of Corollary 3, whence the assignment $\widetilde{\lambda} : \mathcal{R}(\theta(\mathfrak{g})) \to \mathbb{C}$, $r(\theta(\mathfrak{g})) \mapsto$

$r(\lambda(\mathfrak{g}))$, is well defined. Moreover, $\widetilde{\lambda}$ is continuous, for $\widetilde{\lambda}(r(\theta(\mathfrak{g}))) \in \mathrm{sp}(r(\theta(\mathfrak{g})))$. The rest is clear. $\square$

Remark 3. Note that $i_\mu(a) = \sum_{s=1}^{k} \widehat{\theta}(a_1 \cdots a_{s-1})\, i(a_s)\, \mu(a_{s+1} \cdots a_k)$ whenever $a = a_1 \cdots a_k$, $a_i = \pi(u_i)$, $u_i \in \mathfrak{g}$, $i(a_s) \in \mathcal{B}(\mathcal{B}(\wedge\mathfrak{g}, X))$ is the homotopy operator. The latter immediate from (3.3) and (3.4).

Corollary 5. *The assignment* $\sigma_{\mathrm{t}}(\alpha) \to \mathrm{Spec}\left(\mathcal{A}_{\mathfrak{g}}\right)$, $\lambda \mapsto \lambda|^{\mathcal{A}_{\mathfrak{g}}}$, *is a continuous mapping.*

Proof. Fix $a \in \mathcal{A}_{\mathfrak{g}}$. We have to prove that the function $f_a : \sigma_{\mathrm{t}}(\alpha) \to \mathbb{C}$, $f_a(\lambda) = \lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)$, is continuous. If $a = \widehat{\pi}(r(\mathfrak{g}))$ for some $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\pi}$, then f_a is reduced to a usual rational function $\lambda \mapsto r(\lambda(\mathfrak{g}))$ by virtue of Proposition 6. Therefore f_a is continuous. In general case, $\widehat{\theta}(a) = \lim_k \widehat{\theta}(a_k)$ of some sequence $a_k = \widehat{\pi}(r_k(\mathfrak{g}))$, $\{r_k(\mathfrak{g})\} \subseteq \mathcal{R}_{\mathfrak{g},\pi}$, by Definition 4. Then $f_a(\lambda) = \widetilde{\lambda}\left(\widehat{\theta}(a)\right) = \lim_k \widetilde{\lambda}\left(\widehat{\theta}(a_k)\right) = \lim_k \lambda|^{\mathcal{A}_{\mathfrak{g}}}(a_k) = \lim_k f_k(\lambda)$ for each point $\lambda \in \sigma_{\mathrm{t}}(\alpha)$, where $f_k \in C(\sigma_{\mathrm{t}}(\alpha))$, $f_k = f_{a_k}$, $k \in \mathbb{N}$. Moreover, bearing in mind that the norm of all characters (in particular, $\widetilde{\lambda}$, $\lambda \in \sigma_{\mathrm{t}}(\alpha)$) of a Banach algebra are at most one, we infer that

$$\sup_{\lambda \in \sigma_{\mathrm{t}}(\alpha)} |f_a(\lambda) - f_k(\lambda)| = \sup_{\lambda \in \sigma_{\mathrm{t}}(\alpha)} \left|\widetilde{\lambda}\left(\widehat{\theta}(a) - \widehat{\theta}(a_k)\right)\right| \leqslant \sup_{\lambda \in \sigma_{\mathrm{t}}(\alpha)} \left\|\widetilde{\lambda}\right\| \left\|\widehat{\theta}(a) - \widehat{\theta}(a_k)\right\|$$

$$\leqslant \left\|\widehat{\theta}(a) - \widehat{\theta}(a_k)\right\| \to 0, \quad n \to \infty.$$

Thus f_a as a uniform limit of the sequence $\{f_k\}$ of continuous functions on the compact space $\sigma_{\mathrm{t}}(\alpha)$ [1, Ch.4, Section 25] is turning into a continuous mapping, that is, $f_a \in C(\sigma_{\mathrm{t}}(\alpha))$. $\square$

The image of a Slodkowski spectrum $\sigma(\alpha)$, $\sigma \in \mathfrak{S}$, under the mapping from Corollary 5 is denoted by $\sigma(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}$, thus $\sigma(\alpha)|^{\mathcal{A}_{\mathfrak{g}}} \subseteq \mathrm{Spec}\left(\mathcal{A}_{\mathfrak{g}}\right)$.

Definition 5. Let $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha)$. An element $a \in \mathcal{A}_{\mathfrak{g}}$ is said to be splitting over $\mathfrak{g}$-module X if for each $\lambda \in \sigma_{\mathrm{t}}(\alpha)$ there exists $n \in \mathbb{N}$ (called splitting power with respect to λ) and an operator $i_{n,\lambda}(a) \in \mathcal{B}(\mathcal{B}(\wedge\mathfrak{g}, X))$ such that

$$\left(\widehat{\theta}(a) - \lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)\right)^n = d(\lambda)\, i_{n,\lambda}(a) + i_{n,\lambda}(a)\, d(\lambda).$$

An element $a \in \mathcal{A}_{\mathfrak{g}}$ is said to be weak splitting over $\mathfrak{g}$-module X if for each $\lambda \in \sigma_{\mathrm{t}}(\alpha)$ the actions of $\widehat{\theta}(a) - \lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)$ on the cohomologies $H^k C^\bullet(\alpha - \lambda)$, $k \in \mathbb{Z}_+$ (see Corollary 3), are nilpotent. The set of all (resp., weak) splitting over $\mathfrak{g}$-module X elements is denoted by $\mathcal{A}_{\mathfrak{g}}(\alpha)$ (resp., $\mathcal{A}_{\mathfrak{g}}\langle\alpha\rangle$).

Obviously, $\mathcal{A}_{\mathfrak{g}}(\alpha) \subseteq \mathcal{A}_{\mathfrak{g}}\langle\alpha\rangle$, and $\mathrm{im}(\widehat{\pi}) \subseteq \mathcal{A}_{\mathfrak{g}}(\alpha)$ by virtue of Proposition 6. Moreover, all rational functions have splitting powers equal 1 with respect to all characters λ

taken from $\sigma_{\mathrm{t}}(\alpha)$. Let us note that a subset in $\mathcal{A}_{\mathfrak{g}}(\alpha)$ of those elements having splitting powers equal 1 with respect to all $\lambda \in \sigma_{\mathrm{t}}(\alpha)$ is a subalgebra in $\mathcal{A}_{\mathfrak{g}}$ containing $\mathrm{im}(\widehat{\pi})$. Indeed, for a such couple $a, b \in \mathcal{A}_{\mathfrak{g}}(\alpha)$, we have

$$\begin{aligned}\widehat{\theta}(ab) - \lambda|^{\mathcal{A}_{\mathfrak{g}}}(ab) &= \widehat{\theta}(a)\left(\widehat{\theta}(b) - \lambda|^{\mathcal{A}_{\mathfrak{g}}}(b)\right) + \lambda|^{\mathcal{A}_{\mathfrak{g}}}(b)\left(\widehat{\theta}(a) - \lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)\right)\\ &= d(\lambda)\left(\widehat{\theta}(a)\, i_{1,\lambda}(b) + \lambda|^{\mathcal{A}_{\mathfrak{g}}}(b)\, i_{1,\lambda}(a)\right)\\ &\quad + \left(\widehat{\theta}(a)\, i_{1,\lambda}(b) + \lambda|^{\mathcal{A}_{\mathfrak{g}}}(b)\, i_{1,\lambda}(a)\right) d(\lambda)\end{aligned}$$

by virtue of Corollary 3. It follows that the splitting power of ab equals 1.

Later in Section 6, we shall consider various examples of dominating algebras. We prove that all of them are comprised by splitting elements which are automatically weak splitting ones as confirmed above. We have no example of a weak splitting element which is not splitting one. It seems that they would be found in a complexioned versions of the considered examples, for instance assuming D to be arbitrary (not necessary Stein) domain in Section 6.4. Meanwhile, one might confirm that the weak splitting elements play key role to have more generalized finite-dimensional spectral mapping theorem (see below Theorem 6).

Now we investigate a stability property of splitting elements under homomorphisms.

Lemma 11. *Let* $\tau : \mathfrak{a} \to \mathfrak{b}$ *be an epimorphism of finite-dimensional Lie algebras*, (X, α) *a Banach* $\mathfrak{b}$*-module and let* $\mathcal{P}$ *be an associative subalgebra in* $\mathcal{B}(\mathcal{B}(\wedge\mathfrak{a}, X))$ *such that for each* $G \in \mathcal{P}$ *there corresponds* $G_\tau \in \mathcal{B}(\mathcal{B}(\wedge\mathfrak{b}, X))$ *such that* $G(\omega \cdot \wedge\tau) = G_\tau(\omega) \cdot \wedge\tau$ *for all* $\omega \in \mathcal{B}(\wedge\mathfrak{b}, X)$, *where* $\wedge\tau \in \mathcal{B}(\wedge\mathfrak{a}, \wedge\mathfrak{b})$ *is the exterior power of* τ. *Then correspondence* $\mathcal{P} \to \mathcal{B}(\mathcal{B}(\wedge\mathfrak{b}, X))$, $G \mapsto G_\tau$, *is a bounded algebra homomorphism.*

Proof. Since $\wedge\tau$ is a surjective map, it follows that $\omega \cdot \wedge\tau = \omega' \cdot \wedge\tau$ iff $\omega = \omega'$ for some $\omega, \omega' \in \mathcal{B}(\wedge\mathfrak{b}, X)$. Hence, if $G_\tau(\omega) \cdot \wedge\tau = G(\omega \cdot \wedge\tau) = G'_\tau(\omega) \cdot \wedge\tau$ for some $G_\tau, G'_\tau \in \mathcal{B}(\mathcal{B}(\wedge\mathfrak{b}, X))$, and for all $\omega \in \mathcal{B}(\wedge\mathfrak{b}, X)$, then $G_\tau = G'_\tau$. Thus we have a well-defined linear map $G \mapsto G_\tau$. Moreover, if $G, G' \in \mathcal{P}$, then $\left(G_\tau G'_\tau\right)(\omega) \cdot \wedge\tau = G_\tau\left(G'_\tau(\omega)\right) \cdot \wedge\tau = G\left(G'_\tau(\omega) \cdot \wedge\tau\right) = G\left(G'(\omega \cdot \wedge\tau)\right) = \left(GG'\right)(\omega \cdot \wedge\tau)$. It follows that $\left(GG'\right)_\tau = G_\tau G'_\tau$, that is, the map $\mathcal{P} \to \mathcal{B}(\mathcal{B}(\wedge\mathfrak{b}, X))$, $G \mapsto G_\tau$, is an algebra homomorphism. It remains to prove that this homomorphism is bounded. Since $\wedge\mathfrak{b}$ is the quotient (normed) space of $\wedge\mathfrak{a}$, it follows that $\left\|G_\tau(\omega)\left(\underline{v}\right)\right\| = \left\|G_\tau(\omega)\left((\wedge\tau)\left(\underline{u}\right)\right)\right\| = \left\|(G_\tau(\omega) \cdot \wedge\tau)\left(\underline{u}\right)\right\| \leqslant \|G(\omega \cdot \wedge\tau)\| \left\|\underline{u}\right\|$, where $\underline{v} = (\wedge\tau)\left(\underline{u}\right)$, that is, $\|G_\tau(\omega)\| \leqslant \|G(\omega \cdot \wedge\tau)\|$. Then $\|G_\tau(\omega)\| \leqslant \|G\| \|\omega\| \|\wedge\tau\|$ for all ω. Finally, $\|G_\tau\| \leqslant \|G\| \|\wedge\tau\|$, so, the map $G \mapsto G_\tau$ is bounded. $\square$

Lemma 12. *Let* $\tau : \mathfrak{g} \to \mathfrak{h}$ *be an epimorphism of finite-dimensional nilpotent Lie algebras*, (X, β) *a Banach* $\mathfrak{b}$*-module*, $\alpha = \beta \cdot \tau$, *and let* $\theta_\alpha : \mathfrak{g} \to \mathcal{B}(\mathcal{B}(\wedge\mathfrak{g}, X))$ *and* $\theta_\beta : \mathfrak{h} \to \mathcal{B}(\mathcal{B}(\wedge\mathfrak{h}, X))$ *be the Lie representations induced by modules* (X, α) *and* (X, β), *respectively. There exists a bounded algebra homomorphism* $\widehat{\tau} : \mathcal{A}_{\theta_\alpha} \to \mathcal{A}_{\theta_\beta}$

such that $\widehat{\tau}(\theta_\alpha(u)) = \theta_\beta(\tau(u))$ *for all* $u \in \mathfrak{g}$, *where* $\mathcal{A}_{\theta_\alpha} \succ (X, \alpha)$ *and* $\mathcal{A}_{\theta_\beta} \succ (X, \beta)$ *are the dominating algebras* (*see Example* 2).

Proof. Let $u \in \mathfrak{g}$. Since τ is a Lie homomorphism, it follows that $\wedge\tau \cdot T_{\mathfrak{g}}(u) = T_{\mathfrak{h}}(\tau(u)) \cdot \wedge\tau$, where $T_{\mathfrak{g}}(u) \in \mathcal{B}(\wedge\mathfrak{g})$ (resp., $T_{\mathfrak{h}}(v) \in \mathcal{B}(\wedge\mathfrak{h})$, $v \in \mathfrak{h}$) is the operator extending $\mathrm{ad}(u)$ (resp., $\mathrm{ad}(v)$). Then

$$\begin{aligned}\theta_\alpha(u)(\omega \cdot \wedge\tau) &= \left(L_{\alpha(u)} - R_{T_{\mathfrak{g}}(u)}\right)(\omega \cdot \wedge\tau) = \beta(\tau(u)) \cdot \omega \cdot \wedge\tau - \omega \cdot T_{\mathfrak{h}}(\tau(u)) \cdot \wedge\tau \\ &= \left(\left(L_{\beta(\tau(u))} - R_{T_{\mathfrak{h}}(\tau(u))}\right)\omega\right) \cdot \wedge\tau = \theta_\beta(\tau(u))(\omega) \cdot \wedge\tau,\end{aligned}$$

$\omega \in \mathcal{B}(\wedge\mathfrak{h}, X)$. Further, note that one uniquely defines a map $\widetilde{\tau} : \mathcal{R}_{\mathfrak{g},\alpha} \to \mathcal{R}_{\mathfrak{h},\beta}$ extending τ such that $\widehat{\beta} \cdot \widetilde{\tau} = \widehat{\alpha}$, by virtue of Lemma 5. Moreover, $\mathcal{R}_{\mathfrak{g},\theta_\alpha} = \mathcal{R}_{\mathfrak{g},\alpha}$ and $\mathcal{R}_{\mathfrak{h},\theta_\beta} = \mathcal{R}_{\mathfrak{h},\beta}$ due to Corollary 4. Let us prove that

$$r(\theta_\alpha(\mathfrak{g}))(\omega \cdot \wedge\tau) = \widetilde{r}\left(\theta_\beta(\mathfrak{h})\right)(\omega) \cdot \wedge\tau \tag{4.2}$$

for all rational functions $r(\theta_\alpha(\mathfrak{g})) \in \mathcal{A}_{\theta_\alpha}$ (here $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\theta_\alpha}$), where $\widetilde{r}(\mathfrak{h}) = \widetilde{\tau}(r(\mathfrak{g}))$. At first, note that $\mathrm{sp}(r(\theta_\alpha(\mathfrak{g}))) = \mathrm{sp}\left(\widetilde{r}\left(\theta_\beta(\mathfrak{h})\right)\right)$. Indeed, using Corollary 3, we infer that $\mathrm{sp}(a) = \mathrm{sp}\left(\alpha|^{\mathcal{A}_{\theta_\alpha}}(a)\right)$, $a \in \mathcal{A}_{\theta_\alpha}$, and $\mathrm{sp}(b) = \mathrm{sp}\left(\beta|^{\mathcal{A}_{\theta_\beta}}(b)\right)$, $b \in \mathcal{A}_{\theta_\beta}$. Then

$$\begin{aligned}\mathrm{sp}(r(\theta_\alpha(\mathfrak{g}))) &= \mathrm{sp}\left(\alpha|^{\mathcal{A}_{\theta_\alpha}} r(\theta_\alpha(\mathfrak{g}))\right) = \mathrm{sp}(r(\alpha(\mathfrak{g}))) = \mathrm{sp}(\widehat{\alpha}(r(\mathfrak{g}))) = \mathrm{sp}\left(\widehat{\beta}(\widetilde{r}(\mathfrak{h}))\right) \\ &= \mathrm{sp}(\widetilde{r}(\beta(\mathfrak{h}))) = \mathrm{sp}\left(\beta|^{\mathcal{A}_{\theta_\beta}} \widetilde{r}\left(\theta_\beta(\mathfrak{h})\right)\right) = \mathrm{sp}\left(\widetilde{r}\left(\theta_\beta(\mathfrak{h})\right)\right).\end{aligned}$$

Further, since $\theta_\alpha(u)(\omega \cdot \wedge\tau) = \theta_\beta(\tau(u))(\omega) \cdot \wedge\tau$, $u \in \mathfrak{g}$, it follows that (4.2) is valid for all polynomials $r(\mathfrak{g})$. Moreover, using the equality for spectra of operators participating in both parts of (4.2), we deduce that if (4.2) is true for some invertible rational function $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\theta_\alpha}$ then it is also true for its inverse $r^{-1}(\mathfrak{g})$:

$$\omega \cdot \wedge\tau = \left(\widetilde{r}\left(\theta_\beta(\mathfrak{h})\right)\widetilde{r}^{-1}\left(\theta_\beta(\mathfrak{h})\right)\omega\right) \cdot \wedge\tau = r(\theta_\alpha(\mathfrak{g}))\left(\left(\widetilde{r}^{-1}\left(\theta_\beta(\mathfrak{h})\right)\omega\right) \cdot \wedge\tau\right),$$

which implies that $r^{-1}(\theta_\alpha(\mathfrak{g}))(\omega \cdot \wedge\tau) = \widetilde{r}^{-1}\left(\theta_\beta(\mathfrak{h})\right)(\omega) \cdot \wedge\tau$. By induction on the order of rational functions (see Section 2.3), we establish (4.2) for all $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\theta_\alpha}$.

