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# Fréchet Sheaves and Taylor Spectrum for Supernilpotent Lie Algebra of Operators

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*To my parents*

**Abstract.** In the paper we investigate the transversality property of the Fréchet algebras of formally radical functions in elements of a nilpotent Lie algebra and its relationship to the Taylor spectrum of a family of bounded linear operators generating a supernilpotent Lie algebra.

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## 1. Introduction

Let $\mathcal{L}(X)$ be the Banach algebra of all bounded linear operators acting on a complex Banach space $X$ and let $T = (T_1, \ldots, T_m)$ be a family of operators in $\mathcal{L}(X)$ generating a finite-dimensional nilpotent Lie subalgebra $\mathfrak{g}_T \subseteq \mathcal{L}(X)$. It is known [21] that the Lie ideal $[\mathfrak{g}_T, \mathfrak{g}_T]$ of commutators consists of quasinilpotent operators. We say that $T$ generates *a supernilpotent Lie algebra* $\mathfrak{g}_T$ if $[\mathfrak{g}_T, \mathfrak{g}_T]$ consists of nilpotent operators. Obviously, each mutually commuting operator tuple generates automatically a supernilpotent Lie algebra. Moreover, if $X$ is a finite-dimensional space then each nilpotent Lie subalgebra in $\mathcal{L}(X)$ is supernilpotent. The class of noncommutative supernilpotent Lie algebras of operators on an infinite-dimensional Banach space $X$ is sufficiently wider than the class of commutative Lie algebras. The well developed joint spectral theory for an operator tuple $T$ generating a nilpotent Lie algebra has been proposed in papers [1], [6], [7], [12]. In particular, we have a well defined Taylor spectrum $\sigma(T)$ of the operator tuple $T$ which possesses the spectral mapping property with respect to the noncommutative polynomials. The relevant (noncommutative) holomorphic functional calculus about the Taylor spectrum $\sigma(T)$ remains inexplicit. Some developments toward this problem have been done in [3], [4], [5], [6], [10], [17]. The aim to conduct Taylor's program [20] (see [11] for the commutative case) on the general framework of the "noncommutative holomorphic functional calculus" for a nilpotent Lie algebra of bounded linear operators. The basic result of Taylor's approach is to create a subtle connection between the joint spectral theory and topological homology (see [20], [11] for the commutative case). Namely, the resolvent set with respect to the Taylor spectrum can be described in terms of the transversality behavior of the sheaf $\mathcal{O}$ of germs of holomorphic functions in several complex variables. The operator families generating supernilpotent Lie algebras have great advantageous to implement that connection in the noncommutative case. One may use the Fréchet algebra sheaf $\mathfrak{T}_{\mathfrak{g}}$ of germs of formally-radical functions in elements of a positively graded nilpotent Lie algebra (the noncommutative variable space) $\mathfrak{g}$ proposed in [8] (see also [5]). As a sheaf of the Fréchet spaces, $\mathfrak{T}_{\mathfrak{g}}$ has a relatively simple structure, namely it is just the projective tensor product

$$\mathfrak{T}_{\mathfrak{g}} = \mathcal{O}\widehat{\otimes}\mathbb{C}\left[\left[\omega_1, \ldots, \omega_k\right]\right]$$

of the sheaf $\mathcal{O}$ of germs of usual holomorphic functions over $\mathbb{C}^m$ and the constant sheaf $\mathbb{C}\left[\left[\omega_1, \ldots, \omega_k\right]\right]$ of all formal power series in $k$-variables, where $m = \dim\left(\mathfrak{g}/\left[\mathfrak{g}, \mathfrak{g}\right]\right)$ and $k = \dim\left(\left[\mathfrak{g}, \mathfrak{g}\right]\right)$. The algebraic structure on $\mathfrak{T}_{\mathfrak{g}}(D)$ for a polydisk $D$ is uniquely lifted from the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ which is its proper subalgebra (see Section 3). Confirm that similar construction is used in the noncommutative algebraic geometry in [16]. In particular, $\mathfrak{T}_{\mathfrak{g}} = \mathcal{O}$ whenever $\mathfrak{g}$ is a commutative Lie algebra. If $T$ is a $m$-tuple of operators on $X$ generating a supernilpotent Lie algebra then $X$ turns out to be a left Banach module over the Fréchet algebra $\mathcal{F}_{\mathfrak{g}} = \mathfrak{T}_{\mathfrak{g}}(\mathbb{C}^m)$ of all global sections of the sheaf $\mathfrak{T}_{\mathfrak{g}}$, that is, all entire formally radical functions act on $X$. Moreover, $\mathfrak{T}_{\mathfrak{g}}(D)$ possesses the Koszul resolution [5], [8], which is a free $\mathfrak{T}_{\mathfrak{g}}(D)$-bimodule resolution.

In the present paper we prove the crucial result of Taylor's approach on the connection between Taylor spectrum and transversality for an operator family generating a supernilpotent Lie algebra. Namely, let $T$ be a $m$-tuple of bounded linear operators on a Banach space generating a supernilpotent Lie algebra in $\mathcal{L}(X)$. Then $X$ turns out to be a Banach left $\mathcal{F}_{\mathfrak{g}}$-module for a certain nilpotent Lie algebra $\mathfrak{g}$ of noncommutative variables, and the resolvent set $\mathbb{C}^m\backslash\sigma(T)$ with respect to the Taylor spectrum $\sigma(T)$ consists of those $\lambda \in \mathbb{C}^m$ such that $\mathfrak{T}_{\mathfrak{g}}(D) \perp_{\mathcal{F}_{\mathfrak{g}}} X$ (that is, $\operatorname{Tor}_k^{\mathcal{F}_{\mathfrak{g}}}\left(\mathfrak{T}_{\mathfrak{g}}(D), X\right) = \{0\}$ for all $k$) for a certain small polydisk $D$ containing $\lambda$. Moreover,

$$\mathfrak{T}_{\mathfrak{g}}(D) \perp_{\mathcal{F}_{\mathfrak{g}}} X \quad \Longleftrightarrow \quad D \cap \sigma(T) = \emptyset.$$

This result plays a key role in the solution of the noncommutative holomorphic functionl calculus problem for a supernilpotent Lie algebra of operators [9]. Confirm that we are using homology theory for the topological algebras developed by

J. Taylor [19] and A. Ya. Helemskii [14]. The proof is based upon a noncommutative version of the known result on analytically parametrized complexes of Banach spaces [18] by J. Taylor.

## 2. Preliminaries

All considered linear spaces are complex and algebras are assumed to be unital and associative. Taking a linear space $X$, $\wedge X = \oplus_{k\geq 0} \wedge^k X$ is the exterior algebra of $X$. If $\underline{u} = u_1 \wedge \ldots \wedge u_k \in \wedge^k X$ is a $k$-vector then we use the following denotation $\underline{u}_i = u_1 \wedge \ldots \wedge \widehat{u_i} \wedge \ldots \wedge u_k$, for $(k-1)$-vector, where $\widehat{u_i}$ means the omission of the variable $u_i$. If we throw out two variables $u_i$ and $u_j$, $i < j$, from the expression of $\underline{u}$, the obtained vector is denoted by $\underline{u}_{ij}$. The space of all $X$-valued polynomials in $s$ variables is denoted by $X[\omega_1, \ldots, \omega_s]$, whereas $X[[\omega_1, \ldots, \omega_s]]$ denotes the space of all $X$-valued formal power series in $s$ variables, so, each its element $f$ has the unique formal power series expansion $f = \sum_{J\in\mathbb{Z}_+^s} x_J \omega^J$, where $x_J \in X$, $\omega^J = \omega_1^{j_1} \cdots \omega_s^{j_s}$. If $X$ and $Y$ are Fréchet spaces then the space of all continuous linear mappings $X \to Y$ is denoted by $\mathcal{L}(X, Y)$, we also write $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$. We use the conventional notation $X\widehat{\otimes}Y$ for the projective tensor product of these spaces. If $\{p_t : t \in \Lambda\}$ is a defining countable seminorm family in $X$ then the space $X[[\omega_1, \ldots, \omega_s]]$ turns out to be a Fréchet space with its defining seminorm family $\left\{q_{t,K} : (t, K) \in \Lambda \times \mathbb{Z}_+^s\right\}$, where $q_{t,K}(f) = \max\{p_t(x_J) : J \leq K\}$. One may easily verify that the topology generated by the latter seminorm family is merely the direct product topology of $X^{\mathbb{Z}_+^s}$. In particular,

$$X[[\omega_1, \ldots, \omega_s]] = X\widehat{\otimes}\mathbb{C}[[\omega_1, \ldots, \omega_s]],$$

and if $X$ is a nuclear space then so is the space $X[[\omega_1, \ldots, \omega_s]]$.

The Jacobson radical of a (Fréchet) algebra $A$ is denoted by $\operatorname{Rad} A$. The left (respectively, right) multiplication operator on $A$ is denoted by $L_a$ (respectively, $R_a$), that is, $L_a(x) = ax$ and $R_a(x) = xa$ for all $a, x \in A$. The unit of $A$ is denoted by $1_A$. A Fréchet algebra $A$ with its distinguished continuous character (multiplicative linear functional) $\varepsilon_A : A \to \mathbb{C}$ is called *an augmented algebra*. Further, a Fréchet space $X$ is said to be a left *Fréchet $A$-module* if $X$ has a structure of a left $A$-module such that the mapping $A \times X \to X$, $(a, x) \mapsto a \cdot x$ is jointly (or separately) continuous. By analogy, it is defined a right (bi)module over $A$. The category (usually we refer as a class) of all left Fréchet $A$-modules is denoted by $A$-mod. On the same manner mod-$A$ (respectively, $A$-mod-$A$) denotes the category of all right (respectively, bi)modules. If $A$ is an augmented algebra with its distinguished continuous character $\varepsilon_A : A \to \mathbb{C}$ then the one-dimensional space $\mathbb{C}$ turns into a $A$-module via pullback along $\varepsilon_A$ called the *trivial $A$-module* and denoted by $\mathbb{C}(\varepsilon_A)$.

The universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{g}$ is denoted by $\mathcal{U}(\mathfrak{g})$. The algebra $\mathcal{U}(\mathfrak{g})$ turns out to be a topological algebra equipped with the finest locally convex topology. The space of all Lie characters of a Lie algebra $\mathfrak{g}$ is denoted by $\Delta(\mathfrak{g})$. The space of all characters of $\mathcal{U}(\mathfrak{g})$ is identified with $\Delta(\mathfrak{g})$, that is, each Lie character $\lambda \in \Delta(\mathfrak{g})$ has a unique extension up to a character on $\mathcal{U}(\mathfrak{g})$ denoted by $\lambda$, too. Take a basis $e = (e_1, \ldots, e_n)$ in a Lie algebra $\mathfrak{g}$. For a $n$-tuple $J = (j_1, \ldots, j_n) \in \mathbb{Z}_+^n$ of nonnegative integers we put $e^J = e_1^{j_1} \cdots e_n^{j_n}$ to indicate the ordered monomial in $\mathcal{U}(\mathfrak{g})$ taken by the basis $e$. By Poincare-Birkhoff-Witt theorem (see [2, 2.2.1]), the set $\{e^J\} \subseteq \mathcal{U}(\mathfrak{g})$ of all ordered monomials is an algebraic basis in $\mathcal{U}(\mathfrak{g})$. For each $k$, let introduce the "insertion" operator $\Delta_{e_k} \in \mathcal{L}(\mathcal{U}(\mathfrak{g}))$ by the rule

$$\Delta_{e_k}\left(e_1^{j_1} \cdots e_n^{j_n}\right) = e_1^{j_1} \cdots e_k^{j_k+1} \cdots e_n^{j_n}$$

for all ordered monomials $e^J$.