Now we apply Lemma 11 for the subalgebra $\mathcal{P} = \mathcal{R}(\theta_\alpha(\mathfrak{g})) \subseteq \mathcal{A}_{\theta_\alpha}$. There exists a bounded algebra homomorphism $\gamma : \mathcal{P} \to \mathcal{A}_{\theta_\alpha}$ such that $\gamma(r(\theta_\alpha(\mathfrak{g}))) = \widetilde{r}\left(\theta_\beta(\mathfrak{h})\right)$. In particular, $\gamma(\theta_\alpha(u)) = \theta_\beta(\tau(u))$ for all $u \in \mathfrak{g}$. Its extension $\widehat{\tau}$ up to the closure $\mathcal{A}_{\theta_\alpha}$ of $\mathcal{P}$ is the required algebra homomorphism. □

Theorem 4. *Let* $\tau : \mathfrak{g} \to \mathfrak{h}$ *be an epimorphism of finite-dimensional nilpotent Lie algebras*, (X, β) *a Banach* $\mathfrak{b}$-*module and let* $\alpha = \beta \cdot \tau$. *Then* $\widehat{\tau}\left(\mathcal{A}_{\theta_\alpha}\langle\alpha\rangle\right) \subseteq \mathcal{A}_{\theta_\beta}\langle\beta\rangle$, *where* $\widehat{\tau} : \mathcal{A}_{\theta_\alpha} \to \mathcal{A}_{\theta_\beta}$ *is the bounded algebra homomorphism such that* $\widehat{\tau}(\theta_\alpha(u)) = \theta_\beta(\tau(u))$, $u \in \mathfrak{g}$.

Proof. At first, note that the homomorphism $\widehat{\tau}$ exists and it is unique due to Lemma 12. Let $G = \ker(\tau)$. Using the argument carried out in the proof of Proposition 3, one can reduce assertion to the case $[G, \mathfrak{g}] = \{0\}$. Indeed, if $[G, \mathfrak{g}] \neq \{0\}$ then τ splits into the product $\tau_1\tau_2\cdots\tau_{s+1}$ of Lie epimorphisms $\tau_i : \mathfrak{g}_i \to \mathfrak{g}_{i-1}$ such that $[\ker(\tau_i), \mathfrak{g}_i] = \{0\}$, where $\mathfrak{g}^{(s+1)} = \{0\}$, $\mathfrak{g}_i = \mathfrak{g}/G_i$, $G_0 = G$, $G_i = [\mathfrak{g}, G_{i-1}]$, $i \geqslant 1$. There exists a bounded algebra homomorphism $\widehat{\tau}_i : \mathcal{A}_i \to \mathcal{A}_{i-1}$ extending τ_i by virtue of Lemma 12, where $\mathcal{A}_i = \mathcal{A}_{\theta_{\beta_i}}$ is the dominating over $\mathfrak{g}_i$-module (X, β_i) from Example 2, where $\beta_i = \beta\tau_1\tau_2\cdots\tau_i$. Moreover, $\widehat{\tau} = \widehat{\tau}_1\cdots\widehat{\tau}_{s+1}$ by Lemma 12. If the assertion has been proved for the case $[G, \mathfrak{g}] = \{0\}$ then we would obtain that $\widehat{\tau}(\mathcal{A}_{\theta_\alpha}\langle\alpha\rangle) = \widehat{\tau}(\mathcal{A}_{s+1}\langle\beta_{s+1}\rangle) \subseteq \widehat{\tau}_1\cdots\widehat{\tau}_s(\mathcal{A}_s\langle\beta_s\rangle) \subseteq \cdots \subseteq \widehat{\tau}_1(\mathcal{A}_1\langle\beta_1\rangle) \subseteq \mathcal{A}_{\theta_\beta}\langle\beta\rangle$.

So, assume that $[G, \mathfrak{g}] = \{0\}$ and take $a \in \mathcal{A}_{\theta_\alpha}\langle\alpha\rangle$. By Definition 5, we have to prove that the operator $\widehat{\tau}(a) - \lambda|^{\mathcal{A}_{\theta_\beta}}(\widehat{\tau}(a))$ is nilpotent on all cohomologies $H^kC^\bullet(\beta - \lambda)$, for each $\lambda \in \sigma_{\mathrm{t}}(\beta)$. By Proposition 3, $\mu = \lambda\cdot\tau \in \sigma_{\mathrm{t}}(\alpha)$. Moreover, $\mu|^{\mathcal{A}_{\theta_\alpha}}(r(\theta_\alpha(\mathfrak{g}))) = r(\mu(\mathfrak{g}))$, $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\alpha} = \mathcal{R}_{\mathfrak{g},\theta_\alpha}$, and $\lambda|^{\mathcal{A}_{\theta_\beta}}(r(\theta_\beta(\mathfrak{h}))) = r(\lambda(\mathfrak{h}))$, $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{h},\beta} = \mathcal{R}_{\mathfrak{h},\theta_\beta}$, due to Proposition 6. It follows that $\mathcal{R}_{\mathfrak{g},\alpha} \subseteq \mathcal{R}_{\mathfrak{g},\mu}$ and $\mathcal{R}_{\mathfrak{h},\beta} \subseteq \mathcal{R}_{\mathfrak{h},\lambda}$. By Lemma 4, the Lie epimorphism τ has unique extension $\widetilde{\tau} : \mathcal{R}_{\mathfrak{g},\mu} \to \mathcal{R}_{\mathfrak{h},\lambda}$ such that $\widehat{\lambda}\cdot\widetilde{\tau} = \widehat{\mu}$, moreover, $\widetilde{\tau}(\mathcal{R}_{\mathfrak{g},\alpha}) \subseteq \mathcal{R}_{\mathfrak{h},\beta}$ and $\widehat{\beta}\cdot\widetilde{\tau} = \widehat{\alpha}$. Then

$$
\begin{aligned}
\mu|^{\mathcal{A}_{\theta_\alpha}}(r(\theta_\alpha(\mathfrak{g}))) &= r(\mu(\mathfrak{g})) = \widehat{\mu}(r(\mathfrak{g})) = \widehat{\lambda}(\widetilde{\tau}(r(\mathfrak{g}))) = \widehat{\lambda}(\widetilde{r}(\mathfrak{h})) = \widetilde{r}(\lambda(\mathfrak{h})) \\
&= \lambda|^{\mathcal{A}_{\theta_\beta}}(\widetilde{r}(\theta_\beta(\mathfrak{h}))) = \lambda|^{\mathcal{A}_{\theta_\beta}}\cdot\widehat{\tau}(r(\theta_\alpha(\mathfrak{g}))),
\end{aligned}
$$

where $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\alpha}$, $\widetilde{r}(\mathfrak{h}) = \widetilde{\tau}(r(\mathfrak{g}))$. Thus $\mu|^{\mathcal{A}_{\theta_\alpha}}(b) = \lambda|^{\mathcal{A}_{\theta_\beta}}\cdot\widehat{\tau}(b)$ for all $b \in \mathcal{R}(\theta_\alpha(\mathfrak{g})) \subseteq \mathcal{A}_{\theta_\alpha}$. Using the continuity of $\widehat{\tau}$ (Lemma 12), we infer that $\mu|^{\mathcal{A}_{\theta_\alpha}} = \lambda|^{\mathcal{A}_{\theta_\beta}}\cdot\widehat{\tau}$. It follows that $\widehat{\tau}(a) - \lambda|^{\mathcal{A}_{\theta_\beta}}(\widehat{\tau}(a)) = \widehat{\tau}(a - \mu|^{\mathcal{A}_{\theta_\alpha}}(a))$, whence one suffices to prove that if actions on the cohomologies $H^kC^\bullet(\alpha)$ of a certain $a \in \mathcal{A}_{\theta_\alpha}\langle\alpha\rangle$ are nilpotent then actions on the cohomologies $H^kC^\bullet(\beta)$ of $\widehat{\tau}(a)$ are also nilpotent. To prove that, we use the exact complexes

$$
0 \to \mathcal{B}(G_s, C^\bullet(\beta)) \xrightarrow{\tau_s^X} C_s^\bullet(\alpha) \xrightarrow{\iota^X} C_{s+1}^\bullet(\alpha) \to 0 \tag{4.3}
$$

of $\mathfrak{g}$-modules suggested in Lemma 8, where $s \in \mathbb{Z}_+$. With $\mathcal{A}_{\theta_\alpha} \succ (X, \alpha)$ (Example 2) in mind and taking into account that $C_s^\bullet(\alpha)$ is a subcomplex of $C^\bullet(\alpha)$, we infer that $C_s^\bullet(\alpha) \in \overline{\mathcal{A}_{\theta_\alpha}\text{-}mod}$ by virtue of Proposition 2 and Corollary 3. As follows from Proposition 2 again, that ι^X is a $\mathcal{A}_{\theta_\alpha}$-morphism of complexes. Moreover, $\mathcal{B}(G_s, C^\bullet(\beta)) \in \overline{\mathcal{A}_{\theta_\beta}\text{-}mod}$ furnished $\mathcal{A}_{\theta_\beta}$-module structure by the left regular representation (see Corollary 3), whence $\mathcal{B}(G_s, C^\bullet(\beta)) \in \overline{\mathcal{A}_{\theta_\alpha}\text{-}mod}$ along the algebra homomorphism $\widehat{\tau} : \mathcal{A}_{\theta_\alpha} \to \mathcal{A}_{\theta_\beta}$ suggested in Lemma 12. Actually, (4.3) is a short exact sequence of $\mathcal{A}_{\theta_\alpha}$-module complexes. Indeed, $\tau^X_{s,k} \cdot L_{\widehat{\tau}(b)} = b \cdot \tau^X_{s,k}$, $b \in \theta_\alpha(\mathfrak{g})$, by Lemmas 8, 12. It follows that the latter equality is valid for all $b \in \mathcal{A}_{\theta_\alpha}$, that is, (4.3) is a sequence of $\mathcal{A}_{\theta_\alpha}$-module complexes. As a corollary of this reasoning, we conclude that the following diagram

$$\begin{array}{ccccccccccc} \cdots & \to & H^{k-2}_{s+1} & \to & \mathcal{B}\left(G_s, H^k_\beta\right) & \to & H^k_s & \to & H^{k-1}_{s+1} & \to & \cdots \\ & & \uparrow a & & \uparrow L_{\widehat{\tau}(a)} & & \uparrow a & & \uparrow a & & \\ \cdots & \to & H^{k-2}_{s+1} & \to & \mathcal{B}\left(G_s, H^k_\beta\right) & \to & H^k_s & \to & H^{k-1}_{s+1} & \to & \cdots \end{array} \tag{4.4}$$

is commutative, where the first (and second) row is the long cohomology sequence associated by the short exact sequence (4.3) of complexes, the columns are the operators on cohomologies induced by $a \in \mathcal{A}_{\theta_\alpha}\langle\alpha\rangle$ and $\widehat{\tau}(a)$. We have to prove that all induced operators $\widehat{\tau}(a) \in \mathcal{B}\left(H^k_\beta\right)$, $k \in \mathbb{Z}_+$, are nilpotent. We proceed by induction on k. Bearing in mind that $C^\bullet_0(\alpha) = C^\bullet(\alpha)$, the assertion for $k = 0$ directly follows from (4.4) by setting $s = 0$ (here $H^0_\beta = H^0_0 = H^0_\alpha$).

Now let $k > 0$. By induction hypothesis, all operators $\widehat{\tau}(a) \in \mathcal{B}\left(H^i_\beta\right)$, $i \leqslant k-1$, are nilpotent. Applying Lemma 1 to the left side of the commutative diagram (4.4) for $s = 0$, we see that one suffices to prove the nilpotency of the operator $a \in \mathcal{B}\left(H^{k-2}_1\right)$. By induction on the pair (i, j), let us prove that all operators $a \in \mathcal{B}\left(H^j_{d-i}\right)$, $0 \leqslant i \leqslant d-1$, $0 \leqslant j \leqslant k-2$, are nilpotent, where $d = \dim(G)$. Note that $C^\bullet(\beta) = \mathcal{B}(G_d, C^\bullet(\beta)) = C^\bullet_d(\alpha)$ (to within an isomorphism in $\mathcal{A}_{\theta_\alpha}$-*mod*) by virtue of (4.3), whence all $a \in \mathcal{B}\left(H^t_d\right)$, $0 \leqslant t \leqslant k-1$, are nilpotent operators. Fix (i, j), $i > 0$. By induction hypothesis, the operators $a \in \mathcal{B}\left(H^{j-1}_{d-i+1}\right)$ and $\widehat{\tau}(a) \in \mathcal{B}\left(H^j_d\right)$ ($H^j_d = H^j_\beta$) are nilpotent. Then $L_{\widehat{\tau}(a)} \in \mathcal{B}\left(\mathcal{B}\left(G_{d-i}, H^j_\beta\right)\right)$ is also nilpotent, and using the right side of the commutative diagram (4.4) for $s = d-i$, and Lemma 1, we deduce that $a \in \mathcal{B}\left(H^j_{d-i}\right)$ is a nilpotent operator. In particular, $a \in \mathcal{B}\left(H^{k-2}_1\right)$ is nilpotent. Therefore $\widehat{\tau}(a) \in \mathcal{B}\left(H^k_\beta\right)$ is a nilpotent operator, too. □

Remark 4. Let $\tau : \mathfrak{g} \to \mathfrak{h}$ be an epimorphism of nilpotent Lie algebras, (X, β) a Banach $\mathfrak{b}$-module, $\alpha = \beta \cdot \tau$ and let $\mathcal{A}_\mathfrak{g} \succ (X, \alpha)$. Then $\widehat{\tau\theta_\alpha}\left(\mathcal{A}_\mathfrak{g}\langle\alpha\rangle\right) \subseteq \mathcal{A}_{\theta_\beta}\langle\beta\rangle$. Indeed, $\widehat{\theta_\alpha}\left(\mathcal{A}_\mathfrak{g}\langle\alpha\rangle\right) \subseteq \mathcal{A}_{\theta_\alpha}\langle\alpha\rangle$ by Definition 5, and $\widehat{\tau}\left(\mathcal{A}_{\theta_\alpha}\langle\alpha\rangle\right) \subseteq \mathcal{A}_{\theta_\beta}\langle\beta\rangle$ by Theorem 4.