Now, let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra with its vanishing lower central series $\{\mathfrak{g}^{(s)} : s \geq 1\}$, where $\mathfrak{g}^{(1)} = \mathfrak{g}$, $\mathfrak{g}^{(s)} = [\mathfrak{g}, \mathfrak{g}^{(s-1)}]$ if $s > 1$. A basis $e = (e_1, \ldots, e_n)$ in $\mathfrak{g}$ is said to be a *triangular basis* if it obeys to the lower central series. Thus $[e_i, e_j] = \sum_{k>j} c_{ij}^k e_k$ whenever $i < j$. If $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_c$ is graded with the positive integers $1, \ldots, c$ then each basis subordinated to the latter decomposition (called *a graded basis*) is a triangular one. For a triangular basis $e = (e_1, \ldots, e_n)$ of a nilpotent Lie algebra $\mathfrak{g}$, $e_{\mathrm{r}} = (e_{m+1}, \ldots, e_n)$ will be a basis in $[\mathfrak{g}, \mathfrak{g}]$ for a certain $m$. We say that $e_{\mathrm{r}}$ is a *radical part* of $e$ and $e_{\mathrm{s}} = (e_1, \ldots, e_m)$ is a *semisimple part* of $e$. In that concern, we also write $J_{\mathrm{s}} = (j_1, \ldots, j_m)$ and $J_{\mathrm{r}} = (j_{m+1}, \ldots, j_n)$ if $J = (j_1, \ldots, j_n) \in \mathbb{Z}_+^n$, confirm also that $J = J_{\mathrm{s}} \cup J_{\mathrm{r}}$. The set $\{e^J\}$ of all ordered monomials turns into a linearly ordered set with respect to the relation: $e^I \preceq e^J$ if the first nonzero integer in the tuple $(j_n - i_n, \ldots, j_1 - i_1)$ is positive.

The following lemma on a commutator can easily be proved based upon the induction argument (for the details see [8]).

**Lemma 2.1.** *Let $e = (e_1, \ldots, e_n)$ be a triangular basis of a nilpotent Lie algebra $\mathfrak{g}$, such that $e_{m+1}, \ldots, e_n$ is a basis in $[\mathfrak{g}, \mathfrak{g}]$, $I = (i_1, \ldots, i_m) \in \mathbb{Z}_+^m$, $1 \leq i \leq n$ and let $k = \max\{i, m\}$. Then*

$$\left[e_i, e_{\mathrm{s}}^I\right] = \sum_{t=k+1}^{n} p_{I,t}(e_1, \ldots, e_m)\, e_t,$$

*where $p_{I,t} \in \mathbb{C}[\omega_1, \ldots, \omega_m]$.*

Now, let $A$ be a Banach algebra and let $\mathfrak{g}$ be its finite-dimensional nilpotent Lie subalgebra. The closed associative envelope $B$ of $\mathfrak{g}$ in $A$ is a commutative algebra modulo its Jacobson radical $\operatorname{Rad} B$ thanks to Turovskii Lemma [21]. Therefore $\operatorname{Rad} B$ is the set of all quasinilpotent elements in $B$ which is just the left (or right) closed ideal in $B$ generated by the Lie ideal $[\mathfrak{g}, \mathfrak{g}]$. We say that $\mathfrak{g}$ is *a supernilpotent Lie algebra* in $A$ if each $a \in [\mathfrak{g}, \mathfrak{g}]$ is nilpotent in $A$. Note that if $A$ is a finite-dimensional Banach algebra then each its nilpotent Lie subalgebra is supernilpotent one. Moreover, a commutative Lie subalgebra of a Banach algebra $A$ is supernilpotent, too. If $A = \mathcal{L}(X)$ is the Banach algebra of all bounded linear operators on a Banach space $X$ and $\mathfrak{g}$ is a supernilpotent Lie algebra in $A$ then we say that $\mathfrak{g}$ is *a supernilpotent Lie algebra of operators.* As an example of operators acting on an infinite-dimensional Banach space $X$ generating (noncommutative) supernilpotent Lie subalgebra in $\mathcal{L}(X)$, we suggest the following operators.

*Example.* Let $X = \ell_p$ $(p \geq 1)$ be the Banach space of all $p$-th degree absolutely convergent sequences and let $\{e_n\}_{n\in\mathbb{N}}$ be its canonical basis. We set $T, S \in \mathcal{L}(X)$,

$$T(e_{2n-1}) = e_{2n+1}, \quad T(e_{2n}) = n^{-1}(n+1)\, e_{2n+2},$$
$$S(e_{2n-1}) = e_{2n-1}, \quad S(e_{2n}) = e_{2n} + e_{2n-1},$$

for all $n \in \mathbb{N}$. Then $[T,S] \neq 0$, $[T,[T,S]] = [S,[T,S]] = 0$ and $[T,S]^2 = 0$. Thus the operators $T$ and $S$ generate a nilpotent Lie algebra with the nilpotent commutators.

The following lemma asserts that the radical part of a supernilpotent Lie algebra can be enclosed into a finite dimensional linear space.

**Lemma 2.2.** *Let $\mathfrak{g}$ be a supernilpotent Lie subalgebra of $A$. Then $[\mathfrak{g},\mathfrak{g}]$ generates a finite-dimensional nilpotent associative subalgebra in $A$.*

*Proof.* Let $B$ be an associative subalgebra in $A$ generated by $[\mathfrak{g},\mathfrak{g}]$. Take a triangular basis $e = (e_1,\dots,e_n)$ in $\mathfrak{g}$ such that $e_{\mathrm{r}} = (e_{m+1},\dots,e_n)$ is a basis in $[\mathfrak{g},\mathfrak{g}]$. Then $e_i^k = 0$ in $A$ for all $i$, $i > m$, and for some $k$. Since monomials $e_{\mathrm{r}}^J$ in $A$ generate $B$ as a linear space, it follows that $\dim(B) < +\infty$.
Now, take a polynomial $x = \sum_J a_J e_{\mathrm{r}}^J \in B$, which is the range of the "free" polynomial $y = \sum_J a_J e_{\mathrm{r}}^J \in \mathcal{U}(\mathfrak{g})$. Let us prove that $x$ is nilpotent. First, introduce a (nilpotent) degree of (non-ordered) monomials $e_{i_1}\cdots e_{i_s}$ in $\mathcal{U}(\mathfrak{g})$ taken by $e$ in $\mathfrak{g}$. We set $\deg(e_i) = \max\left\{k : e_i \in \mathfrak{g}^{(k)}\right\}$ for all elements $e_i$ of the basis $e$. If $v = e_{i_1}\cdots e_{i_s}$ then the degree $\deg(v)$ of $v$ is the sum $\deg(e_{i_1}) + \cdots + \deg(e_{i_s})$. We use the notation $\langle J\rangle$ instead of $\deg\left(e^J\right)$, that is, when $v = e^J$ is an ordered monomial. Let $\mathcal{U}_k(e)$ be a subspace in $\mathcal{U}(\mathfrak{g})$ generated by all ordered monomials $e^J$ of the degree $k$, and let $\mathcal{U}^k(e)$ be a subspace in $\mathcal{U}(\mathfrak{g})$ generated by all (non-ordered) monomials $e_{i_1}\cdots e_{i_s}$ of degree at least $k$. Obviously, $\mathcal{U}^k(e)\cdot\mathcal{U}^s(e) \subseteq \mathcal{U}^{k+s}(e)$. Bearing in mind that the set $\left\{e^J\right\}$ is an algebraic basis in $\mathcal{U}(\mathfrak{g})$, we deduce the following decomposition

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}_0(e) \oplus \cdots \oplus \mathcal{U}_{k-1}(e) \oplus \mathcal{U}^k(e) \qquad \text{for each } k.$$

Note that each subspace $\mathcal{U}_k(e)$ is finite-dimensional with the basis $\left\{e^J : \langle J\rangle = k\right\}$ and $\mathcal{U}^k(e) = \bigoplus_{i\geq k}\mathcal{U}_i(e)$ (see [4]). Undoubtedly, $y \in \mathcal{U}^2(e)$ and $y^p \in \mathcal{U}^{2p}(e)$ for all $p$. But

$$\mathcal{U}^{2p}(e) = \bigoplus_{i\geq 2p}\mathcal{U}_i(e),$$

therefore $y^p$ has a unique expansion by means of the ordered radical monomials $e_{\mathrm{r}}^J \in \mathcal{U}(\mathfrak{g})$, $\langle J\rangle \geq 2p$. But

$$\langle J\rangle = \deg\left(e_{\mathrm{r}}^J\right) = \sum_{i>m} j_i \deg(e_i),$$

whence, $x^p = 0$, whenever $2p \geq (n-m)\,kc$, where $c$ is the nilpotence power of $\mathfrak{g}$ (that is, $\mathfrak{g}^{(c)} \neq \{0\}$, $\mathfrak{g}^{(c+1)} = \{0\}$). Thus, there exists $p$, such that $x^p = 0$ for all $x \in B$. Thereby $B$ is a nilpotent subalgebra. □

Finally, take $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$, $r = (r_1, \ldots, r_n) \in \overline{\mathbb{R}}^n_+$, and let $D_{a,r}$ be a polydisk in $\mathbb{C}^n$ of multiradius $r$ centered at $a$. If $a = 0$ then we write $D_r$ instead of $D_{0,r}$. If $X$ is a Banach space then the space of all $X$-valued holomorphic functions on an open set $U$ is denoted by $\mathcal{O}(U, X)$. For $X = \mathbb{C}$, we write $\mathcal{O}(U)$ instead of $\mathcal{O}(U, \mathbb{C})$. Remind that $\mathcal{O}(U, X)$ is a Fréchet space and $\mathcal{O}(U)$ is a Fréchet algebra with respect to the compact-open topology, and $\mathcal{O}(U, X) = \mathcal{O}(U)\,\widehat{\otimes} X$ [13] (see also [14, Ch. 2]). If $U = D_{a,r}$ and $t \in \mathbb{R}^n_+$ with $0 < t < r$, then the seminorm set

$$\left\| \sum_J x_J (z-a)^J \right\|_t = \sum_J \|x_J\|_X \, t^J, \quad t \in \Lambda, \tag{2.1}$$

on $\mathcal{O}(D_{a,r}, X)$ are equivalent to one associated by the compact-open topology due to the known Cauchy inequality.

## 3. Formally-Radical Functions in Elements of a Nilpotent Lie Algebra

In this section we remind the basic properties of the formally radical functions in elements of a nilpotent Lie algebra investigated in [5], [8].

Everywhere below we fix a finite dimensional positively graded nilpotent Lie algebra $\mathfrak{g}$ and its basis $e = (e_1, \ldots, e_n)$ which obeys to that grading. Since each nilpotent Lie algebra can be presented as a quotient of a positively graded nilpotent Lie algebra, that will motivate our choice $\mathfrak{g}$ as the noncommutative variable space. Indeed, let $\mathfrak{h}$ be a finite dimensional nilpotent Lie algebra generated by $x_1, \ldots, x_m$. Consider the quotient $\mathfrak{g}$ of the free Lie algebra generated by $m$ elements $e_1, \ldots, e_m$ modulo the appropriate order of its lower central series. Evidently, $\mathfrak{g}$ admits positive grading with numbers $1, \ldots, c$, where $c$ is the nilpotence degree of $\mathfrak{h}$. Moreover, there exists a Lie epimorphism $\tau : \mathfrak{g} \to \mathfrak{h}$ such that $\tau(e_i) = x_i$, $1 \leq i \leq m$.

Let $\mathfrak{r}_e \subseteq \mathcal{U}(\mathfrak{g})$ be a subset of all radical monomials $e_{\mathrm{r}}^{J_{\mathrm{r}}} = e_{m+1}^{j_{m+1}} \cdots e_n^{j_n}$ ($1 = e_{\mathrm{r}}^{J_{\mathrm{r}}}$ for $J_{\mathrm{r}} = (0, \ldots, 0)$), $J_{\mathrm{r}} \in \mathbb{Z}^{n-m}_+$. Evidently, $\mathfrak{r}_e$ is a subset of the linearly ordered set $\{e^J\}$ of all ordered monomials (see Section 2). If $D_r$ is a polydisk in the character space $\Delta(\mathfrak{g})(= \mathbb{C}^m)$ of multiradius $r$ centered at the origin then the Fréchet algebra $\mathcal{F}_{\mathfrak{g}}(D_r)$ of all *formally radical functions in elements of* $\mathfrak{g}$ is defined as the Fréchet space $\mathcal{O}(D_r)\,[[e_{m+1}, \ldots, e_n]]$ of all formal power series over the Fréchet space $\mathcal{O}(D_r)$ in several radical variables $e_{\mathrm{r}}$. Thus each $f \in \mathcal{F}_{\mathfrak{g}}(D_r)$ has a unique formal power series expansion

$$f = \sum_{J_{\mathrm{r}}} f_{J_{\mathrm{r}}} e_{\mathrm{r}}^{J_{\mathrm{r}}},$$

where $f_{J_{\mathrm{r}}} \in \mathcal{O}(D_r)$. If $p = \sum_J x_J e^J \in \mathcal{U}(\mathfrak{g})$ then $p = \sum_{J_{\mathrm{r}}} p_{J_{\mathrm{r}}} e_{\mathrm{r}}^{J_{\mathrm{r}}}$, where $p_{J_{\mathrm{r}}} = \sum_{I_{\mathrm{s}}} x_{I_{\mathrm{s}} \cup J_{\mathrm{r}}} e^{I_{\mathrm{s}}}$. If we identify $p_{J_{\mathrm{r}}}$ with a polynomial $\sum_{I_{\mathrm{s}}} x_{I_{\mathrm{s}} \cup J_{\mathrm{r}}} z^{I_{\mathrm{s}}}$ in $\mathcal{O}(D_r)$ then $p = \sum_{J_{\mathrm{r}}} p_{J_{\mathrm{r}}} e_{\mathrm{r}}^{J_{\mathrm{r}}} \in \mathcal{F}_{\mathfrak{g}}(D_r)$. Thus $\mathcal{U}(\mathfrak{g})$ is a dense subspace in $\mathcal{F}_{\mathfrak{g}}(D_r)$. Moreover, the multiplication on $\mathcal{U}(\mathfrak{g})$ can uniquely be lifted up to the jointly continuous (noncommutative) multiplication $*$ on $\mathcal{F}_{\mathfrak{g}}(D_r)$ called the nilpotent convolution [8]. Whence $\mathcal{F}_{\mathfrak{g}}(D_r)$ is a Fréchet algebra and $\mathcal{U}(\mathfrak{g})$ is its dense subalgebra. Moreover, the space of all continuous characters on $\mathcal{F}_{\mathfrak{g}}(D_r)$ is identified with the polydisk $D_r$ [8]. If $X$ is a Banach space then the space $\mathcal{F}_{\mathfrak{g}}(D_r, X)$ of *$X$-valued formally radical functions in elements of* $\mathfrak{g}$ is defined as the projective tensor product $\mathcal{F}_{\mathfrak{g}}(D_r) \widehat{\otimes} X$. It is not so hard to prove that the set $\{e^J\}$ of all ordered monomials in $\mathcal{U}(\mathfrak{g})$ is an absolute $X$-valued basis in $\mathcal{F}_{\mathfrak{g}}(D_r, X)$ [8], that is, each $\overline{f} \in \mathcal{F}_{\mathfrak{g}}(D_r, X)$ has a unique power series expansion $\overline{f} = \sum_J x_J e^J$, $x_J \in X$, as absolutely convergent in $\mathcal{F}_{\mathfrak{g}}(D_r, X)$ power series. Without any doubt,

$$\mathcal{F}_{\mathfrak{g}}(D_r, X) = \mathcal{O}(D_r, X)[[e_{\mathrm{r}}]]$$

to within a topological isomorphism.

Now, take a point $a \in \Delta(\mathfrak{g})$ and let $\mathfrak{g} - a$ be a Lie subalgebra in $\mathcal{U}(\mathfrak{g})$ comprising all elements $u - a(u)$, $u \in \mathfrak{g}$. We set $\mathcal{F}_{\mathfrak{g}}(D_{a,r}) = \mathcal{F}_{\mathfrak{g}-a}(D_r)$ for a polydisk $D_{a,r} \subseteq \Delta(\mathfrak{g})$ of multiradius $r$ centered at $a$. If $D_{b,v} \subset D_{a,r} \subseteq \Delta(\mathfrak{g})$ are polydisks then we have a well defined restriction mapping

$$P_{b,v}^{a,r} : \mathcal{F}_{\mathfrak{g}}(D_{a,r}) \to \mathcal{F}_{\mathfrak{g}}(D_{b,v}), \quad P_{b,v}^{a,r}\left(\sum_{J_{\mathrm{r}}} f_{J_{\mathrm{r}}} e_{\mathrm{r}}^{J_{\mathrm{r}}}\right) = \sum_{J_{\mathrm{r}}} \left(f_{J_{\mathrm{r}}}|_{D_{b,v}}\right) e_{\mathrm{r}}^{J_{\mathrm{r}}},$$

where $f_{J_{\mathrm{r}}}|_{D_{b,v}}$ is the usual restriction of the holomorphic function $f_{J_{\mathrm{r}}} \in \mathcal{O}(D_{a,r})$. This is a continuous algebra homomorphism [8]. Take arbitrary points $a, b, c$ in $\Delta(\mathfrak{g})$, $f \in \mathcal{F}_{\mathfrak{g}}(D_{a,r})$, $g \in \mathcal{F}_{\mathfrak{g}}(D_{b,v})$, and let $D_{c,q} \subset D_{a,r} \cap D_{b,v}$. Then $f = \sum_{J_{\mathrm{r}}} f_{J_{\mathrm{r}}} e_{\mathrm{r}}^{J_{\mathrm{r}}}$, $g = \sum_{J_{\mathrm{r}}} g_{J_{\mathrm{r}}} e_{\mathrm{r}}^{J_{\mathrm{r}}}$, where $f_{J_{\mathrm{r}}} \in \mathcal{O}(D_{a,r})$, $g_{J_{\mathrm{r}}} \in \mathcal{O}(D_{b,v})$. Assume that $P_{c,q}^{a,r}(f) = P_{c,q}^{b,v}(g)$, where $P_{c,q}^{a,r} : \mathcal{F}_{\mathfrak{g}}(D_{a,r}) \to \mathcal{F}_{\mathfrak{g}}(D_{c,q})$ and $P_{c,q}^{b,v} : \mathcal{F}_{\mathfrak{g}}(D_{b,v}) \to \mathcal{F}_{\mathfrak{g}}(D_{c,q})$ are the algebra homomorphisms. Then $f_{J_{\mathrm{r}}}|_{D_{c,q}} = g_{J_{\mathrm{r}}}|_{D_{c,q}}$ for all $J_{\mathrm{r}}$. Therefore, $f_{J_{\mathrm{r}}}|_{D_{a,r} \cap D_{b,v}} = g_{J_{\mathrm{r}}}|_{D_{a,r} \cap D_{b,v}}$. In this situation, we write $f|_{D_{a,r} \cap D_{b,v}} = g|_{D_{a,r} \cap D_{b,v}}$. Let $U$ be a non-empty open subset in $\mathbb{C}^m$. Then $U$ has a countable cover $U = \cup_i D_i$ by open polydisks $D_i = D_{a_i, r_i}$. Let $\mathcal{F}_{\mathfrak{g}}(U)$ be a subspace of the topological direct product $\prod_i \mathcal{F}_{\mathfrak{g}}(D_i)$ comprising all compatible elements $\{f_i\}_i$, that is, $f_i|_{D_i \cap D_j} = f_j|_{D_i \cap D_j}$ for all $i$, $j$. Since the "restriction" mappings $P_b^a$ are continuous, it follows that $\mathcal{F}_{\mathfrak{g}}(U)$ is a closed subspace, thereby, $\mathcal{F}_{\mathfrak{g}}(U)$ is a Fréchet space. The nilpotent convolution is extended up to $\mathcal{F}_{\mathfrak{g}}(U)$ by the canonical way:

$$\{f_i\}_i * \{g_i\}_i = \{f_i * g_i\}_i .$$

Then $(f_i * g_i)|_{D_i \cap D_j} = f_i|_{D_i \cap D_j} * g_i|_{D_i \cap D_j} = f_j|_{D_i \cap D_j} * g_j|_{D_i \cap D_j} = (f_j * g_j)|_{D_i \cap D_j}$ for all $i$,$j$. Thus $\mathcal{F}_{\mathfrak{g}}(U)$ is a Fréchet algebra with respect to the nilpotent convolution. Moreover,

$$\mathcal{F}_{\mathfrak{g}}(U) = \mathcal{O}(U)[[e_{\mathrm{r}}]]$$

as the Fréchet space [8]. In particular, the space $\mathcal{F}_{\mathfrak{g}}(U)$ does not depend upon the particular choice of a polydisk cover $\{D_i\}$ of $U$. Therefore, if $U = D_{a,r}$ is a polydisk then $\mathcal{F}_{\mathfrak{g}}(U) = \mathcal{F}_{\mathfrak{g}}(D_{a,r})$. If $X$ is a Banach space then

$$\mathcal{F}_{\mathfrak{g}}(U)\,\widehat{\otimes}X = \mathcal{O}(U,X)\,\widehat{\otimes}\mathbb{C}[[e_{\mathrm{r}}]] = \mathcal{O}(U,X)[[e_{\mathrm{r}}]] \tag{3.1}$$

to within a topological isomorphism of the Fréchet spaces. In particular, each $\overline{f} \in \mathcal{F}_{\mathfrak{g}}(U)\,\widehat{\otimes}X$ has a unique expansion $\overline{f} = \sum_{J_{\mathrm{r}}} \overline{f}_{J_{\mathrm{r}}} e_{\mathrm{r}}^{J_{\mathrm{r}}}$ as formal power series, where $\overline{f}_{J_{\mathrm{r}}} \in \mathcal{O}(U,X)$.