4.3. The forward spectral mapping property

Now let $f : \Delta(\mathfrak{g}) \to \Delta(\mathfrak{F})$ be arbitrary continuous extension of the continuous map $\sigma_\mathrm{t}(\alpha) \to \Delta(\mathfrak{F})$, $\lambda \mapsto \lambda|^{\mathcal{A}_\mathfrak{g}}|_\mathfrak{F}$ from Corollary 5.

Lemma 13. *Let $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha)$, $\mathfrak{F}$ a normed Lie subalgebra, $\mathfrak{U}$ an ultrafilter and let $\widetilde{\alpha} = \alpha|^{\mathcal{A}_{\mathfrak{g}}}$. If $\widehat{\mathfrak{F}} \in Proj$ then f is a prespectral mapping with respect to the bicomplex $\mathcal{B}(\mathfrak{g}, \mathfrak{F}, X_{\mathfrak{U}})$ connecting $\mathcal{C}^{\bullet}(\alpha_{\mathfrak{U}})$ and $\mathcal{C}^{\bullet}(\widetilde{\alpha}_{\mathfrak{U}}|_{\mathfrak{F}})$.*

Proof. At first, note that the complexes $\mathcal{C}^{\bullet}(\alpha_{\mathfrak{U}})$ and $\mathcal{C}^{\bullet}(\widetilde{\alpha}_{\mathfrak{U}}|_{\mathfrak{F}})$ are π-spectrally connected by means of the bicomplex $\mathcal{B}(\mathfrak{g}, \mathfrak{F}, X_{\mathfrak{U}})$ due to Proposition 5. Now take $\lambda \in \Delta(\mathfrak{g})$. If $\lambda \notin \sigma_{\mathrm{t}}(\alpha)$ then noting is left to prove (see Definition 2). So, assume that $H^m C^{\bullet}(\alpha_{\mathfrak{U}} - \lambda)$ is a nontrivial Banach space, $\mu = f(\lambda)$, and let $\beta_{\widetilde{\mathfrak{U}\mu}} : H^m C^{\bullet}(\alpha_{\mathfrak{U}} - \lambda) \to H^m \mathcal{B}\left(\widehat{\mathfrak{F}}, C^{\bullet}(\alpha_{\mathfrak{U}} - \lambda)\right)$ be the differential of the mth vertical cohomology complex of the bicomplex $\mathcal{B}_{\lambda,\mu}(\mathfrak{g}, \mathfrak{F}, X_{\mathfrak{U}})$. Using the same argument as in the proof of Theorem 8.6 from [6], we obtain that $\beta_{\widetilde{\mathfrak{U}\mu}} = 0$, thereby f is a prespectral mapping. □

Theorem 5. *Let $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha)$, $\mathfrak{F}$ a normed Lie subalgebra in $\mathcal{A}_{\mathfrak{g}}$ and let $\sigma \in \mathfrak{S}^{\pi}$. If $\widehat{\mathfrak{F}} \in Proj$ then $\sigma(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}} \subseteq \sigma_u\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$.*

Proof. Let $\mathfrak{U}$ be an ultrafilter and let $\widetilde{\alpha} = \alpha|^{\mathcal{A}_{\mathfrak{g}}}$. By (3.2), $\mathcal{C}^{\bullet}(\alpha)_{\mathfrak{U}} = \mathcal{C}^{\bullet}(\alpha_{\mathfrak{U}})$. Moreover, the parametrized Banach space complexes $\mathcal{C}^{\bullet}(\alpha_{\mathfrak{U}})$ and $\mathcal{C}^{\bullet}(\widetilde{\alpha}_{\mathfrak{U}}|_{\mathfrak{F}})$ are π-spectrally connected by means of $\Delta(\mathfrak{g}) \times \Delta(\mathfrak{F})$-Banach bicomplex $\mathcal{B}(\mathfrak{g}, \mathfrak{F}, X_{\mathfrak{U}})$ due to Proposition 5. Now let $f : \Delta(\mathfrak{g}) \to \Delta(\mathfrak{F})$ be arbitrary continuous extension of the map $\sigma_{\mathrm{t}}(\alpha) \to \Delta(\mathfrak{F})$, $\lambda \mapsto \lambda|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}$. Then f is a prespectral mapping with respect to the bicomplex $\mathcal{B}(\mathfrak{g}, \mathfrak{F}, X_{\mathfrak{U}})$ connecting $\mathcal{C}^{\bullet}(\alpha)_{\mathfrak{U}}$ and $\mathcal{C}^{\bullet}(\widetilde{\alpha}_{\mathfrak{U}}|_{\mathfrak{F}})$ by Lemma 13. By Theorem 2, $f(\sigma(\alpha)) = f(\sigma(\mathcal{C}^{\bullet}(\alpha))) \subseteq \sigma(\mathcal{C}^{\bullet}(\widetilde{\alpha}_{\mathfrak{U}}|_{\mathfrak{F}})) = \sigma\left(\mathcal{C}^{\bullet}\left((\widetilde{\alpha}|_{\mathfrak{F}})_{\mathfrak{U}}\right)\right) \subseteq \sigma_u(\widetilde{\alpha}|_{\mathfrak{F}})$ (see Definition 3). □

Corollary 6. *Let $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha)$ and let $\mathfrak{F}$ be a finite-dimensional Lie subalgebra in $\mathcal{A}_{\mathfrak{g}}$. Then $\sigma(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}} \subseteq \sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$ for all $\sigma \in \mathfrak{S}_{\delta} \cup \mathfrak{S}^{\pi}$.*

Proof. Taking into account that $\mathfrak{F} \in Proj$, the inclusion for spectra $\sigma \in \mathfrak{S}^{\pi}$ immediately follows from (3.2) and Theorem 5.

Now fix $\sigma = \sigma_{\delta,k} \in \mathfrak{S}_{\delta}$. By Lemma 10, $\mathcal{A}_{\mathfrak{g}}^{op} \succ (X^*, \alpha^*)$. Using (3.1) and Theorem 5, we obtain that

$$\sigma(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}} = \sigma_{\delta,k}(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}} = \sigma^{\pi,k}(\alpha^*)|^{\mathcal{A}_{\mathfrak{g}}^{op}}|_{\mathfrak{F}^{op}} \subseteq \sigma_u^{\pi,k}\left(\alpha^*|^{\mathcal{A}_{\mathfrak{g}}^{op}}|_{\mathfrak{F}^{op}}\right) = \sigma^{\pi,k}\left(\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)^*\right)$$
$$= \sigma_{\delta,k}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right) = \sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right),$$

that is, $\sigma(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}} \subseteq \sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$. □

Corollary 7. *Let $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha)$ and let $\mathfrak{F}$ be a normed Lie subalgebra in $\mathcal{A}_{\mathfrak{g}}$. If $\widehat{\mathfrak{F}} \in Proj$ then $\sigma_{\delta,k}(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}} \subseteq \sigma_u^{\pi,k}\left(\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)^*\right)$, $k \in \mathbb{Z}_+$. In particular, $\sigma_{\delta,k}(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}} \subseteq \sigma_{\delta,k}^{u}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$ whenever X is superreflexive* [16].

Proof. Using Lemma 10 and Theorem 5, we infer that

$$\sigma_{\delta,k}(\alpha)\,|^{\mathcal{A}_\mathfrak{g}}|_{\mathfrak{F}} = \sigma^{\pi,k}\left(\alpha^*\right)|^{\mathcal{A}_\mathfrak{g}^{op}}|_{\mathfrak{F}^{op}} \subseteq \sigma_u^{\pi,k}\left(\gamma^*\right),$$

where $\gamma = \alpha|^{\mathcal{A}_\mathfrak{g}}|_{\mathfrak{F}}$. If X is superreflexive then $(X^*)_\mathfrak{U} = (X_\mathfrak{U})^*$ for a countably incomplete ultrafilter $\mathfrak{U}$ [16, Corollary 7.2]. In particular, $(\gamma^*)_\mathfrak{U} = \left(\gamma_\mathfrak{U}\right)^*$ and $\sigma_u^{\pi,k}(\gamma^*) = \bigcup_\mathfrak{U} \sigma^{\pi,k}\left((\gamma^*)_\mathfrak{U}\right) = \bigcup_\mathfrak{U} \sigma^{\pi,k}\left(\left(\gamma_\mathfrak{U}\right)^*\right) = \bigcup_\mathfrak{U} \sigma_{\delta,k}\left(\gamma_\mathfrak{U}\right) = \sigma^u_{\delta,k}(\gamma)$ by Definition 3. □

4.4. The backward spectral mapping property

Now we investigate the problem whether or not a continuous extension $f : \Delta(\mathfrak{g}) \to \Delta(\mathfrak{F})$ of the map $\sigma_\mathrm{t}(\alpha) \to \Delta(\mathfrak{F})$, $\lambda \mapsto \lambda|^{\mathcal{A}_\mathfrak{g}}|_{\mathfrak{F}}$, is a spectral mapping with respect to the $\Delta(\mathfrak{g}) \times \Delta(\mathfrak{F})$-Banach bicomplex $\mathcal{B}(\mathfrak{g}, \mathfrak{F}, X_\mathfrak{U})$ connecting the complexes $\mathcal{C}^\bullet(\alpha_\mathfrak{U})$ and $\mathcal{C}^\bullet(\widetilde{\alpha}_\mathfrak{U}|_{\mathfrak{F}})$, where $\widetilde{\alpha} = \alpha|^{\mathcal{A}_\mathfrak{g}}$.

Fix $m \in \mathbb{Z}_+$ and let $\alpha_\lambda = \alpha - \lambda$, $\lambda \in \Delta(\mathfrak{g})$. Then

$$0 \to H^m C^\bullet(\alpha_\lambda) \xrightarrow{\beta_\mu^\sim} H^m C^1\left(\widehat{\mathfrak{F}}, C^\bullet(\alpha_\lambda)\right) \to \cdots \to H^m C^s\left(\widehat{\mathfrak{F}}, C^\bullet(\alpha_\lambda)\right) \xrightarrow{\beta_\mu^\sim} \cdots \tag{4.5}$$

is the mth vertical cohomology complex of the bicomplex $\mathcal{B}_{\lambda,\mu}(\mathfrak{g}, \mathfrak{F}, X)$. By Corollary 3, $\ker d^m(\lambda)$ is a closed $\mathcal{A}_\mathfrak{g}$ (in particular, $\mathfrak{F}$)-submodule in $C^m(\mathfrak{g}, X)$, where $d^m(\lambda)$ is the differential of (0th row) the complex $C^\bullet(\alpha_\lambda)$. The complex $C^\bullet\left(\widehat{\theta}|_{\mathfrak{F}} - \mu|\ker d^m(\lambda)\right)$ generated by the $\mathfrak{F}$-module $\left(\ker d^m(\lambda), \widehat{\theta}|_{\mathfrak{F}} - \mu\right)$ is a Banach space complex of $\mathfrak{F}$-modules and it is a subcomplex of the mth column of $\mathcal{B}_{\lambda,\mu}(\mathfrak{g}, \mathfrak{F}, X)$. The $\mathfrak{F}$-module structure on this complex is defined by the θ-type representation (see Section 3.2)

$$\Theta_{s,\mu} : \mathfrak{F} \to \mathcal{B}\left(\mathcal{B}\left(\wedge^s \mathfrak{F}, \ker d^m(\lambda)\right)\right), \quad \Theta_{s,\mu}(a) = L_{\left(\widehat{\theta}-\mu\right)(a)} - R_{T_s(a)},$$

extending $\widehat{\theta}|_{\mathfrak{F}} - \mu$, and let $I_s(a)$ ($a \in \mathfrak{F}$) be the homotopy operator on the Banach space complex $C^\bullet\left(\widehat{\theta}|_{\mathfrak{F}} - \mu|\ker d^m(\lambda)\right)$. Using Corollary 3, one can easily verify that

$$L_{\widehat{\theta}(a)}\delta_\lambda^{m-1} = \delta_\lambda^{m-1} L_{\widehat{\theta}(a)}, \quad R_{T_s(a)}\delta_\lambda^{m-1} = \delta_\lambda^{m-1} R_{T_s(a)}$$

and $I_s(a)\,\delta_\lambda^{m-1} = \delta_\lambda^{m-1} I_s(a)$, where δ_λ^{m-1} is the row differential of $\mathcal{B}_{\lambda,\mu}(\mathfrak{g}, \mathfrak{F}, X)$. In particular, $\Theta_{s,\mu}(a)\,\delta_\lambda^{m-1} = \delta_\lambda^{m-1}\Theta_{s,\mu}(a)$ and the image $\mathrm{im}\left(\delta_\lambda^{m-1}\right)$ is invariant under all operators $L_{\widehat{\theta}(a)}$, $R_{T_s(a)}$, $\Theta_{s,\mu}(a)$ and $I_s(a)$, $a \in \mathfrak{F}$, thereby they induce operators on the cohomologies

$$L^\sim_{\widehat{\theta}(a)}, \Theta^\sim_{s,\mu}(a), R^\sim_{T_s(a)} \in \mathcal{B}\left(H^m C^s\left(\mathfrak{F}, C^\bullet(\alpha_\lambda)\right)\right),$$

$$I_s^\sim(a) \in \mathcal{B}\left(H^m C^s\left(\mathfrak{F}, C^\bullet(\alpha_\lambda)\right), H^m C^{s-1}\left(\mathfrak{F}, C^\bullet(\alpha_\lambda)\right)\right),$$

by the canonical way. Using (3.3) and (3.4) for the complex $C^{\bullet}\left(\widehat{\theta}|_{\mathfrak{F}}-\mu|\ker d^{m}(\lambda)\right)$ and by passing to the cohomologies, we obtain that

$$\beta_{\mu}^{\sim}\Theta_{s,\mu}^{\sim}(a)=\Theta_{s,\mu}^{\sim}(a)\,\beta_{\mu}^{\sim}, \tag{4.6}$$

$$\beta_{\mu}^{\sim}I_{s}^{\sim}(a)+I_{s+1}^{\sim}(a)\,\beta_{\mu}^{\sim}=\Theta_{s,\mu}^{\sim}(a). \tag{4.7}$$

The following lemma describes the operator $\Theta_{s,\mu}^{\sim}(a)$ when $a\in\mathcal{A}_{\mathfrak{g}}(\alpha)$.

Lemma 14. *Assume that* $a\in\mathfrak{F}\cap\mathcal{A}_{\mathfrak{g}}(\alpha)$, *or* $a\in\mathfrak{F}\cap\mathcal{A}_{\mathfrak{g}}\langle\alpha\rangle$ *and* $\dim(\mathfrak{F})<\infty$. *Then* $\Theta_{s,\mu}^{\sim}(a)=\lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)-\mu(a)-R_{T_{s}(a)}^{\sim}+N$, *where N is a nilpotent operator. In particular,* $0\notin\operatorname{sp}\left(\Theta_{s,\mu}^{\sim}(a)\right)$ *whenever* $\operatorname{sp}(\operatorname{ad}(a)|_{\mathfrak{F}})=\{0\}$ *and* $\lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)\neq\mu(a)$.