Finally, if $\cdots \longleftarrow X_{n-1} \overset{T_{n-1}}{\longleftarrow} X_n \overset{T_n}{\longleftarrow} X_{n+1} \longleftarrow \cdots$ is an exact sequence of Fréchet spaces then the sequence

$$\cdots \longleftarrow \mathcal{F}_{\mathfrak{g}}(U)\,\widehat{\otimes}X_{n-1} \overset{1\otimes T_{n-1}}{\longleftarrow} \mathcal{F}_{\mathfrak{g}}(U)\,\widehat{\otimes}X_n \overset{1\otimes T_n}{\longleftarrow} \mathcal{F}_{\mathfrak{g}}(U)\,\widehat{\otimes}X_{n+1} \longleftarrow \cdots$$

remains exact. Indeed, it is well known [14, 2.4.16] that the sequence

$$\cdots \longleftarrow \mathcal{O}(U,X_{n-1}) \overset{1\otimes T_{n-1}}{\longleftarrow} \mathcal{O}(U,X_n) \overset{1\otimes T_n}{\longleftarrow} \mathcal{O}(U,X_{n+1}) \longleftarrow \cdots$$

remains exact. It remains to note that the functor $\circ\widehat{\otimes}\mathbb{C}[[\omega_1,\ldots,\omega_{n-m}]]$ applied to the latter complex does not change its exactness [19], for $\mathbb{C}[[\omega_1,\ldots,\omega_{n-m}]]$ is a nuclear Fréchet space.

## 4. Parametrization

In this section we generalize Taylor's result on analytically parametrized Banach space complexes [18, Theorem 2.2] for noncommutative polynomials. Since we deal with the operators on $\mathcal{F}_{\mathfrak{g}}(U)$ which can be expressed in terms of infinite triangular matrices, first we introduce triangular matrices in its general framework.

### 4.1. Triangular Matrices

Let $X_n, Y_n$, $n \in \mathbb{N}$, be Fréchet spaces and let $X = \prod_n X_n$, $Y = \prod_n Y_n$ be their topological direct products. Note that each element $x \in X$ has a unique expansion $x = \sum_n \iota_{X,n}(\pi_{X,n}(x))$ as unconditionally convergent (in $X$) series, where $\iota_{X,n} \in \mathcal{L}(X_n, X)$ (respectively, $\pi_{X,n} \in \mathcal{L}(X, X_n)$) is a canonical embedding (respectively, projection) of the topological direct product. For each linear mapping $S : X \to Y$ there corresponds its infinite matrix $[S_{mn}]$ with $S_{mn} = \pi_{Y,m}\cdot S\cdot\iota_{X,n}$. Undoubtedly, if $S \in \mathcal{L}(X,Y)$ then $S_{mn} \in \mathcal{L}(X_n, Y_m)$ for all $n$, $m$. We say that $S$ is a *triangular operator* if its matrix is lower (or upper) triangular, that is, $S_{mn} = 0$ whenever $n > m$ (or $n < m$).

Now, let $[S_{mn}]$ be a lower triangular matrix, where $S_{mn} : X_n \to Y_m$ are certain linear mappings. The latter matrix defines a linear mapping $S : X \to Y$ by the rule

$$S(x) = \sum_m \iota_{Y,m}\left(\sum_{n=1}^{m} S_{mn}(\pi_{X,n}(x))\right).$$

Evidently, the matrix of the latter mapping $S$ is reduced to the original matrix $[S_{mn}]$.

**Lemma 4.1.** *Let $[S_{mn}]$ be a lower triangular matrix with $S_{mn} \in \mathcal{L}(X_n, Y_m)$. Then the linear mapping $S : X \to Y$ induced by the matrix is continuous. If $Z = \prod_n Z_n$, $[T_{mn}]$ is a lower triangular matrix with $T_{mn} \in \mathcal{L}(Y_n, Z_m)$, and $T : Y \to Z$ is the induced linear mapping, then the matrix of $TS$ is the product of the matrices $[T_{mn}]$ and $[S_{mn}]$.*

*Proof.* To demonstrate the continuity of $S$, one suffices to prove that $\pi_{Y,m} S$ is continuous for all $m$. In order to show that $\pi_{Y,m} S$ is continuous, it suffices to write $\pi_{Y,m} S = \sum_{n \le m} S_{mn} \pi_{X,n}$ and to observe that the sum of continuous mappings is continuous.
Further, $S \in \mathcal{L}(X, Y)$ and $T \in \mathcal{L}(Y, Z)$ are triangular operators due to the just established fact. Moreover,

$$\begin{aligned} TS(x) &= \sum_m T\iota_{Y,m} \left( \sum_{n=1}^{m} S_{mn} (\pi_{X,n}(x)) \right) = \sum_m \sum_{k \ge m} \iota_{Z,k} \sum_{n=1}^{m} T_{km} S_{mn} (\pi_{X,n}(x)) \\ &= \sum_k \sum_{m=1}^{k} \iota_{Z,k} \sum_{n=1}^{m} T_{km} S_{mn} (\pi_{X,n}(x)) \\ &= \sum_k \iota_{Z,k} \sum_{n=1}^{k} \sum_{m=1}^{k} T_{km} S_{mn} (\pi_{X,n}(x)) = \sum_k \iota_{Z,k} \sum_{n=1}^{k} G_{km} (\pi_{X,n}(x)) \\ &= G(x), \end{aligned}$$

where $[G_{km}]$ is the product of the matrices $[T_{nm}]$ and $[S_{nm}]$, $G$ is the triangular operator associated with $[G_{km}]$. □

**Proposition 4.2.** *Let $X = \prod_n X_n$, $Y = \prod_n Y_n$, $Z = \prod_n Z_n$, with the Fréchet spaces $X_n, Y_n, Z_n$, $n \in \mathbb{N}$, and let $[S_{mn}]$, $[T_{mn}]$ be lower triangular operator matrices with $S_{mn} \in \mathcal{L}(X_n, Y_m)$, $T_{mn} \in \mathcal{L}(Y_n, Z_m)$. Let us assume that the sequence*

$$Z \xleftarrow{T} Y \xleftarrow{S} X \tag{4.1}$$

*is a chain complex, where $S$ and $T$ are the triangular operators induced by the matrices $[S_{mn}]$ and $[T_{mn}]$, respectively. If all $Z_n \xleftarrow{T_{nn}} Y_n \xleftarrow{S_{nn}} X_n$ are exact sequences then so is* (4.1).

*Proof.* Take $y \in \ker(T)$. Then $Ty = \sum_m \iota_{Z,m} \left( \sum_{n=1}^{m} T_{mn} (\pi_{Y,n}(y)) \right) = 0$. In particular, $T_{11} \pi_{Y,1}(y) = 0$. By assumption, there exists $x_1 \in X_1$ such that $S_{11} x_1 = \pi_{Y,1}(y)$. But

$$S(\iota_{X,1} x_1) = \sum_m \iota_{Y,m} S_{m1} x_1, \quad y - S\iota_{X,1} x_1 = \sum_{m \ge 2} \iota_{Y,m} (y_m - S_{m1} x_1).$$

By induction on $n$, let us prove that there exist elements $x_m \in X_m$ such that

$$y - \sum_{k=1}^{n} S\iota_{X,k} x_k = \sum_{m > n} \iota_{Y,m} \left( y_m - \sum_{k=1}^{n} S_{mk} x_k \right).$$

By induction hypothesis,

$$y - \sum_{k=1}^{n-1} S\iota_{X,k}x_k = \sum_{m>n-1} \iota_{Y,m}\left(y_m - \sum_{k=1}^{n-1} S_{mk}x_k\right)$$

and

$$T\left(y - \sum_{k=1}^{n-1} S\iota_{X,k}x_k\right) = Ty - \sum_{k=1}^{n-1} TS\iota_{X,k}x_k = 0.$$

It follows that $T_{nn}\left(y_n - \sum_{k=1}^{n-1} S_{nk}x_k\right) = 0$. Then again $y_n - \sum_{k=1}^{n-1} S_{nk}x_k = S_{nn}x_n$ for a certain $x_n \in X_n$, and

$$\begin{aligned} y - \sum_{k=1}^{n} S\iota_{X,k}x_k &= \sum_{m\geq n} \iota_{Y,m}\left(y_m - \sum_{k=1}^{n-1} S_{mk}x_k\right) - \sum_{m\geq n} \iota_{Y,m}S_{mn}x_n \\ &= \sum_{m\geq n+1} \iota_{Y,m}\left(y_m - \sum_{k=1}^{n} S_{mk}x_k\right). \end{aligned}$$

Now, let $x = \sum_k \iota_{X,k}x_k \in X$. Using Lemma 4.1, we deduce that

$$\begin{aligned} \pi_{Y,n}(y - Sx) &= \pi_{Y,n}\left(y - \sum_k S\iota_{X,k}x_k\right) \\ &= \pi_{Y,n}\left(y - \sum_{k=1}^{n} S\iota_{X,k}x_k\right) + \pi_{Y,n}\left(\sum_{k\geq n+1} S\iota_{X,k}x_k\right) \\ &= \pi_{Y,n}\left(\sum_{m\geq n+1} \iota_{Y,m}\left(y_m - \sum_{k=1}^{n} S_{mk}x_k\right)\right) + \sum_{k\geq n+1} \pi_{Y,n}S\iota_{X,k}x_k \\ &= \sum_{k\geq n+1} S_{nk}x_k = 0 \end{aligned}$$

for all $n$. Thus $Sx = y$. □

**Corollary 4.3.** *Let* $[S_{mn}]$ *be a lower triangular matrix with* $S_{mn} \in \mathcal{L}(X_n, X_m)$. *If all operators* $S_{nn} \in \mathcal{L}(X_n)$ *are invertible then the triangular operator* $S \in \mathcal{L}(X)$ *induced by the matrix* $[S_{mn}]$ *is invertible.*

*Proof.* One suffices to set $Y_n = X_n$ and $Z_n = 0$ in Proposition 4.2 and apply the open mapping theorem for the Fréchet spaces. □

### 4.2. The Global Complex Associated by the Local Complexes

Consider the Banach spaces $X$, $Y$, $Z$ and let

$$\mathcal{T} = \sum_{i=0}^{n} e_i \otimes T_i \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{L}(Y, Z), \quad \mathcal{S} = \sum_{i=0}^{n} e_i \otimes S_i \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{L}(X, Y)$$

be (noncommutative) operator-valued polynomials, where $e_0 = 1$. We associate the sequence

$$\mathcal{F}_{\mathfrak{g}}(U)\,\widehat{\otimes} Z \xleftarrow{T_U} \mathcal{F}_{\mathfrak{g}}(U)\,\widehat{\otimes} Y \xleftarrow{S_U} \mathcal{F}_{\mathfrak{g}}(U)\,\widehat{\otimes} X \tag{4.2}$$

with the continuous operators

$$T_U = \sum_{i=0}^{n} R_{e_i} \otimes T_i \quad \text{and} \quad S_U = \sum_{i=0}^{n} R_{e_i} \otimes S_i,$$

where $U$ is an open subset in $\Delta(\mathfrak{g})$, $R_{e_i} \in \mathcal{L}(\mathcal{F}_{\mathfrak{g}}(U))$ is the right multiplication operator. Assume that (4.2) is a chain complex, that is, $T_U S_U = 0$. Note that

$$\begin{aligned} T_U S_U &= \sum_{i=0}^{n} R_{e_i^2} \otimes T_i S_i + \sum_{0\le i<j} R_{e_j e_i} \otimes (T_i S_j + T_j S_i) + \sum_{0<i<j} \sum_{k>j} c_{ij}^k R_{e_k} \otimes T_j S_i \\ &= \sum_{i=0}^{n} R_{e_i^2} \otimes T_i S_i + \sum_{0<i<j} R_{e_j e_i} \otimes (T_i S_j + T_j S_i) + \sum_{j=1}^{m} R_{e_j} \otimes (T_0 S_j + T_j S_0) \\ &\quad + \sum_{k=m+1}^{n} R_{e_k} \otimes \left( T_0 S_k + T_k S_0 + \sum_{0<i<j<k} c_{ij}^k T_j S_i \right). \end{aligned}$$

Thus, the condition $T_U S_U = 0$ is equivalent to the following system of the operator equations:

$$\begin{gathered} T_i S_i = 0, \quad 0 \le i \le n, \\ T_i S_j + T_j S_i = 0, \quad 0 < i < j \le n, \\ T_i S_0 + T_0 S_i = 0, \quad 0 < i \le m, \\ T_k S_0 + T_0 S_k + \textstyle\sum_{1\le i<j<k} c_{ij}^k T_j S_i = 0, \quad m+1 \le k \le n. \end{gathered} \tag{4.3}$$

Indeed, $T_U S_U = 0$ implies that $T_U S_U (1 \otimes x) = 0$ for all $x \in X$.

Now, let

$$\mathcal{T}_{\mathrm{s}} = \sum_{i=0}^{m} z_i \otimes T_i \in \mathcal{L}(Y, Z)[z_1, \dots, z_m], \quad \mathcal{S}_{\mathrm{s}} = \sum_{i=0}^{m} z_i \otimes S_i \in \mathcal{L}(X, Y)[z_1, \dots, z_m],$$

be the operator-valued polynomials, where $z_0 = 1$. Consider the following

$$Z \xleftarrow{\mathcal{T}_{\mathrm{s}}(z)} Y \xleftarrow{\mathcal{S}_{\mathrm{s}}(z)} X \tag{4.4}$$

(polynomially) parametrized on $\Delta(\mathfrak{g})(= \mathbb{C}^m)$ sequence of Banach spaces. Using (4.3), we obtain that

$$\begin{aligned} \mathcal{T}_{\mathrm{s}}(z)\,\mathcal{S}_{\mathrm{s}}(z) &= \sum_{i=0}^{m} z_i T_i \sum_{i=0}^{m} z_i S_i \\ &= \sum_{i=0}^{m} z_i^2 T_i S_i + \sum_{0<i<j} z_i z_j (T_i S_j + T_j S_i) + \sum_{i=1}^{m} z_i (T_i S_0 + T_0 S_i) \\ &= 0 \end{aligned}$$

for each point $z \in \Delta(\mathfrak{g})$. Thus (4.4) is a parametrized chain complex (see [18, Section 2]). We say that the *global complex* (4.2) is associated by the *local complexes* (4.4) by means of the operator-valued polynomials $\mathcal{T}$ and $\mathcal{S}$.

Now, take $\overline{f} \in \mathcal{F}_{\mathfrak{g}}(U) \widehat{\otimes} Y$. Since $\mathcal{F}_{\mathfrak{g}}(U) \widehat{\otimes} Y = \mathcal{O}(U,Y)[[e_{\mathrm{r}}]]$ (see (3.1)), it follows that $\overline{f}$ has a unique expansion $\overline{f} = \sum_{J_{\mathrm{r}}} \overline{f}_{J_{\mathrm{r}}} e_{\mathrm{r}}^{J_{\mathrm{r}}}$ as formal power series with $\overline{f}_{J_{\mathrm{r}}} \in \mathcal{O}(U,Y)$. Fix a countable polydisk cover $\{D_i\}$ of the domain $U$, and let

$$\mathcal{F}_{U,Y}(v) = \{\overline{g}v = \{\overline{g}|_{D_i} v\}_i : \overline{g} \in \mathcal{O}(U,Y)\}$$

be a closed subspace in $\mathcal{F}_{\mathfrak{g}}(U) \widehat{\otimes} Y$, $v \in \mathfrak{r}_e$. Confirm again that $\mathfrak{r}_e$ is the linearly ordered countable set (Section 2) of all ordered radical monomials taken by the basis $e$. Then $\mathcal{F}_{\mathfrak{g}}(U) \widehat{\otimes} Y = \prod_{v \in \mathfrak{r}_e} \mathcal{F}_{U,Y}(v)$ (see (3.1)). Moreover, the linear mapping

$$T_U : \prod_{v \in \mathfrak{r}_e} \mathcal{F}_{U,Y}(v) \to \prod_{v \in \mathfrak{r}_e} \mathcal{F}_{U,Z}(v)$$

is represented by its operator matrix $\left(T_U^{(wv)}\right)$, where $T_U^{(wv)} = \pi_{Z,w} T_U \iota_{Y,v} \in \mathcal{L}(\mathcal{F}_{U,Y}(v), \mathcal{F}_{U,Z}(w))$, $\iota_{Y,v} : \mathcal{F}_{U,Y}(v) \to \mathcal{F}_{\mathfrak{g}}(U) \widehat{\otimes} Y$ is the canonical embedding and $\pi_{Z,w} : \mathcal{F}_{\mathfrak{g}}(U) \widehat{\otimes} Z \to \mathcal{F}_{U,Z}(w)$ is the canonical projection (see Subsection 4.1). The following lemma asserts that $T_U$ is a triangular operator.

**Lemma 4.4.** *For each radical monomial $v \in \mathfrak{r}_e$, $T_U^{(wv)} = 0$ for all $w$, $w \prec v$. Moreover, $T_U^{(wv)} = 0$ for all $w$, $w \succeq v$, except finitely many of them, and the diagonal operator $T_U^{(v)} = T_U^{(vv)}$ acts by the rule $T_U^{(v)}\left(\overline{f}v\right) = \Delta_U\left(\overline{f}\right)v$, where*

$$\Delta_U : \mathcal{O}(U,Y) \to \mathcal{O}(U,Z), \quad \Delta_U\left(\overline{f}\right)(z) = \mathcal{T}_{\mathrm{s}}(z)\overline{f}(z),$$

*is a continuous linear mapping. In particular, the matrix $\left(T_U^{(wv)}\right)$ of $T_U$ is lower triangular.*

*Proof.* First, we reduce the situation to the polydisk case. Take a simple function $f \otimes y \in \mathcal{O}(U,Y)$ with $f \in \mathcal{O}(U)$, $y \in Y$. Then $(f \otimes y)v = fv \otimes y \in \mathcal{F}_{U,Y}(v)$ and

$$\begin{aligned} T_U(fv \otimes y) &= \sum_{k=0}^{n} fv * e_k \otimes T_k(y) = \sum_{k=0}^{n} \{f|_{D_i} v * e_k\}_i \otimes T_k(y) \\ &= \left\{\sum_{k=0}^{n} f|_{D_i} v * e_k \otimes T_k(y)\right\}_i = \{T_{D_i}(f|_{D_i} v \otimes y)\}_i, \end{aligned}$$

so, the family $\{T_{D_i}(f|_{D_i} v \otimes y)\}_i$ is compatible. It follows that $\left\{T_{D_i}\left(\overline{f}|_{D_i} v\right)\right\}_i$ is a compatible family and $T_U\left(\overline{f}v\right) = \left\{T_{D_i}\left(\overline{f}|_{D_i} v\right)\right\}_i$ for each absolutely convergent series $\overline{f} = \sum_k f_k \otimes y_k \in \mathcal{O}(U,Y)$. Indeed,

$$\begin{aligned} T_U\left(\overline{f}v\right) &= \sum_k T_U\left(f_k v \otimes y_k\right) = \sum_k \left\{T_{D_i}\left(f_k|_{D_i} v \otimes y_k\right)\right\}_i \\ &= \left\{\sum_k T_{D_i}\left(f_k|_{D_i} v \otimes y_k\right)\right\}_i = \left\{T_{D_i}\left(\sum_k f_k|_{D_i} v \otimes y_k\right)\right\}_i \\ &= \left\{T_{D_i}\left(\overline{f}|_{D_i} v\right)\right\}_i . \end{aligned}$$