Proof. By definition of $\Theta_{s,\mu}^{\sim}(a)$, one suffices to prove that $N=L_{\widehat{\theta}(a)}^{\sim}-\lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)$ is a nilpotent operator on the cohomology $H^{m}C^{s}(\mathfrak{F},C^{\bullet}(\alpha_{\lambda}))$. Take $\Phi\in C^{s}(\mathfrak{F},\ker d^{m}(\lambda))$. By Definition 5, $\left(\widehat{\theta}(a)-\lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)\right)^{n}=d^{m-1}(\lambda)\,i_{n,\lambda}(a)+i_{n,\lambda}(a)\,d^{m}(\lambda)$ for a certain $n\in\mathbb{N}$ and some operator $i_{n,\lambda}(a)\in\mathcal{B}(\mathcal{B}(\wedge\mathfrak{g},X))$ whenever $a\in\mathfrak{F}\cap\mathcal{A}_{\mathfrak{g}}(\alpha)$. Then

$$\left(L_{\widehat{\theta}(a)}-\lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)\right)^{n}\Phi=d^{m-1}(\lambda)\,i_{n,\lambda}(a)\,\Phi+i_{n,\lambda}(a)\,d^{m}(\lambda)\,\Phi=\delta_{\lambda}^{m-1}\left(L_{i_{n,\lambda}(a)}\Phi\right),$$

thereby $\left(L_{\widehat{\theta}(a)}^{\sim}-\lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)\right)^{n}\Phi^{\sim}=\left(\delta_{\lambda}^{m-1}\left(L_{i_{n,\lambda}(a)}\Phi\right)\right)^{\sim}=0$. If $a\in\mathfrak{F}\cap\mathcal{A}_{\mathfrak{g}}\langle\alpha\rangle$ and $\dim(\mathfrak{F})<\infty$, then

$$\left(\left(L_{\widehat{\theta}(a)}-\lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)\right)^{n}\Phi\right)(\underline{u})=\left(\widehat{\theta}(a)-\lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)\right)^{n}(\Phi(\underline{u}))\in\operatorname{im}d^{m-1}(\lambda)$$

for all $\underline{u}\in\wedge^{s}\mathfrak{F}$, and therefore $\left(L_{\widehat{\theta}(a)}-\lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)\right)^{n}\Phi=d^{m-1}(\lambda)\cdot\Psi$ for some $\Psi\in C^{s}\left(\mathfrak{F},C^{m-1}(\mathfrak{g},X)\right)$. It follows again that $\left(L_{\widehat{\theta}(a)}^{\sim}-\lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)\right)^{n}\Phi^{\sim}=0$.

Now let us assume that $\operatorname{sp}(\operatorname{ad}(a)|_{\mathfrak{F}})=\{0\}$ and $z_{a}=\lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)-\mu(a)\neq0$. Then operator of the adjoint representation $\operatorname{ad}(a)\in\mathcal{B}\left(\widehat{\mathfrak{F}}\right)$ is quasinilpotent. It follows that all $T_{s}(a)\in\mathcal{B}\left(\wedge^{s}\widehat{\mathfrak{F}}\right)$, $s\in\mathbb{Z}_{+}$, are also quasinilpotent by [6, Lemma 6.1]. The latter involves that $\operatorname{sp}\left(R_{T_{s}(a)}\right)=\{0\}$ for all s. But, $R_{T_{s}(a)}\in\mathcal{B}(C^{s}(\mathfrak{F},\ker d^{m}(\lambda)))$ is a Banach space operator, so, the series $G_{s}(a)=\sum_{k=0}^{\infty}\left(z_{a}^{-1}R_{T_{s}(a)}\right)^{k}$ converges absolutely in $\mathcal{B}(C^{s}(\mathfrak{F},\ker d^{m}(\lambda)))$ and $\delta_{\lambda}^{m-1}G_{s}(a)=G_{s}(a)\,\delta_{\lambda}^{m-1}$. Moreover, the operator

$G_s^{\sim}(a) \in \mathcal{B}(H^m C^s(\mathfrak{F}, \mathcal{C}^\bullet(\alpha_\lambda)))$ commutes with N. Indeed,

$$\left[G_s^{\sim}(a), N\right] = \left[G_s(a), L_{\widehat{\theta}(a)}\right]^{\sim} = \left(\sum_{k=0}^{\infty} z_a^{-1}\left[R_{T_s(a)}^k, L_{\widehat{\theta}(a)}\right]\right)^{\sim} = 0^{\sim}.$$

Note also that $z_a^{-1}G_s(a) = \left(z_a - R_{T_s(a)}\right)^{-1}$ and $z_a^{-1}G_s^{\sim}(a) = \left(z_a - R_{T_s(a)}^{\sim}\right)^{-1}$. Finally,

$$\begin{aligned} z_a^{-1}G_s^{\sim}(a)\,\Theta_{s,\mu}^{\sim}(a) &= z_a^{-1}\Theta_{s,\mu}^{\sim}(a)\,G_s^{\sim}(a) = z_a^{-1}G_s^{\sim}(a)\left(z_a - R_{T_s(a)}^{\sim} + N\right) \\ &= 1 + z_a^{-1}G_s^{\sim}(a)\,N. \end{aligned}$$

It is clear that $1 + z_a^{-1}G_s^{\sim}(a)\,N$ is invertible and

$$\left(1 + z_a^{-1}G_s^{\sim}(a)\,N\right)^{-1} = \sum_{k=0}^{n-1}(-1)^k\left(z_a^{-1}G_s^{\sim}(a)\,N\right)^k.$$

Thus $0 \notin \operatorname{sp}\left(\Theta_{s,\mu}^{\sim}(a)\right)$. □

Proposition 7. *Let $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha)$ and let S be a subset in $\mathcal{A}_{\mathfrak{g}}(\alpha)$ (resp., $\mathcal{A}_{\mathfrak{g}}\langle\alpha\rangle$) generating a quasinilpotent normed (resp., finite-dimensional) Lie subalgebra $\mathfrak{F} \subseteq \mathcal{A}_{\mathfrak{g}}$. If $\widehat{\mathfrak{F}} \in Proj$ then $f : \Delta(\mathfrak{g}) \to \Delta(\mathfrak{F})$ is a spectral mapping with respect to the bicomplex $\mathcal{B}(\mathfrak{g}, \mathfrak{F}, X_{\mathfrak{U}})$ connecting $\mathcal{C}^\bullet(\alpha_{\mathfrak{U}})$ and $\mathcal{C}^\bullet(\widetilde{\alpha}_{\mathfrak{U}}|_{\mathfrak{F}})$, where $\widetilde{\alpha} = \alpha|^{\mathcal{A}_{\mathfrak{g}}}$.*

Proof. At first, note that if $S \subseteq \mathcal{A}_{\mathfrak{g}}\langle\alpha\rangle$ generates a finite-dimensional Lie subalgebra $\mathfrak{F} \subseteq \mathcal{A}_{\mathfrak{g}}$ then $\mathfrak{F} = \widehat{\mathfrak{F}} \in Proj$. We have already proved (see Lemma 13) that f is a prespectral mapping whenever $\widehat{\mathfrak{F}} \in Proj$. So, it remains (see Definition 2) to prove that all vertical cohomology complexes (4.5) of the bicomplex $\mathcal{B}_{\lambda,\mu}(\mathfrak{g}, \mathfrak{F}, X_{\mathfrak{U}})$ are exact, whenever $f(\lambda) \neq \mu$. Note that $f(\lambda) = \lambda|^{\mathcal{A}_{\mathfrak{g}}}$ and if $f(\lambda) \neq \mu$ then $\lambda|^{\mathcal{A}_{\mathfrak{g}}}(a) \neq \mu(a)$ for a certain $a \in S$, because of S is a set of topological Lie generators of $\mathfrak{F}$ (or $\widehat{\mathfrak{F}}$). By assumption, $a \in \mathcal{A}_{\mathfrak{g}}(\alpha)$ (resp., $a \in \mathcal{A}_{\mathfrak{g}}\langle\alpha\rangle$) and $\operatorname{sp}(\operatorname{ad}(a)|_{\mathfrak{F}}) = \{0\}$. Using Lemma 14, we conclude that all operators $\Theta_{s,\mu}^{\sim}(a)$, $s \in \mathbb{Z}_+$, acting on the vertical cohomology complexes of the bicomplex $\mathcal{B}_{\lambda,\mu}(\mathfrak{g}, \mathfrak{F}, X_{\mathfrak{U}})$ are invertible. But, the latter implies that all vertical cohomology complexes are exact by virtue of (4.6) and (4.7). □

As follows from Proposition 7, to prove the backward spectral mapping property for normed Lie subalgebras of the dominating algebra one remains to establish the Projection Property suggested in Theorem 3.

Proposition 8. *Let $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha)$, $\mathfrak{U}$ an ultrafilter and let $\mathfrak{F}$ be a normed Lie subalgebra in $\mathcal{A}_{\mathfrak{g}}$. If $\widehat{\mathfrak{F}} \in Proj$ then $\sigma(\mathfrak{g}, \mathfrak{F}, X_{\mathfrak{U}})|_{\{0\}\times\mathfrak{F}} = \sigma\left(\alpha_{\mathfrak{U}}|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$ for all $\sigma \in \mathfrak{S}$.*

The proof is based on the same argument as in [6, Theorem 9.6].

Theorem 6. *Let $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha)$ and let S be a subset in $\mathcal{A}_{\mathfrak{g}}(\alpha)$ (resp., $\mathcal{A}_{\mathfrak{g}}\langle\alpha\rangle$) generating a quasinilpotent normed (resp., finite-dimensional) Lie subalgebra $\mathfrak{F} \subseteq \mathcal{A}_{\mathfrak{g}}$. If $\widehat{\mathfrak{F}} \in Proj$ then*

$$\sigma_u\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right) = \sigma(\alpha)\,|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}, \quad \sigma \in \mathfrak{S}^{\pi}.$$

Proof. The inclusion $\sigma(\alpha)\,|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}} \subseteq \sigma_u\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$ was proved in Theorem 5. The reverse inclusion follows from Propositions 7, 8, and Theorem 3. □

Corollary 8. *Let $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha)$, and let $\mathfrak{F} \subseteq \mathcal{A}_{\mathfrak{g}}\langle\alpha\rangle$ be a finite-dimensional nilpotent Lie subalgebra. Then*

$$\sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right) = \sigma(\alpha)\,|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}, \quad \sigma \in \mathfrak{S}.$$

Proof. If $\sigma \in \mathfrak{S}^{\pi}$ then result follows from Theorem 6. To prove the equality for spectra $\sigma = \sigma_{\delta,k} \in \mathfrak{S}_{\delta}$ we use the same argument carried out in the proof of Corollary 6. Namely, using Lemma 10 and Theorem 6, we obtain that

$$\begin{aligned}\sigma_{\delta,k}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right) &= \sigma^{\pi,k}\left(\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)^{*}\right) = \sigma_u^{\pi,k}\left(\alpha^{*}|^{\mathcal{A}_{\mathfrak{g}}^{op}}|_{\mathfrak{F}^{op}}\right)\\ &\subseteq \sigma^{\pi,k}\left(\alpha^{*}\right)|^{\mathcal{A}_{\mathfrak{g}}^{op}}|_{\mathfrak{F}^{op}} = \sigma_{\delta,k}(\alpha)\,|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}.\end{aligned}$$

By Corollary 6, $\sigma_{\delta,k}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right) = \sigma_{\delta,k}(\alpha)\,|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}$.

Finally, note that chain and cochain complexes generated by a representation of a finite-dimensional nilpotent Lie algebra are isomorphic [13, Proposition 3.1], whence $\sigma^{\delta,k}(\alpha) = \sigma_{\delta,k}(\alpha)$ and $\sigma_{\pi,k}(\alpha) = \sigma^{\pi,k}(\alpha)$. By assumption, $\mathfrak{F}$ is a nilpotent Lie subalgebra, therefore $\sigma^{\delta,k}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right) = \sigma_{\delta,k}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$ and $\sigma_{\pi,k}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right) = \sigma^{\pi,k}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$. □

Corollary 9. *Let $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha)$ and let S be a subset in $\mathcal{A}_{\mathfrak{g}}(\alpha)$ generating a quasinilpotent normed Lie subalgebra $\mathfrak{F} \subseteq \mathcal{A}_{\mathfrak{g}}$ such that $\widehat{\mathfrak{F}} \in Proj$. Then*

$$\sigma_{\delta,k}(\alpha)\,|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}} = \sigma_u^{\pi,k}\left(\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)^{*}\right), \ k \in \mathbb{Z}_{+}.$$

In particular, $\sigma^u_{\delta,k}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right) = \sigma_{\delta,k}(\alpha)\,|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}$ whenever X is superreflexive.

Proof. The inclusion $\sigma_{\delta,k}(\alpha)\,|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}} \subseteq \sigma_u^{\pi,k}\left(\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)^{*}\right)$ was proved in Corollary 7. The reverse inclusion follows from Theorem 6. Namely,

$$\sigma_u^{\pi,k}\left(\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)^{*}\right) = \sigma_u^{\pi,k}\left(\alpha^{*}|^{\mathcal{A}_{\mathfrak{g}}^{op}}|_{\mathfrak{F}^{op}}\right) \subseteq \sigma^{\pi,k}\left(\alpha^{*}\right)|^{\mathcal{A}_{\mathfrak{g}}^{op}}|_{\mathfrak{F}^{op}} = \sigma_{\delta,k}(\alpha)\,|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}.$$

If X is superreflexive then

$$\sigma_u^{\pi,k}\left(\left(\alpha|^{\mathcal{A}_\mathfrak{g}}|_\mathfrak{F}\right)^*\right) = \sigma_{\delta,k}^u\left(\alpha|^{\mathcal{A}_\mathfrak{g}}|_\mathfrak{F}\right)$$

(see Corollary 7). Therefore $\sigma_{\delta,k}(\alpha)|^{\mathcal{A}_\mathfrak{g}}|_\mathfrak{F} = \sigma_{\delta,k}^u\left(\alpha|^{\mathcal{A}_\mathfrak{g}}|_\mathfrak{F}\right)$. □

Remark 5. To obtain the classical form

$$f(\sigma(a)) = \sigma(f(a))$$

of the spectral mapping formulae suggested above one suffices to carry out the reasoning as in [5].

5. The finite-dimensional spectral mapping theorem for a family of functions generating solvable Lie subalgebra

In this section, we investigate the spectral mapping theorem for a Lie subalgebras $\mathfrak{F} \subseteq \mathcal{A}_\mathfrak{g}$ having finite-dimensional images. Using the Cartan subalgebras we could restore the Fainshtein version (see [13,14]) of a finite-dimensional spectral mapping theorem with respect to a more general (than polynomials) functions in noncommuting variables.

5.1. Cartan–Slodkowski spectra

Let $\mathfrak{A}$ be a finite-dimensional solvable Lie algebra, (X, β) be a Banach $\mathfrak{A}$-module, $\mathcal{H}$ a Cartan subalgebra in $\mathfrak{A}$ and let $\mathfrak{A} = \mathcal{H} \oplus \mathcal{H}_+$ be the Cartan decomposition of $\mathfrak{A}$ with respect to $\mathcal{H}$ [1, Ch. 1, Section 5]. One defines the Cartan–Slodkowski (resp., Cartan–Taylor) spectra [1, Ch. 4, Section 27], [7], of β as

$$\Sigma(\beta) = \left\{\lambda \in \mathfrak{A}^* : \lambda|_\mathcal{H} \in \sigma(\beta|_\mathcal{H}), \lambda(\mathcal{H}_+) = \{0\}\right\}, \quad \sigma \in \mathfrak{S}.$$

It is obvious that $\Sigma(\beta)$ coincides with the spectrum $\sigma(\beta)$ for a nilpotent Lie algebra $\mathfrak{A}$. If $\mathfrak{A} \subseteq \mathcal{B}(X)$ then we set $\Sigma(\mathfrak{A}) = \Sigma(*id_\mathfrak{A})$. One can prove that $\Sigma(\beta) \subseteq \Delta(\mathfrak{A})$ and $\Sigma(\beta)$ does not depend on the choice of a Cartan subalgebra $\mathcal{H}$ (see [7]). The family of all set-valued functions Σ defined on the class of solvable Lie algebra representations is denoted by $\mathfrak{R}\mathfrak{S}$.

The following projection property was proved in [1, Ch. 4, Section 27] for the Cartan–Taylor spectrum and was extended to all Cartan–Slodkowski spectra in [7].

Proposition 9. *Let $\mathfrak{A}$ be a finite-dimensional solvable Lie algebra, and let (X, β) be a Banach $\mathfrak{A}$-module. If $\mathfrak{L}$ is a Lie subalgebra in $\mathfrak{A}$ then $\Sigma(\beta|_\mathfrak{L}) = \Sigma(\beta)|_\mathfrak{L}$ for all $\Sigma \in \mathfrak{R}\mathfrak{S}$.*

The following assertion is the Cartan–Slodkowski version of the stability property for the Slodkowski spectra suggested in Proposition 3.