Now fix a polydisk $D_i = D_{a_i,r_i}$, take $\overline{h} \in \mathcal{O}(D_i,Y)(= \mathcal{O}(D_i)\widehat{\otimes}Y)$ and $v \in \mathfrak{r}_e$. For simplicity, we assume that $a_i = 0$, that is, $D_i$ is a polydisk centered at zero. Let $\overline{h} = \sum_{J_s} e_s^{J_s} \otimes y_{J_s}$ be the expansion of $\overline{h}$ in $\mathcal{O}(D_i)\widehat{\otimes}Y \subseteq \mathcal{F}_{\mathfrak{g}}(D_i,Y)$. Then $\overline{h}v = \sum_{J_s} e_s^{J_s} v \otimes y_{J_s} \in \mathcal{F}_{D_i,Y}(v)$, and

$$\Delta_{D_i}\left(\overline{h}\right)v = \sum_{k=0}^{m}\sum_{J_s} \Delta_{e_k}\left(e_s^{J_s}\right) v \otimes T_k\left(y_{J_s}\right),$$

where $\Delta_{e_k}$ is the insertion operator introduced in Section 2. Moreover, the range of the continuous linear mapping

$$\begin{aligned} &A_{D_i} : \mathcal{F}_{D_i,Y}(v) \to \mathcal{F}_{\mathfrak{g}}(D_i)\widehat{\otimes}Z, \\ &A_{D_i}\left(\overline{h}v\right) = \sum_{k=m+1}^{n}\sum_{J_s} e_s^{J_s} v e_k \otimes T_k\left(y_{J_s}\right) + \sum_{k=0}^{m}\sum_{J_s} e_s^{J_s}\left[v,e_k\right] \otimes T_k\left(y_{J_s}\right), \end{aligned}$$

belongs to the finite sum $\sum_{w \succ v} \mathcal{F}_{D_i,Z}(w)$ ($ve_k$ and $[v,e_k]$ are of linear combinations of some ordered radical monomials $w$, $w \succ v$), that is, $\pi_{Z,w}A_{D_i} = 0$ for all $w$ except finitely many of them. Further, one can easily check that

$$\begin{aligned} T_{D_i}\left(\overline{h}v\right) &= \sum_{k=0}^{n}\sum_{J_s} e_s^{J_s} v e_k \otimes T_k\left(y_{J_s}\right) = \Delta_{D_i}\left(\overline{h}\right)v + A_{D_i}\left(\overline{h}v\right) \\ &\quad + \sum_{k=0}^{m}\sum_{J_s} e_1^{j_1}\cdots e_k^{j_k}\left[e_{k+1}^{j_{k+1}}\cdots e_m^{j_m}, e_k\right] v \otimes T_k\left(y_{J_s}\right). \end{aligned}$$

By using Lemma 2.1, we infer

$$\left[e_{k+1}^{j_{k+1}}\cdots e_m^{j_m}, e_k\right] = \sum_{t=m+1}^{n} p_{j_{k+1}\ldots j_m,t}\left(e_{k+1},\ldots,e_m\right)e_t.$$

We set

$$G_{D_i}\left(\overline{h}v\right) = T_{D_i}\left(\overline{h}v\right) - \Delta_{D_i}\left(\overline{h}\right)v - A_{D_i}\left(\overline{h}v\right).$$

Then $G_{D_i} : \mathcal{F}_{D_i,Y}(v) \to \mathcal{F}_{\mathfrak{g}}(D_i)\widehat{\otimes}Z$ is a continuous linear mapping. Since each $e_t v$ ($t > m$) is a finite sum of some radical monomials $w \in \mathfrak{r}_e$, $w \succ v$, it follows that $\pi_{Z,w}G_{D_i} = 0$ for all $w$, $w \preceq v$, and $\pi_{Z,w}G_{D_i} = 0$ for all $w$, $w \succ v$, except finitely many of them. Moreover, $G_{D_i}(\overline{h}v) = \sum_{t=m+1}^{n} \overline{h}_t e_t v$, where

$$\overline{h}_t = \sum_{k=0}^{m} \sum_{J_s} e_1^{j_1} \cdots e_k^{j_k} p_{j_{k+1}\ldots j_m,s}(e_{k+1}, \ldots, e_m) \otimes T_k(y_{J_s}).$$

Thus,

$$T_{D_i}(\overline{h}v) = \Delta_{D_i}(\overline{h})v + A_{D_i}(\overline{h}v) + G_{D_i}(\overline{h}v) = \Delta_{D_i}(\overline{h})v + \sum_{w \in M(v)} \Delta_{D_i}^{(w)}(\overline{h})w,$$

where $\Delta_{D_i}^{(w)} \in \mathcal{L}(\mathcal{O}(D_i, Y), \mathcal{O}(D_i, Z))$ and $M(v)$ is a finite subset in $\mathfrak{r}_e$ comprising some $w$, $w \succ v$.

Let us assume that $\overline{h}|_{D_i \cap D_j} = \overline{g}|_{D_i \cap D_j}$ for some $\overline{h} \in \mathcal{O}(D_i, Y)$, $\overline{g} \in \mathcal{O}(D_j, Y)$. Then we have

$$\begin{aligned}\Delta_{D_i}(\overline{h})|_{D_i \cap D_j} v + \sum_{w \in M(v)} \Delta_{D_i}^{(w)}(\overline{h})|_{D_i \cap D_j} w &= T_{D_i}(\overline{h}v)|_{D_i \cap D_j} \\ = T_{D_j}(\overline{g}v)|_{D_i \cap D_j} = \Delta_{D_j}(\overline{g})|_{D_i \cap D_j} v &+ \sum_{w \in M(v)} \Delta_{D_j}^{(w)}(\overline{g})|_{D_i \cap D_j} w.\end{aligned}$$

It follows that $\Delta_{D_i}^{(w)}(\overline{h})|_{D_i \cap D_j} = \Delta_{D_j}^{(w)}(\overline{g})|_{D_i \cap D_j}$ for all $w \in M(v)$. Thus

$$T_U(\overline{f}v) = T_U^{(v)}(\overline{f}v) + \sum_{w \in M(v)} T_U^{(wv)}(\overline{f}v),$$

where $\overline{f}v \in \mathcal{F}_{U,Y}(v)$, and

$$T_U^{(v)}(\overline{f}v) = \left\{\Delta_{D_i}(\overline{f}|_{D_i})v\right\}_i = \Delta_U(\overline{f})v, \quad T_U^{(wv)}(\overline{f}v) = \left\{\Delta_{D_i}^{(w)}(\overline{f}|_{D_i})w\right\}_i.$$

Therefore, $\pi_{Z,w} T \iota_{Y,v} = 0$ for all $w$, $w \prec v$, and $\pi_{Z,w} T \iota_{Y,v} = 0$ for all $w$, $w \succeq v$, except finitely many of them. Thereby, $T$ is a triangular operator with its diagonal operators $T_U^{(v)}$, $v \in \mathfrak{r}_e$. □

**Theorem 4.5.** *Let $U$ be a (pseudo)convex domain in $\Delta(\mathfrak{g})$. If the local complexes* (4.4) *are exact for all $z \in U$ then so is the global complex* (4.2).

*Proof.* First, note that (4.4) is an analytically parametrized on the domain $U$ Banach space complex (see [18]). Let

$$\mathcal{O}(U, Z) \xleftarrow{T_s} \mathcal{O}(U, Y) \xleftarrow{S_s} \mathcal{O}(U, X) \tag{4.5}$$

be a sequence associated with (4.4), where

$$T_s = \sum_{i=0}^{m} R_{z_i} \otimes T_i \quad \text{and} \quad S_s = \sum_{i=0}^{m} R_{z_i} \otimes S_i.$$

Since

$$T_s(\overline{f})(z) = \mathcal{T}_s(z)\overline{f}(z) \quad \text{and} \quad S_s(\overline{g})(z) = \mathcal{S}_s(z)\overline{g}(z) \qquad \text{for all } z \in U,$$

from (4.3) it follows that the sequence (4.5) is a chain complex, i.e., $T_s S_s = 0$. Moreover, (4.5) turns out to be an exact sequence whenever the local complexes (4.4) are exact for all $z \in U$ thanks to Taylor's theorem on analytically parametrized complexes (see [11, Corollary 2.1.9], [18, Theorem 2.2]). By Lemma 4.4, the operators

$$T : \mathcal{F}_{\mathfrak{g}}(U)\widehat{\otimes}Y \to \mathcal{F}_{\mathfrak{g}}(U)\widehat{\otimes}Z \quad \text{and} \quad S : \mathcal{F}_{\mathfrak{g}}(U)\widehat{\otimes}X \to \mathcal{F}_{\mathfrak{g}}(U)\widehat{\otimes}Y$$

have lower triangular operator matrices with their diagonal operators $T_v\left(\overline{f}v\right) = T_{\mathrm{s}}\left(\overline{f}\right)v$ and $S_v(\overline{g}v) = S_{\mathrm{s}}(\overline{g})v$ respectively, where $\overline{f} \in \mathcal{O}(U,Y)$, $\overline{g} \in \mathcal{O}(U,X)$, $v \in \mathfrak{r}_e$. Using Proposition 4.2, we conclude that (4.2) is an exact chain complex. □

## 5. Taylor Spectrum and Transversality

In this section we prove the main result of this paper: the resolvent set with respect to Taylor spectrum of a supernilpotent Lie algebra of operators can be described in terms of transversality of the Fréchet algebras of formally radical functions in elements of a nilpotent Lie algebra.

First, we need some basic definitions of topological homology.

### 5.1. Resolutions, Transversality and Taylor Spectrum

Let $A$ be a Fréchet algebra. The projective tensor product (over $A$) of Fréchet modules $X \in$ mod-$A$ and $Y \in A$-mod is denoted by $X\underset{A}{\widehat{\otimes}}Y$. By definition, $X\underset{A}{\widehat{\otimes}}Y$ is the quotient space of $X\widehat{\otimes}Y$ with respect to the closed subspace generated by the elements $x \cdot a \otimes y - x \otimes a \cdot y$, $x \in X$, $y \in Y$, $a \in A$. A module $X \in A$-mod is said to be *a free $A$-module* if $X = A\widehat{\otimes}E$ for a certain Fréchet space $E$. The left module structure on $A\widehat{\otimes}E$ is given by the rule: $a \cdot (b \otimes e) = ab \otimes e$, $a, b \in A$, $e \in E$. A module $X \in A$-mod is said to be *a projective $A$-module* if it is a module summand of a certain free $A$-module. A chain complex

$$(\mathcal{X}, d) : \cdots \longleftarrow X_{n-1} \overset{d_{n-1}}{\longleftarrow} X_n \overset{d_n}{\longleftarrow} X_{n+1} \longleftarrow \cdots$$

in the category $A$-mod is said to be *admissible* if it splits as a complex of Fréchet spaces. *A projective resolution* of an $A$-module $X$ is a complex $(\mathcal{P}, d)$ of left $A$-modules with $\mathcal{P}_n = \{0\}$ for $n < 0$, together with a morphism $\epsilon : \mathcal{P}_0 \to X$ such that the augmented complex

$$0 \leftarrow X \overset{\epsilon}{\longleftarrow} \mathcal{P}_0 \overset{d_0}{\longleftarrow} \mathcal{P}_1 \overset{d_1}{\longleftarrow} \cdots$$

is admissible, and all $\mathcal{P}_n$ are projective modules. If $F : A$-mod $\to B$-mod is an additive functor then by $F_n$ we denote the $n$-th projective derived functor of $F$, where $B$ is a Fréchet algebra. By its very definition, $F_n(X)$ is just the $n$-th homology of the complex $(F(\mathcal{P}), F(d))$ for a projective resolution $(\mathcal{P}, d)$ of the module $X$. Taking into account that all projective resolutions of a module are homotopy equivalent (see [14, 3.2.3]), we conclude that $F_n(X)$ does not depend on the particular choice of a projective resolution $(\mathcal{P}, d)$ of $X$. If $F = X\underset{A}{\widehat{\otimes}}\circ$, then we write $\mathrm{Tor}_n^A(X, \circ)$ instead of the $n$-th projective derived functor, as usual. We set $F \perp X$ if $F_n(X) = \{0\}$, $n \geq 0$. If $F = X \underset{A}{\widehat{\otimes}} \circ$ then we write $X \perp_A Y$ (see [20]) for $Y \in A$-mod if $F \perp Y$. In this case we say that the modules $X$ and $Y$ are in *the transversality relation.* Note that $X \perp_A Y$ iff $(\circ \underset{A}{\widehat{\otimes}} Y) \perp X$ (see [14, 3.4.26]).

Now, let $\mathfrak{g}$ be a finite-dimensional Lie algebra, $X$ a Fréchet space and let $\alpha : \mathfrak{g} \to \mathcal{L}(X)$ be a Lie representation, that is, $X$ is a Fréchet $\mathfrak{g}$-module. The following complex

$$0 \leftarrow X \xleftarrow{d_0} X \otimes \mathfrak{g} \xleftarrow{d_1} \cdots \xleftarrow{d_{p-1}} X \otimes \wedge^p \mathfrak{g} \xleftarrow{d_p} \cdots,$$

is called the *Koszul complex* of the pair $(X, \alpha)$, where

$$d_{p-1}(x \otimes \underline{u}) = \sum_{i=1}^{p} (-1)^{i+1} \alpha(u_i) x \otimes \underline{u}_i + \sum_{i<j} (-1)^{i+j-1} x \otimes [u_i, u_j] \wedge \underline{u}_{ij},$$

$\underline{u} = u_1 \wedge \ldots \wedge u_p \in \wedge^p \mathfrak{g}$, and it is denoted by $\mathrm{Kos}(X, \alpha)$. Obviously, $\alpha - \lambda : \mathfrak{g} \to \mathcal{L}(X)$ is a Lie representation for each $\lambda \in \Delta(\mathfrak{g})$. Recall that the *Taylor spectrum* $\sigma(\mathfrak{g}, X)$ of a $\mathfrak{g}$-module $X$ is defined as a set of those $\lambda \in \Delta(\mathfrak{g})$ such that the Koszul complex $\mathrm{Kos}(X, \alpha - \lambda)$ fails to be exact (see [6], [12]).

Let $U$ be a domain in $\Delta(\mathfrak{g})$ and let $\alpha : \mathfrak{g} \to \mathcal{L}(X)$ be a Lie representation of $\mathfrak{g}$ in a Banach space $X$. Then $\mathcal{F}_{\mathfrak{g}}(U) \widehat{\otimes} X$ turns out to be a $\mathfrak{g}$-module via the representation

$$\rho_{U,X} : \mathfrak{g} \to \mathcal{L}\left(\mathcal{F}_{\mathfrak{g}}(U) \widehat{\otimes} X\right), \quad \rho_{U,X}(u)(f \otimes x) = f \otimes \alpha(u) x - f * u \otimes x,$$

$f \in \mathcal{F}_{\mathfrak{g}}(U), x \in X, u \in \mathfrak{g}$, that is, $\rho_{U,X}(u) = 1 \otimes L_{\alpha(u)} - R_u \otimes 1$, $u \in \mathfrak{g}$. Consider the "global" Koszul complex

$$\mathrm{Kos}\left(\mathcal{F}_{\mathfrak{g}}(U) \widehat{\otimes} X, \rho_{U,X}\right) : 0 \leftarrow \mathcal{F}_{\mathfrak{g}}(U) \widehat{\otimes} X \xleftarrow{T_U} \mathcal{F}_{\mathfrak{g}}(U) \widehat{\otimes} X \otimes \mathfrak{g} \xleftarrow{T_U} \cdots$$

with the differential

$$\begin{aligned} T_U(f \otimes x \otimes \underline{u}) = & \sum_{i=1}^{k+1} (-1)^{i+1} \rho_{U,X}(u_i)(f \otimes x) \otimes \underline{u}_i \\ & + \sum_{i<j} (-1)^{i+j-1} f \otimes x \otimes [u_i, u_j] \wedge \underline{u}_{i,j}, \end{aligned}$$

where $\underline{u} = u_1 \wedge \ldots \wedge u_{k+1}$. One can easily check that

$$T_U(f \otimes x \otimes \underline{u}) = f \otimes T_0(x \otimes \underline{u}) + \sum_{i=1}^{k+1} (-1)^i f * u_i \otimes x \otimes \underline{u}_i, \tag{5.1}$$

where $T_0$ is the differential of the complex $\mathrm{Kos}(X, \alpha)$. Fix the graded basis $e = (e_1, \ldots, e_n)$ and introduce the operators $T_i \in \mathcal{L}(X \otimes \wedge \mathfrak{g})$,

$$T_i\left(x \otimes e_{j_1} \wedge \ldots \wedge e_{j_{k+1}}\right) = \begin{cases} (-1)^p x \otimes e_{j_1} \wedge \ldots \wedge \widehat{e_{j_p}} \wedge \ldots \wedge e_{j_{k+1}} & \text{if } j_p = i, \\ 0 & \text{if } j_p \neq i \text{ for all } p, \end{cases}$$

where $1 \leq i \leq n$. Then (5.1) can be rewritten as

$$T_U\left(f \otimes x \otimes \underline{e}\right) = f \otimes T_0\left(x \otimes \underline{e}\right) + \sum_{i=1}^{n} f * e_i \otimes T_i\left(x \otimes \underline{e}\right),$$

that is, the differential $T_U \in \mathcal{L}\left(\mathcal{F}_{\mathfrak{g}}\left(U\right) \widehat{\otimes} \left(X \otimes \wedge \mathfrak{g}\right)\right)$ has the following description

$$T_U = \sum_{i=0}^{n} R_{e_i} \otimes T_i$$

(see (4.2)). Now, we introduce the "local" parametrized Koszul complexes

$$\operatorname{Kos}\left(X, \alpha - z\right), \qquad z \in U.$$

The differential of the complex $\operatorname{Kos}\left(X, \alpha - z\right)$ is

$$\mathcal{T}_{\mathrm{s}}\left(z\right) = \sum_{i=0}^{m} z_i T_i$$

$(z_0 = 1)$ which is the value of the polynomial

$$\mathcal{T}_{\mathrm{s}} = \sum_{i=0}^{m} z_i \otimes T_i \in \mathcal{L}\left(X \otimes \wedge \mathfrak{g}\right)\left[z_1, \ldots, z_m\right]$$

at the point $z$. Thus the global complex $\operatorname{Kos}\left(\mathcal{F}_{\mathfrak{g}}\left(U\right) \widehat{\otimes} X, \rho_{U,X}\right)$ is associated with the local complexes $\operatorname{Kos}\left(X, \alpha - z\right)$, $z \in U$, by means of the noncommutative operator-valued polynomials

$$\mathcal{T} = \sum_{i=0}^{n} e_i \otimes T_i \in \mathcal{U}\left(\mathfrak{g}\right) \otimes \mathcal{L}\left(X \otimes \wedge \mathfrak{g}\right)$$

(see Subsection 4.2).

Now, we prove the main result of this paper: the Taylor spectrum of a left Banach $\mathcal{F}_{\mathfrak{g}}$-module can be completely determined in terms of the transversality.

**Theorem 5.1.** *Let $U$ be a (pseudo)convex domain in $\Delta\left(\mathfrak{g}\right)$ and let $X$ be a Banach $\mathfrak{g}$-module. If the local complexes* $\operatorname{Kos}\left(X, \alpha - z\right)$, *$z \in U$, are exact then so is the global Koszul complex* $\operatorname{Kos}\left(\mathcal{F}_{\mathfrak{g}}\left(U\right) \widehat{\otimes} X, \rho_{U,X}\right)$. *Moreover, if $X \in \mathcal{F}_{\mathfrak{g}}$*-mod *is a Banach module then*

$$\mathcal{F}_{\mathfrak{g}}\left(U\right) \perp_{\mathcal{F}_{\mathfrak{g}}} X \iff U \cap \sigma\left(\mathfrak{g}, X\right) = \emptyset$$

*whenever $U$ is a polydisk in $\Delta\left(\mathfrak{g}\right)$.*

*Proof.* The exactness of all local complexes $\operatorname{Kos}\left(X, \alpha - z\right)$, $z \in U$, imply the exactness of the global complex $\operatorname{Kos}\left(\mathcal{F}_{\mathfrak{g}}\left(U\right) \widehat{\otimes} X, \rho_{U,X}\right)$ thanks to Theorem 4.5. Now, assume that $U = D_{a,r}$ is a polydisk. For brevity, assume that $a = 0$, and put $D = D_{a,r}$. The complex $\operatorname{Kos}\left(\mathcal{F}_{\mathfrak{g}}\left(D\right) \widehat{\otimes} \mathcal{F}_{\mathfrak{g}}\left(D\right), \rho_D\right)$ augmented by the multiplication mapping

$$\pi_D : \mathcal{F}_{\mathfrak{g}}\left(D\right) \widehat{\otimes} \mathcal{F}_{\mathfrak{g}}\left(D\right) \rightarrow \mathcal{F}_{\mathfrak{g}}\left(D\right), \quad f \otimes g \mapsto fg,$$

is a free $\mathcal{F}_{\mathfrak{g}}(D)$-bimodule resolution of the Fréchet algebra $\mathcal{F}_{\mathfrak{g}}(D)$ (see [5], [8]), where $\rho_D = \rho_{D,\mathcal{F}_{\mathfrak{g}}(D)}$. Namely, the chain complex

$$0 \leftarrow \mathcal{F}_{\mathfrak{g}}(D) \overset{\pi_D}{\longleftarrow} \operatorname{Kos}\left(\mathcal{F}_{\mathfrak{g}}(D) \widehat{\otimes} \mathcal{F}_{\mathfrak{g}}(D), \rho_D\right) \tag{5.2}$$

is admissible. Since all the members of the complex are free right $\mathcal{F}_{\mathfrak{g}}(D)$-modules, it follows that the complex splits as a complex in mod-$\mathcal{F}_{\mathfrak{g}}(D)$. In particular, so is the complex $0 \leftarrow \mathcal{F}_{\mathfrak{g}} \overset{\pi_{\mathbb{C}^m}}{\longleftarrow} \operatorname{Kos}\left(\mathcal{F}_{\mathfrak{g}} \widehat{\otimes} \mathcal{F}_{\mathfrak{g}}, \rho_{\mathbb{C}^m}\right)$ in mod-$\mathcal{F}_{\mathfrak{g}}$ (just put $D = \mathbb{C}^m$). Applying the functor $\circ \underset{\mathcal{F}_{\mathfrak{g}}}{\widehat{\otimes}} X$ to the latter complex, we obtain that the complex

$$0 \leftarrow X \overset{\pi_X}{\longleftarrow} \operatorname{Kos}\left(\mathcal{F}_{\mathfrak{g}} \widehat{\otimes} X, \rho_{\mathbb{C}^m,X}\right) \tag{5.3}$$