Proposition 10. *Let $\tau : \mathfrak{A} \to \mathfrak{B}$ be an epimorphism of finite-dimensional solvable Lie algebras, and let (X, β) be a $\mathfrak{B}$-module. Then $\Sigma(\beta \cdot \tau) = \Sigma(\beta) \cdot \tau$ for all $\Sigma \in \mathfrak{R}\mathfrak{S}$.*

Proof. Take $\mu \in \Sigma(\beta \cdot \tau)$. Then $\mu|_{\mathcal{H}} \in \sigma(\beta \cdot \tau|_{\mathcal{H}})$ and $\mu(\mathcal{H}_+) = \{0\}$ for a Cartan subalgebra $\mathcal{H} \subseteq \mathfrak{A}$. By Proposition 3, $\mu|_{\mathcal{H}} = \lambda \cdot \tau|_{\mathcal{H}}$ for some $\lambda \in \sigma\left(\beta|_{\tau(\mathcal{H})}\right)$. Moreover, $\lambda = \xi|_{\tau(\mathcal{H})}$ for a certain $\xi \in \Sigma(\beta)$ by virtue of Proposition 9. But $\mathcal{H}_+ \subseteq \left[\mathfrak{A}, \mathfrak{A}\right]$, therefore $\xi \cdot \tau = \mu$. Thus $\Sigma(\beta \cdot \tau) \subseteq \Sigma(\beta) \cdot \tau$.

Conversely, take $\lambda \in \Sigma(\beta)$ and let $\mathcal{H} \subseteq \mathfrak{A}$ be a Cartan subalgebra. Then $(\lambda \cdot \tau)(\mathcal{H}_+) \subseteq \lambda\left(\tau\left(\left[\mathfrak{A}, \mathfrak{A}\right]\right)\right) = \lambda\left(\left[\mathfrak{B}, \mathfrak{B}\right]\right) = \{0\}$ and $(\lambda \cdot \tau)|_{\mathcal{H}} = \lambda|_{\tau(\mathcal{H})} \cdot \tau|_{\mathcal{H}}$. By Proposition 9, $\lambda|_{\tau(\mathcal{H})} \in \sigma\left(\beta|_{\tau(\mathcal{H})}\right)$. Bearing in mind that $\mathcal{H}$ is a nilpotent Lie algebra, we infer that $\sigma((\beta \cdot \tau)|_{\mathcal{H}}) = \sigma\left(\beta|_{\tau(\mathcal{H})}\right) \cdot \tau|_{\mathcal{H}}$ due to Proposition 3, whence $(\lambda \cdot \tau)|_{\mathcal{H}} \in \sigma((\beta \cdot \tau)|_{\mathcal{H}})$ and $\lambda \cdot \tau \in \Sigma(\beta \cdot \tau)$ by its very definition. □

5.2. Spectral mapping theorem

Now let (X, α) be a Banach $\mathfrak{g}$-module, $\mathfrak{h} = \alpha(\mathfrak{g})$, $\beta = \mathrm{id}_{\mathfrak{h}} : \mathfrak{h} \to \mathcal{B}(X)$ the identity representation, $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha)$, and let $\mathcal{A}_{\theta_\alpha}$ (resp., $\mathcal{A}_{\theta_\beta}$) be the full subalgebra in $\mathcal{B}(\mathcal{B}(\wedge\mathfrak{g}, X))$ (resp., $\mathcal{B}(\mathcal{B}(\wedge\mathfrak{h}, X))$) generated by $\theta_\alpha(\mathfrak{g})$ (resp., $\theta_\beta(\mathfrak{h})$). As we noted in Example 2, $\mathcal{A}_{\theta_\alpha} \succ (X, \alpha)$, $\mathcal{A}_{\theta_\beta} \succ (X, \beta)$, and these algebras are connected by the homomorphism $\widehat{\alpha} : \mathcal{A}_{\theta_\alpha} \to \mathcal{A}_{\theta_\beta}$ such that $\widehat{\alpha}(\theta_\alpha(u)) = \theta_\beta(\alpha(u))$, $u \in \mathfrak{g}$, by Lemma 12. One can easily verify that

$$\beta|^{\mathcal{A}_{\theta_\beta}} \widehat{\alpha} \cdot \widehat{\theta}_\alpha = \alpha|^{\mathcal{A}_{\theta_\alpha}} \cdot \widehat{\theta}_\alpha = \alpha|^{\mathcal{A}_{\mathfrak{g}}} \tag{5.1}$$

and the images $\mathrm{im}\left(\alpha|^{\mathcal{A}_{\theta_\alpha}}\right)$, $\mathrm{im}\left(\beta|^{\mathcal{A}_{\theta_\beta}}\right)$ belong to the full subalgebra $\mathcal{A}_\alpha \subseteq \mathcal{B}(X)$ generated by $\mathfrak{h}$.

Lemma 15. *Let $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha)$ and let $\lambda \in \sigma_{\mathrm{t}}(\mathfrak{h})$. There exists $\lambda|^{\mathcal{A}_\alpha} \in \mathrm{Spec}(\mathcal{A}_\alpha)$ extending λ such that $\lambda|^{\mathcal{A}_\alpha} \cdot \beta|^{\mathcal{A}_{\theta_\beta}} = \lambda|^{\mathcal{A}_{\theta_\beta}}$, $\lambda|^{\mathcal{A}_\alpha} \cdot \alpha|^{\mathcal{A}_{\theta_\alpha}} = (\lambda \cdot \alpha)|^{\mathcal{A}_{\theta_\alpha}}$ and $\lambda|^{\mathcal{A}_\alpha} \cdot \alpha|^{\mathcal{A}_{\mathfrak{g}}} = (\lambda \cdot \alpha)|^{\mathcal{A}_{\mathfrak{g}}}$.*

Proof. At first, note that $\mu = \lambda \cdot \alpha \in \sigma_{\mathrm{t}}(\alpha)$ by virtue of Proposition 3. By Proposition 6, there exists $\widetilde{\mu} \in \mathrm{Spec}\left(\mathcal{A}_{\theta_\alpha}\right)$ such that $\widetilde{\mu}(r(\theta_\alpha(\mathfrak{g}))) = r(\lambda(\mathfrak{h}))$ for all rational functions $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\theta_\alpha}$. Let us remind that the image of the map $\widehat{\alpha} : \mathcal{R}_{\mathfrak{g},\theta_\alpha} \to \mathcal{B}(X)$ extending α is the full subalgebra $\mathcal{R}(\mathfrak{h})$. Moreover, $\mathrm{sp}(r(\theta_\alpha(\mathfrak{g}))) = \mathrm{sp}\left(\alpha|^{\mathcal{A}_{\theta_\alpha}}(r(\theta_\alpha(\mathfrak{g})))\right) = \mathrm{sp}(r(\mathfrak{h}))$ by Corollary 3. We set $\lambda|^{\mathcal{A}_\alpha}(r(\mathfrak{h})) = r(\lambda(\mathfrak{h}))$, $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\alpha} = \mathcal{R}_{\mathfrak{g},\theta_\alpha}$. To be correct, note that if $r_1(\mathfrak{h}) = r_2(\mathfrak{h})$ for some $r_1, r_2 \in \mathcal{R}_{\mathfrak{g},\alpha}$, then $r_1(\lambda(\mathfrak{h})) = \widetilde{\mu}(r_1(\theta_\alpha(\mathfrak{g}))) = \widetilde{\mu}((r_1 - r_2)(\theta_\alpha(\mathfrak{g}))) + \widetilde{\mu}(r_2(\theta_\alpha(\mathfrak{g})))$ and $\widetilde{\mu}((r_1 - r_2)(\theta_\alpha(\mathfrak{g}))) \in \mathrm{sp}((r_1 - r_2)(\theta_\alpha(\mathfrak{g}))) \subseteq \mathrm{sp}((r_1 - r_2)(\mathfrak{h})) = \{0\}$, that is, $r_1(\lambda(\mathfrak{h})) = r_2(\lambda(\mathfrak{h}))$. The rest is clear. □

Lemma 16. *Let $\mathfrak{A}$ be a finite-dimensional Lie subalgebra in $\mathcal{A}_{\theta_\beta}$ such that $\mathfrak{A} \subseteq \mathcal{A}_{\theta_\beta}\langle\beta\rangle$. Then $\mathfrak{A}$ is a solvable Lie algebra and $\Sigma\left(\beta|^{\mathcal{A}_{\theta_\beta}}|_{\mathfrak{A}}\right) = \Sigma(\mathfrak{h})\,|^{\mathcal{A}_{\theta_\beta}}|_{\mathfrak{A}}$ for all $\Sigma \in \mathfrak{RS}$.*

Proof. By Lemma 3, $\mathcal{A}_{\theta_\beta}$ is commutative modulo its Jacobson radical $Rad\mathcal{A}_{\theta_\beta}$, whence $[\mathfrak{A}, \mathfrak{A}] \subseteq Rad\mathcal{A}_{\theta_\beta}$. It follows that $[\mathfrak{A}, \mathfrak{A}]$ is a nilpotent Lie algebra, consequently, $\mathfrak{A}$ is solvable. Let $\mathcal{H}$ be a Cartan subalgebra in $\mathfrak{A}$. Then $\mathcal{H}_+ \subseteq [\mathfrak{A}, \mathfrak{A}]$, therefore $\mu = \zeta|_{\mathfrak{A}}$ iff $\mu|_{\mathcal{H}} = \zeta|_{\mathcal{H}}$ for arbitrary $\mu \in \Delta(\mathfrak{A})$ and $\zeta \in \mathrm{Spec}(\mathcal{A}_{\theta_\beta})$. With $\mathcal{A}_{\theta_\beta} \succ (X, \beta)$ in mind, we deduce that $\sigma\left(\beta|^{\mathcal{A}_{\theta_\beta}}|_{\mathcal{H}}\right) = \sigma(\mathfrak{h})\,|^{\mathcal{A}_{\theta_\beta}}|_{\mathcal{H}}$ due to Corollary 8. It follows that $\Sigma\left(\beta|^{\mathcal{A}_{\theta_\beta}}|_{\mathfrak{A}}\right) = \sigma(\mathfrak{h})\,|^{\mathcal{A}_{\theta_\beta}}|_{\mathfrak{A}} = \Sigma(\mathfrak{h})\,|^{\mathcal{A}_{\theta_\beta}}|_{\mathfrak{A}}$. □

Lemma 17. *Let $\mathfrak{A}$ be a finite-dimensional Lie subalgebra in $\mathcal{A}_{\theta_\beta}\langle\beta\rangle$, $\mathfrak{A}_0 = \beta|^{\mathcal{A}_{\theta_\beta}}(\mathfrak{A})$. Then $\Sigma(\mathfrak{A}_0) = \Sigma(\mathfrak{h})\,|^{\mathcal{A}_\alpha}|_{\mathfrak{A}_0}$ for all $\Sigma \in \mathfrak{RS}$.*

Proof. Take $\lambda \in \Sigma(\mathfrak{h}) = \sigma(\mathfrak{h})$ and $y \in \mathfrak{A}_0$, $y = \beta|^{\mathcal{A}_{\theta_\beta}}(a)$, $a \in \mathfrak{A}$. By Lemma 15, $\lambda|^{\mathcal{A}_\alpha}(y) = \lambda|^{\mathcal{A}_\alpha}\beta|^{\mathcal{A}_{\theta_\beta}}(a) = \lambda|^{\mathcal{A}_{\theta_\beta}}(a) = \left(\lambda|^{\mathcal{A}_{\theta_\beta}}|_{\mathfrak{A}}\right)(a)$. But $\mu = \lambda|^{\mathcal{A}_{\theta_\beta}}|_{\mathfrak{A}} \in \Sigma(\mathfrak{h})\,|^{\mathcal{A}_{\theta_\beta}}|_{\mathfrak{A}}$. By Lemma 16, $\mu \in \Sigma\left(\beta|^{\mathcal{A}_{\theta_\beta}}|_{\mathfrak{A}}\right)$. Moreover, $\Sigma\left(\beta|^{\mathcal{A}_{\theta_\beta}}|_{\mathfrak{A}}\right) = \Sigma(\mathfrak{A}_0)\cdot\beta|^{\mathcal{A}_{\theta_\beta}}|_{\mathfrak{A}}$ by virtue of Proposition 10. Then $\mu = \eta\cdot\beta|^{\mathcal{A}_{\theta_\beta}}|_{\mathfrak{A}}$ for some $\eta \in \Sigma(\mathfrak{A}_0)$. Moreover, $\lambda|^{\mathcal{A}_\alpha}(y) = \mu(a) = \eta(y)$, that is, $\lambda|^{\mathcal{A}_\alpha}|_{\mathfrak{A}_0} = \eta \in \Sigma(\mathfrak{A}_0)$.

Conversely, take $\eta \in \Sigma(\mathfrak{A}_0)$. Then $\eta\cdot\beta|^{\mathcal{A}_{\theta_\beta}}|_{\mathfrak{A}} \in \Sigma(\mathfrak{A}_0)\cdot\beta|^{\mathcal{A}_{\theta_\beta}}|_{\mathfrak{A}} \subseteq \Sigma\left(\beta|^{\mathcal{A}_{\theta_\beta}}|_{\mathfrak{A}}\right)$, and by Lemma 16, $\eta\cdot\beta|^{\mathcal{A}_{\theta_\beta}}|_{\mathfrak{A}} = \lambda|^{\mathcal{A}_{\theta_\beta}}|_{\mathfrak{A}}$ for a certain $\lambda \in \sigma(\mathfrak{h})$. Take $y = \beta|^{\mathcal{A}_{\theta_\beta}}|_{\mathfrak{A}}(a)$, $a \in \mathfrak{A}$. Then $\eta(y) = \eta\left(\beta|^{\mathcal{A}_{\theta_\beta}}(a)\right) = \lambda|^{\mathcal{A}_{\theta_\beta}}(a) = \lambda|^{\mathcal{A}_\alpha}\left(\beta|^{\mathcal{A}_{\theta_\beta}}(a)\right) = \lambda|^{\mathcal{A}_\alpha}|_{\mathfrak{A}_0}(y)$, that is, $\eta = \lambda|^{\mathcal{A}_\alpha}|_{\mathfrak{A}_0}$. □

Theorem 7. *Let $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha)$. If $\mathfrak{F}$ is a Lie subalgebra in $\mathcal{A}_{\mathfrak{g}}\langle\alpha\rangle$ such that $\widehat{\theta}(\mathfrak{F})$ is a finite-dimensional Lie subalgebra in $\mathcal{B}(\mathcal{B}(\wedge\mathfrak{g}, X))$, then $\mathfrak{F}_\alpha = \alpha|^{\mathcal{A}_{\mathfrak{g}}}(\mathfrak{F})$ is a solvable Lie subalgebra and $\Sigma(\mathfrak{F}_\alpha) = \Sigma(\alpha(\mathfrak{g}))\,|^{\mathcal{A}_\alpha}|_{\mathfrak{F}_\alpha}$ for all $\Sigma \in \mathfrak{RS}$.*

Proof. Let $\mathfrak{A} = \widehat{\alpha}\left(\widehat{\theta}_\alpha(\mathfrak{F})\right)$. It is a finite-dimensional Lie subalgebra in $\mathcal{A}_{\theta_\beta}$, in particular $\mathfrak{A}$ is solvable. By Theorem 4, $\mathfrak{A} \subseteq \mathcal{A}_{\theta_\beta}\langle\beta\rangle$ (see also Remark 4). Moreover, $\beta|^{\mathcal{A}_{\theta_\beta}}(\mathfrak{A}) = \mathfrak{F}_\alpha$ by (5.1). It remains to use Lemma 17. □

Corollary 10. *Let $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha)$. If $\mathfrak{F}$ is a finite-dimensional Lie subalgebra in $\mathcal{A}_{\mathfrak{g}}\langle\alpha\rangle$ then $\mathfrak{F}_\alpha = \alpha|^{\mathcal{A}_{\mathfrak{g}}}(\mathfrak{F})$ is solvable and $\Sigma(\mathfrak{F}_\alpha) = \Sigma(\alpha(\mathfrak{g}))\,|^{\mathcal{A}_\alpha}|_{\mathfrak{F}_\alpha}$ for all $\Sigma \in \mathfrak{RS}$.*

Note that if $\dim\left(\widehat{\theta}(\mathfrak{F})\right) < \infty$ then automatically $\dim(\mathfrak{F}_\alpha) < \infty$. We do not know whether or not the opposite assertion is valid. Namely, does the finite-dimensionality of the Lie subalgebra $\mathfrak{F}_\alpha \subseteq \mathcal{B}(X)$ imply the finite-dimensionality of the Lie subalgebra $\widehat{\theta}(\mathfrak{F}) \subseteq \mathcal{B}(\mathcal{B}(\wedge\mathfrak{g}, X))$? This problem arose in Fainshtein's investigations [13,14], in the case $\mathcal{A}_\mathfrak{g} = \mathcal{U}(\mathfrak{g})$ (see Example 1). If $\mathfrak{g}$ is a commutative Lie algebra then one can easily verify that the answer is positive. In this case $\dim\left(\widehat{\theta}(\mathfrak{F})\right) = \dim(\mathfrak{F}_\alpha)$. For a noncommutative nilpotent Lie algebra $\mathfrak{g}$ the answer is unknown. We could announce that the answer is also positive for Arens-Michael completions $\mathcal{A}_\mathfrak{g}$ of $\mathcal{U}(\mathfrak{g})$ whenever $\mathfrak{g}$ is a Heisenberg algebra (that is, $\dim([\mathfrak{g},\mathfrak{g}]) = 1$).

6. Applications to noncommutative functional calculi

In this section, we apply spectral mapping results suggested in previous sections to various noncommutative functional calculi.

6.1. The polynomial algebra $\mathcal{U}(\mathfrak{g})$

Let us assume that $\mathcal{A}_\mathfrak{g}$ is the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. Then $\mathcal{A}_\mathfrak{g} \succ (X,\alpha)$ for each Banach $\mathfrak{g}$-module X (see Example 1). Moreover, $\mathcal{A}_\mathfrak{g}(\alpha) = \mathcal{A}_\mathfrak{g}\langle\alpha\rangle = \mathcal{A}_\mathfrak{g}$ by Proposition 6. Using Theorems 6, 7, we obtain the following generalization of the main result in [13] (see also [14]).

Corollary 11. *Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra and let $\mathfrak{F} \subseteq \mathcal{U}(\mathfrak{g})$ be a quasinilpotent normed Lie subalgebra. If $\widehat{\mathfrak{F}} \in Proj$ then*

$$\sigma_u\left(\alpha|^{\mathcal{U}(\mathfrak{g})}|_{\mathfrak{F}}\right) = \sigma(\alpha)\,|^{\mathcal{U}(\mathfrak{g})}|_{\mathfrak{F}}, \quad \sigma \in \mathfrak{S}^\pi.$$

In particular, if $\mathfrak{F} \subseteq \mathcal{U}(\mathfrak{g})$ is a finite-dimensional Lie subalgebra then $\mathfrak{F}$ is nilpotent and

$$\sigma(\alpha)\,|^{\mathcal{U}(\mathfrak{g})}|_{\mathfrak{F}} = \sigma\left(\alpha|^{\mathcal{U}(\mathfrak{g})}|_{\mathfrak{F}}\right), \quad \sigma \in \mathfrak{S}.$$

Moreover, if $\mathfrak{P} \subseteq \mathcal{U}(\mathfrak{g})$ is a Lie subalgebra such that $\dim\left(\theta|^{\mathcal{U}(\mathfrak{g})}(\mathfrak{P})\right) < \infty$ then $\mathfrak{P}_\alpha = \alpha|^{\mathcal{U}(\mathfrak{g})}(\mathfrak{P})$ is a solvable Lie subalgebra and

$$\Sigma(\mathfrak{P}_\alpha) = \Sigma(\alpha(\mathfrak{g}))\,|^{\mathcal{P}_\alpha}|_{\mathfrak{P}_\alpha}, \quad \Sigma \in \mathfrak{R}\mathfrak{S},$$

where $\mathcal{P}_\alpha$ is the associative hull of $\alpha(\mathfrak{g})$ in $\mathcal{B}(X)$.

Note that a finite-dimensional Lie subalgebra in $\mathcal{U}(\mathfrak{g})$ is automatically nilpotent one (see [13]).

6.2. The Banach algebra $\mathcal{A}(\mathfrak{g})$ of convergent power series

The next example is the Banach enveloping algebra (or the algebra of convergent power series) of a normed Lie algebra [9, Section 2].

Let us remind necessary facts (for more detailed information see [9, Sections 2, 4]). Let $\mathfrak{I}$ be a B–L algebra and let $\ell_1(\mathfrak{I})$ be the ℓ_1-direct sum of Banach spaces $\mathfrak{I}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$. It is clear that $\ell_1(\mathfrak{I})$ is the ℓ_1-norm completion of the tensor algebra $T(\mathfrak{I})$, therefore it is turning into Banach algebra. Let I be a two-sided ideal in $T(\mathfrak{I})$ generated by tensors $x \otimes y - y \otimes x - [x, y]$, $x, y \in \mathfrak{I}$, and let J be its closure in $\ell_1(\mathfrak{I})$. The Banach enveloping algebra $\mathcal{A}(\mathfrak{I})$ of B–L algebra $\mathfrak{I}$ is defined as the quotient algebra $\ell_1(\mathfrak{I})/J$. Bearing in mind that $\mathcal{U}(\mathfrak{I}) = T(\mathfrak{I})/I$, one automatically defines a canonical algebra homomorphism $\varphi : \mathcal{U}(\mathfrak{I}) \to \mathcal{A}(\mathfrak{I})$ having the dense range and $\|\varphi|_{\mathfrak{I}}\| \leqslant 1$. One can prove [9, Corollary 2.4] that the dual operator $\varphi^* : \mathcal{A}(\mathfrak{I})^* \to \mathfrak{I}^*$ implements a homeomorphism between the topological spaces $\operatorname{Spec}(\mathcal{A}(\mathfrak{I}))$ and $S_{\mathfrak{I}}$ furnished with the $*$-weak topologies taken in $\mathcal{A}(\mathfrak{I})^*$ and $\mathfrak{I}^*$, respectively, where $S_{\mathfrak{I}} = \mathfrak{I}^*_{(1)} \cap \Delta(\mathfrak{I})$.

Lemma 18. *Let $\mathfrak{I}$ be a B–L algebra and let $a \in \mathcal{A}(\mathfrak{I})$. There exist elements $a_{nk}^{(m)} \in \mathfrak{F}$, $1 \leqslant m \leqslant n$, $n, k \in \mathbb{Z}_+$, such that the series $\sum_{n,k} \prod_{m=1}^{n} a_{nk}^{(m)}$ converges absolutely in $\mathcal{A}(\mathfrak{I})$ to a. Moreover, if $\mathfrak{I}$ is a finite-dimensional nilpotent Lie algebra furnished with a norm p then $\sum_{n,k} \prod_{m=1}^{n} p\left(a_{nk}^{(m)}\right) < \infty$.*

Proof. By definition, $a = \varphi(y)$ for some $y \in \ell_1(\mathfrak{I})$. Then $y = (y_n) = \sum_{n \in \mathbb{Z}_+} y_n$ and $\|y\|_{\ell_1(\mathfrak{I})} = \sum_{n \in \mathbb{Z}_+} \|y_n\|_{\mathfrak{I}^{\widehat{\otimes} n}} < \infty$, where $y_n \in \mathfrak{I}^{\widehat{\otimes} n}$. But, $y_n = \sum_k x_{nk}^{(1)} \otimes \cdots \otimes x_{nk}^{(n)}$, $x_{nk}^{(m)} \in \mathfrak{I}$, such that $\sum_k \prod_{m=1}^{n} \left\| x_{nk}^{(m)} \right\|_{\mathfrak{I}} < \|y_n\|_{\mathfrak{I}^{\widehat{\otimes} n}} + 2^{-n}$. Then $a = \varphi(y) = \sum_{n \in \mathbb{Z}_+} \varphi(y_n) = \sum_{n \in \mathbb{Z}_+} \sum_k \prod_{m=1}^{n} a_{nk}^{(m)}$, where $a_{nk}^{(m)} = \varphi|_{\mathfrak{I}}\left(x_{nk}^{(m)}\right)$. With $\|\varphi|_{\mathfrak{I}}\| \leqslant 1$ in mind, infer that

$$\sum_{n,k} \left\| \prod_{m=1}^{n} a_{nk}^{(m)} \right\|_{\mathcal{A}(\mathfrak{I})} \leqslant \sum_{n,k} \prod_{m=1}^{n} \left\| x_{nk}^{(m)} \right\|_{\mathfrak{I}} \leqslant \sum_{n} \left(\|y_n\|_{\mathfrak{I}^{\widehat{\otimes} n}} + 2^{-n} \right) = 2 + \|y\|_{\ell_1(\mathfrak{I})} < \infty,$$

that is, the series $\sum_{n,k} \prod_{m=1}^{n} a_{nk}^{(m)}$ converges absolutely in $\mathcal{A}(\mathfrak{I})$.

Now assume that $\mathfrak{I}$ is a finite-dimensional nilpotent Lie algebra with a norm p. It is clear that $\mathfrak{I}$ is a B–L algebra. Moreover, the Banach enveloping algebra $\mathcal{A}(\mathfrak{I})$ is a norm completion of $\mathcal{U}(\mathfrak{I})$ by virtue of [9, Theorem 3.1], whence $\varphi : \mathcal{U}(\mathfrak{I}) \to \mathcal{A}(\mathfrak{I})$ is just the identity embedding. In particular, $a_{nk}^{(m)} = x_{nk}^{(m)}$ and as we have proved above $\sum_{n,k} \prod_{m=1}^{n} p\left(x_{nk}^{(m)}\right) < \infty$. $\square$

Theorem 8. *Let $\mathfrak{g}$ be a finite-dimensional normed nilpotent Lie algebra with a norm p, (X, α) a Banach $\mathfrak{g}$-module and let $\mathcal{A}_{\mathfrak{g}} = \mathcal{A}(\mathfrak{g})$ be the Banach enveloping algebra* *of the B–L algebra* $\mathfrak{g}$. *If* $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha)$ *then* $\widehat{\theta}(a) - \lambda|^{\mathcal{A}_{\mathfrak{g}}}(a) = d(\lambda)\, i_\lambda(a) + i_\lambda(a)\, d(\lambda)$ *for all* $a \in \mathcal{A}_{\mathfrak{g}}$ *and* $\lambda \in \sigma_{\mathrm{t}}(\alpha) \cap \operatorname{int}\left(\mathfrak{g}^*_{(1)}\right)$, *where* $\operatorname{int}\left(\mathfrak{g}^*_{(1)}\right)$ *is the topological interior of the unit ball* $\mathfrak{g}^*_{(1)}$ *in* $\mathfrak{g}^*$ *with respect to the dual norm* p^*.

Proof. To indicate the norm p, we write $\mathcal{A}(p)$ instead of $\mathcal{A}(\mathfrak{g})$. Note that $\mathcal{A}(p)$ is the norm completion of $\mathcal{U}(\mathfrak{g})$ by [9, Theorem 3.1]. Take $\lambda \in \sigma_{\mathrm{t}}(\alpha) \cap \operatorname{int}\left(\mathfrak{g}^*_{(1)}\right)$. Then $\lambda \in \Delta(\mathfrak{g})$ and $p^*(\lambda) < 1$. Let us introduce the norm q on $\mathfrak{g}$ by setting $q(x) = \sup\{|\mu(x)| : \mu \in K_\varepsilon\}$, where $K_\varepsilon = \{\mu \in \mathfrak{g}^* : p^*(\mu) \leqslant \varepsilon\}$, where $p^*(\lambda) < \varepsilon < 1$. It is beyond a doubt $q \leqslant \varepsilon p$ and $\mathfrak{g}$ is a B–L algebra with respect to the norm q. Moreover, the identity map on $\mathcal{U}(\mathfrak{g})$ is extended up to an algebra homomorphism $u : \mathcal{A}(p) \to \mathcal{A}(q)$, $\|u\| \leqslant 1$, by [9, Proposition 2.2]. Since $\lambda \in K_\varepsilon$, one follows that $|\lambda(x)| \leqslant q(x)$ and $q^*(\lambda) = \sup\{|\lambda(x)| : q(x) \leqslant 1\} \leqslant 1$. Then $\lambda \in \operatorname{Spec}(\mathcal{A}(q))$, that is, λ is extended up to a character $\lambda|^{\mathcal{A}(q)}$ on the Banach enveloping algebra $\mathcal{A}(q)$ of the B–L algebra $\mathfrak{g}$ furnished with the norm q. Moreover, $\lambda|^{\mathcal{A}(q)} \cdot u = \lambda|^{\mathcal{A}(p)}$, for $u|_{\mathcal{U}(\mathfrak{g})} = \mathrm{id}$. Further, by Proposition 6 and Remark 3,

$$i_\lambda(a_1 \cdots a_n) = \sum_{s=1}^{n} \widehat{\theta}(a_1 \cdots a_{s-1})\, i(a_s)\, \lambda|^{\mathcal{A}(q)}(a_{s+1} \cdots a_n)$$

whenever all $a_s \in \mathfrak{g}$, where $i(a_s) \in \mathcal{B}(\mathcal{B}(\wedge \mathfrak{g}, X))$ is the homotopy operator. Take $a \in \mathcal{A}(p)$. By Lemma 18, $a = \sum_{n,k} \prod_{m=1}^{n} a^{(m)}_{nk}$ is an absolutely convergent (in $\mathcal{A}(p)$) series, where $a^{(m)}_{nk} \in \mathfrak{g}$. With continuity of $\widehat{\theta} : \mathcal{A}(p) \to \mathcal{B}(\mathcal{B}(\wedge \mathfrak{g}, X))$ in mind, we conclude that

$$\begin{aligned}\widehat{\theta}(a) - \lambda|^{\mathcal{A}(p)}(a) &= \sum_{n,k} \left(\widehat{\theta}\left(\prod_{m=1}^{n} a^{(m)}_{nk} \right) - \lambda|^{\mathcal{A}(p)} \left(\prod_{m=1}^{n} a^{(m)}_{nk} \right) \right) \\ &= \sum_{n,k} \left(d(\lambda)\, i_\lambda \left(\prod_{m=1}^{n} a^{(m)}_{nk} \right) + i_\lambda \left(\prod_{m=1}^{n} a^{(m)}_{nk} \right) d(\lambda) \right). \qquad (6.1)\end{aligned}$$

Moreover, $i_\lambda\left(\prod_{m=1}^{n} a^{(m)}_{nk}\right) = \sum_{s=1}^{n} \widehat{\theta}\left(a^{(1)}_{nk} \cdots a^{(s-1)}_{nk}\right) i\left(a^{(s)}_{nk}\right) \lambda|^{\mathcal{A}(q)}\left(a^{(s+1)}_{nk} \cdots a^{(n)}_{nk}\right)$. We set $i_\lambda(a) = \sum_{n,k} i_\lambda\left(\prod_{m=1}^{n} a^{(m)}_{nk}\right)$. Let us prove that the latter series converges. Indeed,

$$\begin{aligned}\|i_\lambda(a)\| &\leqslant \sum_{n,k} \sum_{s=1}^{n} \left\|\widehat{\theta}\right\| \left\| a^{(1)}_{nk} \cdots a^{(s-1)}_{nk} \right\|_{\mathcal{A}(p)} \left\| i\left(a^{(s)}_{nk}\right) \right\| \left\| a^{(s+1)}_{nk} \cdots a^{(n)}_{nk} \right\|_{\mathcal{A}(q)} \\ &\leqslant \left\|\widehat{\theta}\right\| \sum_{n,k} \sum_{s=1}^{n} p\left(a^{(1)}_{nk}\right) \cdots p\left(a^{(s)}_{nk}\right) q\left(a^{(s+1)}_{nk}\right) \cdots q\left(a^{(n)}_{nk}\right)\end{aligned}$$

$$\leqslant \left\|\widehat{\theta}\right\| \sum_{n,k} p\left(a_{nk}^{(1)}\right) \cdots p\left(a_{nk}^{(n)}\right) \sum_{s=1}^{n} \varepsilon^{n-s}$$
$$\leqslant \left\|\widehat{\theta}\right\| (1-\varepsilon)^{-1} \sum_{n,k} p\left(a_{nk}^{(1)}\right) \cdots p\left(a_{nk}^{(n)}\right) < \infty,$$

here we took into account that $\|i(u)\| \leqslant p(u)$, $u \in \mathfrak{g}$, and $\sum_{n,k} p\left(a_{nk}^{(1)}\right) \cdots p\left(a_{nk}^{(n)}\right) < \infty$ by virtue of Lemma 18. Using (6.1), we infer that $\widehat{\theta}(a) - \lambda|^{\mathcal{A}(p)}(a) = d(\lambda)\, i_\lambda(a) + i_\lambda(a)\, d(\lambda)$. $\square$

The following assertion generalizes Lemma 7.5 from [6].

Corollary 12. *Let* $\mathfrak{g}$ *be a finite-dimensional normed nilpotent Lie algebra with a norm* p, (X, α) *a Banach* $\mathfrak{g}$*-module and let* $\mathcal{A}(\mathfrak{g})$ *be the Banach enveloping algebra of B–L algebra* $\mathfrak{g}$. *If* $\rho\left(\alpha\left(\mathfrak{g}_{(1)}\right)\right) < 1$ *then* $\mathcal{A}(\mathfrak{g}) \succ (X, \alpha)$ *and all elements of the algebra* $\mathcal{A}(\mathfrak{g})$ *are splitting over the* $\mathfrak{g}$*-module* X.

Proof. As we noted above that the subalgebra of all polynomials taken by $\mathfrak{g}$ is dense in $\mathcal{A}(\mathfrak{g})$. Moreover, one can prove (see [9, Lemma 5.2]) that $\rho\left(\theta\left(\mathfrak{g}_{(1)}\right)\right) \leqslant \rho\left(\alpha\left(\mathfrak{g}_{(1)}\right)\right)$. It follows that $\mathcal{A}(\mathfrak{g}) \succ (X, \alpha)$ by virtue of [9, Proposition 2.2] and Definition 4.

Now let us prove that $\sigma_{\mathrm{t}}(\alpha) \subseteq \mathrm{int}\left(\mathfrak{g}_{(1)}^*\right)$. Take $\lambda \in \sigma_{\mathrm{t}}(\alpha)$. By Proposition 9, $\lambda(x) \in \mathrm{sp}(\alpha(x))$, $x \in \mathfrak{g}$. Therefore

$$p^*(\lambda) = \sup\{|\lambda(x)| : p(x) \leqslant 1\} \leqslant \sup\{\rho(\alpha(x)) : p(x) \leqslant 1\} \leqslant \rho\left(\alpha\left(\mathfrak{g}_{(1)}\right)\right) < 1,$$

that is, $p^*(\lambda) < 1$ or $\lambda \in \mathrm{int}\left(\mathfrak{g}_{(1)}^*\right)$. By Theorem 8, all elements from the algebra $\mathcal{A}(\mathfrak{g})$ are splitting over the $\mathfrak{g}$-module X. $\square$

Remark 6. Note that $\rho\left(\alpha\left(\mathfrak{g}_{(1)}\right)\right) = \max\left\{\rho(\alpha(x)) : x \in \mathfrak{g}_{(1)}\right\}$ by [27], therefore instead of the condition $\rho\left(\alpha\left(\mathfrak{g}_{(1)}\right)\right) < 1$ in Corollary 12 one can just demand $\rho(\alpha(x)) < 1$ for all $x \in \mathfrak{g}_{(1)}$.

Corollary 13. *Let* $\mathfrak{g}$ *be a normed finite-dimensional nilpotent Lie algebra*, (X, α) *a Banach* $\mathfrak{g}$*-module and let* S *be a subset in the Banach enveloping algebra* $\mathcal{A}(\mathfrak{g})$ *generating a quasinilpotent normed Lie subalgebra* $\mathfrak{F} \subseteq \mathcal{A}(\mathfrak{g})$ *such that* $\widehat{\mathfrak{F}} \in Proj$. *If* $\rho\left(\alpha\left(\mathfrak{g}_{(1)}\right)\right) < 1$ *then* $\mathcal{A}(\mathfrak{g}) \succ (X, \alpha)$ *and*

$$\sigma_u\left(\alpha|^{\mathcal{A}(\mathfrak{g})}|_{\mathfrak{F}}\right) = \sigma(\alpha)\,|^{\mathcal{A}(\mathfrak{g})}|_{\mathfrak{F}}, \quad \sigma \in \mathfrak{S}^\pi.$$

In particular, if $\dim(\mathfrak{F}) < \infty$ *then* $\mathfrak{F}$ *is a nilpotent Lie algebra and*

$$\sigma\left(\alpha|^{\mathcal{A}(\mathfrak{g})}|_{\mathfrak{F}}\right) = \sigma(\alpha)|^{\mathcal{A}(\mathfrak{g})}|_{\mathfrak{F}}, \quad \sigma \in \mathfrak{S}.$$

Moreover, if $\mathfrak{A} \subseteq \mathcal{A}(\mathfrak{g})$ *is a finite-dimensional Lie subalgebra then* $\mathfrak{A}_\alpha = \alpha|^{\mathcal{A}(\mathfrak{g})}(\mathfrak{A})$ *is solvable and*

$$\Sigma(\mathfrak{A}_\alpha) = \Sigma(\alpha(\mathfrak{g}))|^{\mathcal{A}_\alpha}|_{\mathfrak{A}_\alpha}, \quad \Sigma \in \mathfrak{R}\mathfrak{S},$$

where $\mathcal{A}_\alpha$ *is the closed associative hull of* $\alpha(\mathfrak{g})$ *in* $\mathcal{B}(X)$.

Proof. The assertion on the infinite-dimensional spectral mapping theorem directly follows from Theorem 6 and Corollary 12.

Now assume that $\dim(\mathfrak{F}) < \infty$. Demonstrate that $\mathfrak{F}$ should be nilpotent. Consider the following set $M = \{x \in \mathfrak{F} : \mathrm{sp}(\mathrm{ad}(x)|_{\mathfrak{F}}) = \{0\}\}$. Taking into account that $S \subseteq M$, one suffices to prove that M is a Lie subalgebra in $\mathfrak{F}$. Undoubtedly, $\lambda M \subseteq M$ for all $\lambda \in \mathbb{C}$. By Lemma 3, the Banach enveloping algebra $\mathcal{A}(\mathfrak{g})$ is commutative modulo its Jacobson radical $Rad\mathcal{A}(\mathfrak{g})$. It follows that $\mathfrak{F}$ is a solvable Lie algebra and $[M, M] \subseteq M$. By Lie theorem, $M + M \subseteq M$, whence M is a Lie subalgebra. Therefore $\mathfrak{F}$ is nilpotent.

The finite-dimensional spectral mapping theorem for $\mathfrak{F}$ follows from Corollary 8. Finally, the spectral mapping theorem for the Cartan–Slodkowski spectra follows from Corollary 10. □

6.3. The Fréchet algebra $\mathcal{O}_\mathfrak{g}(D)$

The next example is the Fréchet algebra $\mathcal{A}_\mathfrak{g} = \mathcal{O}_\mathfrak{g}(D)$ of all holomorphic functions in elements of $\mathfrak{g}$ on an absolutely convex domain $D \subseteq \Delta(\mathfrak{g})$ (see [9]). By definition [9], $\mathcal{O}_\mathfrak{g}(D)$ is the inverse limit of some projective system of Banach enveloping algebras $\mathcal{A}(\mathfrak{g})$. One proves [9, Section 5] that $\mathcal{O}_\mathfrak{g}(D) \succ (X, \alpha)$ iff $\sigma(\alpha) \subset D$ for a certain $\sigma \in \mathfrak{S}$.

Proposition 11. *Let* $\mathfrak{g}$ *be a finite-dimensional nilpotent Lie algebra,* D *an absolutely convex domain in* $\Delta(\mathfrak{g})$ *and let* (X, α) *be a Banach* $\mathfrak{g}$*-module. If* $\mathcal{O}_\mathfrak{g}(D) \succ (X, \alpha)$ *then all elements of the algebra* $\mathcal{O}_\mathfrak{g}(D)$ *are splitting over the module* X.

Proof. By using [9, Lemma 5.1], we infer that the condition $\mathcal{O}_\mathfrak{g}(D) \succ (X, \alpha)$ implies that $\rho\left(\alpha\left(\mathfrak{g}_{(1)}\right)\right) < 1$ for a certain norm on $\mathfrak{g}$ and the continuous algebra homomorphism $\widehat{\theta} : \mathcal{O}_\mathfrak{g}(D) \to \mathcal{B}(\mathcal{B}(\wedge\mathfrak{g}, X))$ splits into the superposition $\mathcal{O}_\mathfrak{g}(D) \xrightarrow{\pi} \mathcal{A}(\mathfrak{g}) \xrightarrow{\widehat{\theta}} \mathcal{B}(\mathcal{B}(\wedge\mathfrak{g}, X))$ of the canonical projection π of the inverse limit defining $\mathcal{O}_\mathfrak{g}(D)$ and the bounded homomorphism $\widehat{\theta} : \mathcal{A}(\mathfrak{g}) \to \mathcal{B}(\mathcal{B}(\wedge\mathfrak{g}, X))$ taken on the ground of $\mathcal{A}(\mathfrak{g}) \succ (X, \alpha)$ (see Corollary 12). It remains to apply Theorem 8 (note that $\sigma_{\mathrm{t}}(\alpha) \subseteq \mathrm{int}\left(\mathfrak{g}^*_{(1)}\right)$). □

In particular, we obtain a generalization of the finite-dimensional spectral mapping theorem from [9].

Corollary 14. *Let* $\mathfrak{g}$ *be a finite-dimensional nilpotent Lie algebra*, (X, α) *a Banach* $\mathfrak{g}$-*module*, D *an open absolutely convex neighborhood of some Slodkowski spectrum* $\sigma(\alpha)$ *and let* S *a subset in* $\mathcal{O}_{\mathfrak{g}}(D)$ *generating a quasinilpotent normed Lie subalgebra* $\mathfrak{F} \subseteq \mathcal{O}_{\mathfrak{g}}(D)$ *such that* $\widehat{\mathfrak{F}} \in Proj$. *Then* $\mathcal{O}_{\mathfrak{g}}(D) \succ (X, \alpha)$ *and*

$$\sigma_u\left(\alpha|^{\mathcal{O}_{\mathfrak{g}}(D)}|_{\mathfrak{F}}\right) = \sigma(\alpha)\,|^{\mathcal{O}_{\mathfrak{g}}(D)}|_{\mathfrak{F}}, \quad \sigma \in \mathfrak{S}^{\pi}.$$

In particular, if $\dim(\mathfrak{F}) < \infty$ *then* $\mathfrak{F}$ *is a nilpotent Lie subalgebra and*

$$\sigma\left(\alpha|^{\mathcal{O}_{\mathfrak{g}}(D)}|_{\mathfrak{F}}\right) = \sigma(\alpha)\,|^{\mathcal{O}_{\mathfrak{g}}(D)}|_{\mathfrak{F}}, \quad \sigma \in \mathfrak{S}.$$

Moreover, if $\mathfrak{A} \subseteq \mathcal{O}_{\mathfrak{g}}(D)$ *is a finite-dimensional Lie subalgebra then* $\mathfrak{A}_{\alpha} = \alpha|^{\mathcal{O}_{\mathfrak{g}}(D)}(\mathfrak{A})$ *is solvable and*

$$\Sigma(\mathfrak{A}_{\alpha}) = \Sigma(\alpha(\mathfrak{g}))\,|^{\mathcal{A}_{\alpha}}|_{\mathfrak{A}_{\alpha}}, \quad \Sigma \in \mathfrak{R}\mathfrak{S},$$

where $\mathcal{A}_{\alpha}$ *is the closed associative hull of* $\alpha(\mathfrak{g})$ *in* $\mathcal{B}(X)$.

Proof. One suffices to apply Theorem 6, Proposition 11 and Corollaries 8, 10. □

6.4. Formally-radical functions

Our last example is the algebra of all formally-radical functions $\mathcal{F}_{\mathfrak{g}}(D)$ on an open subset $D \subseteq \Delta(\mathfrak{g})$ [10]. One defines a sheaf $\mathcal{F}_{\mathfrak{g}}$ of Fréchet (noncommutative) algebras on $\Delta(\mathfrak{g})$ such that the enveloping algebra $\mathcal{U}(\mathfrak{g})$ is embedded into the algebra $\mathcal{F}_{\mathfrak{g}}(D)$ of all sections over an open subset D as a subalgebra. Note that $\mathcal{F}_{\mathfrak{g}}$ as a sheaf of Fréchet spaces is the projective tensor product $\mathcal{O}\widehat{\otimes}\mathbb{C}[[e_1, \ldots, e_s]]$ of the sheaf $\mathcal{O}$ of germs of holomorphic functions and the constant sheaf [15, Ch. 2] generated by the Fréchet space $\mathbb{C}[[e_1, \ldots, e_s]]$ of all formal power series in s-variables $e = (e_1, \ldots, e_s)$, where $s = \dim\left(\left[\mathfrak{g}, \mathfrak{g}\right]\right)$. Note that e can be interpreted as a basis of $\left[\mathfrak{g}, \mathfrak{g}\right]$. Thus

$$\mathcal{F}_{\mathfrak{g}}(D) = \mathcal{O}(D)[[e_1, \ldots, e_s]] \tag{6.2}$$

is the Fréchet space of all formal power series of s-variables over the space $\mathcal{O}(D)$, whenever D is an open subset in $\Delta(\mathfrak{g})$. In particular, $\mathcal{F}_{\mathfrak{g}} = \mathcal{O}$ if $\mathfrak{g}$ is a commutative Lie algebra. Fix an open set $D \subseteq \Delta(\mathfrak{g})$. The algebra $\mathcal{O}(D)$ is embedded into $\mathcal{F}_{\mathfrak{g}}(D)$ as a closed subspace $\mathcal{S}_{\mathfrak{g}}(D)$ such that

$$\mathcal{F}_{\mathfrak{g}}(D) = \mathcal{S}_{\mathfrak{g}}(D) \oplus Rad\mathcal{F}_{\mathfrak{g}}(D) \tag{6.3}$$

(see [10] and also [9]). Thus the quotient homomorphism $\varphi_D : \mathcal{F}_{\mathfrak{g}}(D) \to \mathcal{O}(D)$ modulo the Jacobson radical $Rad\mathcal{F}_{\mathfrak{g}}(D)$ is a retraction (as a Fréchet space operator) with the right inverse $\varepsilon_D : \mathcal{O}(D) \to \mathcal{F}_{\mathfrak{g}}(D)$ (let us emphasize that ε_D is not an algebra homomorphism).

Now let $X \in \mathbf{BS}$ and let $\alpha : \mathfrak{g} \to \mathcal{B}(X)$ be a Lie representation such that $\alpha\left(\left[\mathfrak{g}, \mathfrak{g}\right]\right)$ consists of nilpotent operators. In this case, we say that $\alpha(\mathfrak{g})$ is a *supernilpotent Lie subalgebra* in $\mathcal{B}(X)$. If D is an open neighborhood (in $\Delta(\mathfrak{g})$) of the Taylor spectrum $\sigma_{\mathrm{t}}(\alpha)$ then there exists a continuous algebra homomorphism $\alpha|^{\mathcal{F}_{\mathfrak{g}}(D)} : \mathcal{F}_{\mathfrak{g}}(D) \to \mathcal{B}(X)$ extending α [10]. That is the generalization of Taylor functional calculus for commuting operators [24].

Below, as a corollary of our framework we suggest spectral mapping theorem with respect to this calculus whenever D is a Stein $\mathcal{F}_{\mathfrak{g}}$-rational domain.

We say that D is a $\mathcal{F}_{\mathfrak{g}}$-rational domain in $\Delta(\mathfrak{g})$ if the full subalgebra $\mathcal{R}(\mathfrak{g}) \subseteq \mathcal{F}_{\mathfrak{g}}(D)$ is dense in $\mathcal{F}_{\mathfrak{g}}(D)$. Since $\varphi_D(\mathcal{R}(\mathfrak{g}))$ is the subalgebra (in $\mathcal{O}(D)$) of usual rational functions on D, it follows that the property being approximated by rational functions of each holomorphic function on the domain D inherits from $\mathcal{F}_{\mathfrak{g}}(D)$, that is, each $\mathcal{F}_{\mathfrak{g}}$-rational domain in $\Delta(\mathfrak{g})$ is $\mathcal{O}$-rational. But, we do not know whether or not a $\mathcal{O}$-rational domain is $\mathcal{F}_{\mathfrak{g}}$-rational. Note that $\mathcal{U}(\mathfrak{g})$ is dense in $\mathcal{F}_{\mathfrak{g}}(D)$ iff the usual polynomials is dense in $\mathcal{O}(D)$.

Theorem 9. *Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra, (X, α) a Banach $\mathfrak{g}$-module such that $\alpha(\mathfrak{g})$ is a supernilpotent Lie subalgebra in $\mathcal{B}(X)$ and let D be a $\mathcal{F}_{\mathfrak{g}}$-rational domain in $\Delta(\mathfrak{g})$ containing the Taylor spectrum $\sigma_{\mathrm{t}}(\alpha)$. Then $\mathcal{F}_{\mathfrak{g}}(D) \succ (X, \alpha)$, and all elements from the algebra $\mathcal{F}_{\mathfrak{g}}(D)$ are splitting over the $\mathfrak{g}$-module X whenever D is additionally a Stein domain.*

Proof. Note that $\mathcal{B}(\wedge\mathfrak{g}, X) = \oplus_{k\in\mathbb{Z}_+} C^k(\mathfrak{g}, X)$ and $\theta = \oplus_{k\in\mathbb{Z}_+} \theta_k$, therefore

$$\sigma_{\mathrm{t}}(\theta) = \bigcup_{k\in\mathbb{Z}_+} \sigma_{\mathrm{t}}(\theta_k) = \sigma_{\mathrm{t}}(\alpha) \cup \bigcup_{k\in\mathbb{N}} \sigma_{\mathrm{t}}(\theta_k) = \sigma_{\mathrm{t}}(\alpha)$$

by virtue of Corollary 2. Moreover, $\theta(a) = L_{\alpha(a)} - R_{T(a)}$ for all $a \in \mathfrak{g}$. By assumption, all operators $\alpha(a) \in \mathcal{B}(X)$, $a \in \left[\mathfrak{g}, \mathfrak{g}\right]$, are nilpotent. Then $\theta(a)$ as the difference of mutually commuting nilpotent operators $L_{\alpha(a)}$ and $R_{T(a)}$ (see [6, Lemma 6.1]) is a nilpotent operator, therefore $\theta(\mathfrak{g})$ is a supernilpotent Lie subalgebra in $\mathcal{B}(\mathcal{B}(\wedge\mathfrak{g}, X))$. Using the functional calculus theorem [10] for the sheaf $\mathcal{F}_{\mathfrak{g}}$ and with respect to the Lie representation $\theta : \mathfrak{g} \to \mathcal{B}(\mathcal{B}(\wedge\mathfrak{g}, X))$, we infer that θ extends up to a continuous algebra homomorphism $\widehat{\theta} : \mathcal{F}_{\mathfrak{g}}(D) \to \mathcal{B}(\mathcal{B}(\wedge\mathfrak{g}, X))$. Moreover, $\widehat{\theta}(\mathcal{R}(\mathfrak{g}))$ is dense in $\widehat{\theta}\left(\mathcal{F}_{\mathfrak{g}}(D)\right)$, thereby $\mathcal{F}_{\mathfrak{g}}(D) \succ (X, \alpha)$ by Definition 4.

Now additionally assume that D is a Stein domain. Take $a \in \mathcal{F}_{\mathfrak{g}}(D)$ and $\lambda \in \sigma_{\mathrm{t}}(\alpha) \subset D$. One suffices (see Definition 5) to prove that

$$\widehat{\theta}(a) - \lambda|^{\mathcal{F}_{\mathfrak{g}}(D)}(a) = d(\lambda)\, i_\lambda(a) + i_\lambda(a)\, d(\lambda)$$

for some $i_\lambda(a) \in \mathcal{B}(\mathcal{B}(\wedge\mathfrak{g}, X))$, where $d(\lambda)$ is the differential of $C^\bullet(\alpha - \lambda)$. By (6.2), the element a has unique expansion $a = \sum_{J\in\mathbb{Z}_+^s} a_J \cdot e^J$ as a formal power series, where $a_J \in \mathcal{S}_\mathfrak{g}(D)$, e is a basis $[\mathfrak{g}, \mathfrak{g}]$, and $a_J \cdot e^J = a_J e_1^{j_1} \cdots e_s^{j_s}$ is the multiplication of relevant elements in the algebra $\mathcal{F}_\mathfrak{g}(D)$, $J = (j_1, \ldots, j_s)$. By (6.3), $a \in Rad\mathcal{F}_\mathfrak{g}(D)$ iff $a = \sum_{|J|>0} a_J \cdot e^J$, where $|J| = \sum_k j_k$. For each tuple $J \in \mathbb{Z}_+^s$, $|J| > 0$, the most nonzero index in J is denoted by m_J and we set $\overline{J} = (j_1, \ldots, j_{m_J - 1}, 0, \ldots, 0) \in \mathbb{Z}_+^s$. Then $\overline{a}_k = \sum_{J\in M_k} a_J \cdot e^{\overline{J}} \in \mathcal{F}_\mathfrak{g}(D)$ (see (6.2)), where $M_k = \{J \in \mathbb{Z}_+^s : |J| > 0, m_J = k\}$, $1 \leqslant k \leqslant s$. First, assume that $a \in Rad\mathcal{F}_\mathfrak{g}(D)$. Using Corollary 3, we infer that

$$\begin{aligned}
\widehat{\theta}(a) - \lambda|^{\mathcal{F}_\mathfrak{g}(D)}(a) &= \widehat{\theta}(a) = \widehat{\theta}\left(\sum_{k=1}^{s}\sum_{J\in M_k} a_J \cdot e^J\right) = \sum_{k=1}^{s}\sum_{J\in M_k} \widehat{\theta}\left(a_J \cdot e^{\overline{J}}\right)\theta(e_k) \\
&= \sum_{k=1}^{s}\sum_{J\in M_k} \widehat{\theta}\left(a_J \cdot e^{\overline{J}}\right)(d(\lambda)\, i(e_k) + i(e_k)\, d(\lambda)) \\
&= \sum_{k=1}^{s}\sum_{J\in M_k} \left(d(\lambda)\,\widehat{\theta}\left(a_J \cdot e^{\overline{J}}\right) i(e_k) + \widehat{\theta}\left(a_J \cdot e^{\overline{J}}\right) i(e_k)\, d(\lambda)\right) \\
&= d(\lambda)\, i(a) + i(a)\, d(\lambda),
\end{aligned}$$

where $i(a) = \sum_{k=1}^{s} \widehat{\theta}(\overline{a}_k)\, i(e_k)$. Further, assume that $a \in \mathcal{S}_\mathfrak{g}(D)$, that is, $a = \varepsilon_D(f)$ for some $f \in \mathcal{O}(D)$. By assumption, D is a Stein domain. By Hefer's decomposition theorem [20, Ch. 5, Section 2.2], $f - f(\lambda) = \sum_{k=1}^{m} g_k \cdot (z_k - \lambda_k)$ for some $g_k \in \mathcal{O}(D)$, where $m = \dim(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$, $\Delta(\mathfrak{g})$ is identified with the complex space $\mathbb{C}^m$, z_k are the coordinate linear functions on $\mathbb{C}^m$ and $\lambda = (\lambda_1, \ldots, \lambda_m)$. Then

$$a - \lambda|^{\mathcal{F}_\mathfrak{g}(D)}(a) = \varepsilon_D(f - f(\lambda)) = \sum_{k=1}^{m} a_k \cdot (x_k - \lambda(x_k)) + b,$$

where $a_k = \varepsilon_D(g_k)$, $x_k = \varepsilon_D(z_k) \in \mathfrak{g}$, $b = \sum_{k=1}^{m} (\varepsilon_D(g_k \cdot (z_k - \lambda_k)) - a_k \cdot (x_k - \lambda(x_k)))$. Note that

$$\begin{aligned}
\varphi_D(b) &= \sum_{k=1}^{m} \left(\varphi_D \varepsilon_D(g_k \cdot (z_k - \lambda_k)) - \varphi_D(a_k \cdot (x_k - \lambda(x_k)))\right) \\
&= \sum_{k=1}^{m} \left(g_k \cdot (z_k - \lambda_k) - \varphi_D(a_k) \cdot \varphi_D(x_k - \lambda(x_k))\right) = 0,
\end{aligned}$$

whence $b \in Rad\mathcal{F}_{\mathfrak{g}}(D)$ and $\widehat{\theta}(b) = d(\lambda)\, i(b) + i(b)\, d(\lambda)$ as we have proved above. Finally,

$$\widehat{\theta}(a) - \lambda|^{\mathcal{F}_{\mathfrak{g}}(D)}(a) = \sum_{k=1}^{m} \widehat{\theta}(a_k) \cdot (\theta(x_k) - \lambda(x_k)) + \widehat{\theta}(b) = d(\lambda)\, i_{\lambda}(a) + i_{\lambda}(a)\, d(\lambda),$$

where $i_{\lambda}(a) = \sum_{k=1}^{m} \widehat{\theta}(a_k)\, i(x_k) + i(b)$. By appealing (6.3), we obtain that all elements of $\mathcal{F}_{\mathfrak{g}}(D)$ are splitting over the $\mathfrak{g}$-module X. □

Corollary 15. *Let* $\mathfrak{g}$ *be a finite-dimensional nilpotent Lie algebra*, (X, α) *a Banach* $\mathfrak{g}$-*module such that* $\alpha(\mathfrak{g})$ *is a supernilpotent Lie subalgebra in* $\mathcal{B}(X)$, D *a Stein* $\mathcal{F}_{\mathfrak{g}}$-*rational domain in* $\Delta(\mathfrak{g})$ *containing the Taylor spectrum* $\sigma_{\mathrm{t}}(\alpha)$ *and let* $S \subseteq \mathcal{F}_{\mathfrak{g}}(D)$ *be a subset generating a quasinilpotent normed Lie subalgebra* $\mathfrak{F} \subseteq \mathcal{F}_{\mathfrak{g}}(D)$ *such that* $\widehat{\mathfrak{F}} \in Proj$. *Then* $\mathcal{F}_{\mathfrak{g}}(D) \succ (X, \alpha)$ *and*

$$\sigma_u\left(\alpha|^{\mathcal{F}_{\mathfrak{g}}(D)}|_{\mathfrak{F}}\right) = \sigma(\alpha)\,|^{\mathcal{F}_{\mathfrak{g}}(D)}|_{\mathfrak{F}}, \quad \sigma \in \mathfrak{S}^{\pi}.$$

In particular, if $\dim(\mathfrak{F}) < \infty$ *then* $\mathfrak{F}$ *is a nilpotent Lie subalgebra and*

$$\sigma\left(\alpha|^{\mathcal{F}_{\mathfrak{g}}(D)}|_{\mathfrak{F}}\right) = \sigma(\alpha)\,|^{\mathcal{F}_{\mathfrak{g}}(D)}|_{\mathfrak{F}}, \quad \sigma \in \mathfrak{S}.$$

Moreover, if $\mathfrak{A} \subseteq \mathcal{F}_{\mathfrak{g}}(D)$ *is a finite-dimensional Lie subalgebra then* $\mathfrak{A}_{\alpha} = \alpha|^{\mathcal{F}_{\mathfrak{g}}(D)}(\mathfrak{A})$ *is solvable and*

$$\Sigma(\mathfrak{A}_{\alpha}) = \Sigma(\alpha(\mathfrak{g}))\,|^{\mathcal{A}_{\alpha}}|_{\mathfrak{A}_{\alpha}}, \quad \Sigma \in \mathfrak{R}\mathfrak{S},$$

where $\mathcal{A}_{\alpha}$ *is the closed full subalgebra in* $\mathcal{B}(X)$ *generated by* $\alpha(\mathfrak{g})$.

Proof. Note that $\mathcal{F}_{\mathfrak{g}}(D)$ is commutative modulo its Jacobson radical therefore if S generates a finite-dimensional quasinilpotent Lie subalgebra then it is automatically nilpotent. By appealing Theorems 6, 7, 9 and Corollary 8, we end the proof. □

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References

[1] D. Beltiţa, M. Şabac, Lie algebras of bounded operators, Operator Theory: Advances and Applications, vol. 120, Birkhauser-Verlag, Basel, 2001.
[2] N. Bourbaki, Spectral Theory, Mir, Moscow, 1972 (in Russian).
[3] N. Bourbaki, Lie Groups and Lie Algebras, Mir, Moscow, 1976 (Chapter 1–3) (in Russian).
[4] A.A. Dosiev, Holomorphic functions of a basis of a nilpotent Lie algebra, Functional Anal. Appl. 34 (4) (2000) 302–304.
[5] A.A. Dosiev, Ultraspectra of a representation of a Banach Lie algebra, Functional Anal. Appl. 35 (3) (2001) 226–229.
[6] A.A. Dosiev, Spectra of infinite parametrized Banach complexes, J. Operator Theory 48 (3) (2002) 585–614.
[7] A.A. Dosiev, Projection property of spectra for solvable Lie algebra representations, Trans. NAS Azerbaijan 23 (1) (2003) 27–32.
[8] A.A. Dosiev, Homological dimensions of the algebra formed by entire functions of elements of a nilpotent Lie algebra, Functional Anal. Appl. 37 (1) (2003) 61–64.
[9] A.A. Dosiev, Algebras of power series in elements of a Lie algebra and Slodkowski spectra, J. Math. Sci. 124 (2) (2004) 4886–4908.
[10] A.A. Dosiev, Fréchet algebra sheaf cohomology and spectral theory, Functional Anal. Appl. (2005), to appear.
[11] A.A. Dosiev, Spectral Mapping Framework, Banach Center Pub., vol. 67 (2005) 411pp.
[12] J. Eschmeier, M. Putinar, Spectral Decomposition and Analytic Sheaves, Clarendon Press, Oxford, 1996.
[13] A.S. Fainshtein, Taylor joint spectrum for families of operators generating nilpotent Lie algebra, J. Operator Theory 29 (1993) 3–27.
[14] A.S. Fainshtein, Corrections to my paper "Taylor joint spectrum for families of operators generating nilpotent Lie algebra", J. Operator Theory 44 (2) (2000) 433–434.
[15] R. Godement, Algebraic Topology and Sheaf Theory, IIL, Moscow, 1961 (in Russian).
[16] S. Heinrich, Ultraproducts in Banach space theory, J. Reine Angew. Math. 313 (1980) 72–104.
[18] S.S. Kutateladze, Fundamentals of Functional Analysis, Kluwer Academic Publishers, Dordrecht, 1996.
[19] A.Yu. Pirkovskii, Stably Flat Completions of Universal Enveloping Algebras, preprint.
[20] R.M. Range, Holomorphic Functions and Integral Representation in Several Complex Variables, Springer, Berlin, 1990.
[21] G.-C. Rota, W.G. Strang, A note on the joint spectral radius, Indag. Math. 22 (1960) 379–381.
[22] Z. Slodkowski, An infinite family of joint spectra, Stud. Math. LXI (1977) 239–255.
[23] J.L. Taylor, The analytic functional calculus for several commuting operators, Acta Math. 125 (1970) 1–48.
[24] J.L. Taylor, A general framework for a multi-operator functional calculus, Adv. Math. 9 (1972) 183–252.
[25] Yu.V. Turovskii, On spectral properties of elements of normed algebras and invariant subspaces, Funktsional. Anal. i Prilozhen. 18 (2) (1984) 77–78.
[26] Yu.V. Turovskii, On spectral properties of some Lie subalgebras and spectral radius of subsets of a Banach algebra, Sb. Spekt. Teor. Operator. i Prilozhen. Baku ELM (1985) 144–181 (in Russian).
[27] Yu.V. Turovskii, V.S. Shulman, Radicals of Banach algebras and some problems of radical Banach algebras, Funktsional. Anal. i Prilozhen. 35 (4) (2001) 88–91.