is admissible, that is, $X$ has the free Koszul resolution in $\mathcal{F}_{\mathfrak{g}}$-mod. With $\mathcal{F}_{\mathfrak{g}} \subseteq \mathcal{F}_{\mathfrak{g}}(D)$ in mind, infer $\mathcal{F}_{\mathfrak{g}}(D) \in \mathcal{F}_{\mathfrak{g}}$-mod-$\mathcal{F}_{\mathfrak{g}}$. Based upon the resolution (5.3), we infer that $\mathcal{F}_{\mathfrak{g}}(D) \perp_{\mathcal{F}_{\mathfrak{g}}} X$ iff the complex $\mathcal{F}_{\mathfrak{g}}(D) \underset{\mathcal{F}_{\mathfrak{g}}}{\widehat{\otimes}} \operatorname{Kos}\left(\mathcal{F}_{\mathfrak{g}} \widehat{\otimes} X, \rho_{\mathbb{C}^m,X}\right)$ is exact. Evidently,

$$\mathcal{F}_{\mathfrak{g}}(D) \underset{\mathcal{F}_{\mathfrak{g}}}{\widehat{\otimes}} \operatorname{Kos}\left(\mathcal{F}_{\mathfrak{g}} \widehat{\otimes} X, \rho_{\mathbb{C}^m,X}\right) = \operatorname{Kos}\left(\mathcal{F}_{\mathfrak{g}}(D) \widehat{\otimes} X, \rho_{D,X}\right)$$

to within a topological isomorphism. Consequently, if $D \cap \sigma(\mathfrak{g}, X) = \emptyset$ then $\operatorname{Kos}(X, \alpha - z)$ is exact for all $z \in D$, which in turn implies that the complex $\operatorname{Kos}\left(\mathcal{F}_{\mathfrak{g}}(D) \widehat{\otimes} X, \rho_{D,X}\right)$ remains exact (Theorem 4.5), that is, $\mathcal{F}_{\mathfrak{g}}(D) \perp_{\mathcal{F}_{\mathfrak{g}}} X$. Conversely, assume that $\mathcal{F}_{\mathfrak{g}}(D) \perp_{\mathcal{F}_{\mathfrak{g}}} X$ and take a point $z \in D$. As we have confirmed in Section 3 the point $z$ determines a continuous character $z : \mathcal{F}_{\mathfrak{g}}(D) \to \mathbb{C}$, that is, $\mathcal{F}_{\mathfrak{g}}(D)$ is an augmented algebra. Consider the trivial $\mathcal{F}_{\mathfrak{g}}(D)$-module $\mathbb{C}(z)$ generated by $z$ (see Section 2). Since the complex (5.2) is admissible as a complex in $\mathcal{F}_{\mathfrak{g}}(D)$-mod, it follows that the complex

$$0 \leftarrow \mathbb{C}(z) \underset{\mathcal{F}_{\mathfrak{g}}(D)}{\widehat{\otimes}} \mathcal{F}_{\mathfrak{g}}(D) \longleftarrow \mathbb{C}(z) \underset{\mathcal{F}_{\mathfrak{g}}(D)}{\widehat{\otimes}} \operatorname{Kos}\left(\mathcal{F}_{\mathfrak{g}}(D) \widehat{\otimes} \mathcal{F}_{\mathfrak{g}}(D), \rho_D\right)$$

is admissible. Taking into account that $\mathbb{C}(z) \widehat{\otimes} \mathcal{F}_{\mathfrak{g}}(D)$ is identified with $\mathcal{F}_{\mathfrak{g}}(D)$, we deduce that the latter complex is the following

$$0 \leftarrow \mathbb{C}(z) \overset{\pi_{\mathbb{C}(z)}}{\longleftarrow} \operatorname{Kos}\left(\mathcal{F}_{\mathfrak{g}}(D), \rho_{\mathbb{C}(z),D}\right)$$

with $\pi_{\mathbb{C}(z)}(f) = z(f)$ and $\rho_{\mathbb{C}(z),D}(u)(f) = u * f - fz(u) = (u - z(u)) * f$. Thus $\operatorname{Kos}\left(\mathcal{F}_{\mathfrak{g}}(D), \rho_{\mathbb{C}(z),D}\right)$ is just the Koszul complex $\operatorname{Kos}(\mathcal{F}_{\mathfrak{g}}(D), L_{\mathfrak{g}} - z)$ associated by the left multiplication representation $L_{\mathfrak{g}}$ and the character $z$ on $\mathcal{F}_{\mathfrak{g}}(D)$, and

$$0 \leftarrow \mathbb{C}(z) \overset{z}{\longleftarrow} \operatorname{Kos}(\mathcal{F}_{\mathfrak{g}}(D), L_{\mathfrak{g}} - z)$$

is an admissible complex of the right $\mathcal{F}_{\mathfrak{g}}$-modules. Each member $\mathcal{F}_{\mathfrak{g}}(D) \otimes \wedge^k \mathfrak{g}$ of the complex $\operatorname{Kos}(\mathcal{F}_{\mathfrak{g}}(D), L_{\mathfrak{g}})$ is a topological direct sum of $\mathcal{F}_{\mathfrak{g}}(D)$ copies. Being $\circ \underset{\mathcal{F}_{\mathfrak{g}}}{\widehat{\otimes}} X$ an additive functor, we deduce that $(\circ \underset{\mathcal{F}_{\mathfrak{g}}}{\widehat{\otimes}} X) \perp \left(\mathcal{F}_{\mathfrak{g}}(D) \otimes \wedge^k \mathfrak{g}\right)$ for all $k$ whenever $\mathcal{F}_{\mathfrak{g}}(D) \perp_{\mathcal{F}_{\mathfrak{g}}} X$ (that is, $(\circ \underset{\mathcal{F}_{\mathfrak{g}}}{\widehat{\otimes}} X) \perp \mathcal{F}_{\mathfrak{g}}(D)$). Then $(\circ \underset{\mathcal{F}_{\mathfrak{g}}}{\widehat{\otimes}} X) \perp \mathbb{C}(z)$ thanks to [14, 3.3.8]. Consequently, $\mathbb{C}(z) \perp_{\mathcal{F}_{\mathfrak{g}}} X$, where $\mathbb{C}(z)$ is considered to be right

$\mathcal{F}_{\mathfrak{g}}$-module. Thus $\operatorname{Tor}_n^{\mathcal{F}_{\mathfrak{g}}}(\mathbb{C}(z), X) = \{0\}$ for all $k \geq 0$.
Using again the same argument for the polydisk $D = \mathbb{C}^n$, we obtain that the complex

$$0 \leftarrow \mathbb{C}(z) \overset{z}{\longleftarrow} \operatorname{Kos}(\mathcal{F}_{\mathfrak{g}}, L_{\mathfrak{g}} - z)$$

of the right $\mathcal{F}_{\mathfrak{g}}$-modules is admissible, that is, $\operatorname{Kos}(\mathcal{F}_{\mathfrak{g}}, L_{\mathfrak{g}} - z)$ is a free resolution of the right $\mathcal{F}_{\mathfrak{g}}$-module $\mathbb{C}(z)$. It follows that $\operatorname{Tor}_n^{\mathcal{F}_{\mathfrak{g}}}(\mathbb{C}(z), X)$, $k \geq 0$, are the homologies of the complex $\operatorname{Kos}(\mathcal{F}_{\mathfrak{g}}, L_{\mathfrak{g}} - z) \underset{\mathcal{F}_{\mathfrak{g}}}{\widehat{\otimes}} X$. But an easy calculation shows that $\operatorname{Kos}(\mathcal{F}_{\mathfrak{g}}, L_{\mathfrak{g}} - z) \underset{\mathcal{F}_{\mathfrak{g}}}{\widehat{\otimes}} X$ is just the Koszul complex $\operatorname{Kos}(X, \alpha - z)$ of the $\mathfrak{g}$-module $X$. Whence the complex $\operatorname{Kos}(X, \alpha - z)$ is exact, which means that $z \notin \sigma(\mathfrak{g}, X)$. In particular, $D \cap \sigma(\mathfrak{g}, X) = \emptyset$. □

Confirm that each Banach $\mathfrak{g}$-module $X$ automatically turns out to be a left module over the Arens-Michael envelope $\mathcal{O}_{\mathfrak{g}}$, that is, all noncommutative entire functions in elements of $\mathfrak{g}$ act on $X$ (see [15, 5.2.21]). To complete our assertion one might solve the problem when $X$ turns out to be a $\mathcal{F}_{\mathfrak{g}}$-module. Moreover, to have a direct application to the operator theory it is better to formulate the result in terms of the operator tuples rather than in terms of the modules.

Let $X$ be a Banach space and let $T = (T_1, \dots, T_m)$ be a family of bounded linear operators on $X$ generating a finite-dimensional nilpotent Lie subalgebra $\mathfrak{g}_T$ in $\mathcal{L}(X)$. If $\mathfrak{g}_T - \lambda$ is a Lie subalgebra in $\mathcal{L}(X)$ generated by the operator family $T - \lambda = (T_1 - \lambda_1, \dots, T_m - \lambda_m)$, $\lambda_i = \lambda(T_i)$, $1 \leq i \leq m$, $\lambda \in \Delta(\mathfrak{g}_T)$, then the *Taylor spectrum* $\sigma(T)$ *of the operator family* $T$ is defined as a set of those $\lambda$ for which the Koszul complex of the $\mathfrak{g}_T - \lambda$-module $X$ fails to be exact (see [12]). As we have shown in Section 3 the Lie algebra $\mathfrak{g}_T$ is an epimorphic image of a positively graded nilpotent Lie algebra $\mathfrak{g}$ generated by $m$-elements $e_1, \dots, e_m$, that is, there exists a Lie epimorphism $\tau : \mathfrak{g} \to \mathfrak{g}_T$ such that $\tau(e_i) = T_i$, $1 \leq i \leq m$. Therefore the space $X$ turns out to be a left $\mathcal{O}_{\mathfrak{g}}$-module. Take a triangular basis $e$ in $\mathfrak{g}$ generated by $e_1, \dots, e_m$ and consider the Fréchet algebra $\mathcal{F}_{\mathfrak{g}} = \mathcal{F}_{\mathfrak{g}}(\Delta(\mathfrak{g}))$ of all formally-radical entire functions in elements of $\mathfrak{g}$.

**Proposition 5.2.** *The Banach space $X$ turns out to be a left $\mathcal{F}_{\mathfrak{g}}$-module iff $\mathfrak{g}_T$ is a supernilpotent Lie subalgebra in $\mathcal{L}(X)$. Moreover, $\sigma(T) = \sigma(\mathfrak{g}, X)$.*

*Proof.* Let us assume that $X \in \mathcal{F}_{\mathfrak{g}}$-mod. Note that the topology of $\mathcal{F}_{\mathfrak{g}}$ is defined by the system of seminorms $\left\{\|\cdot\|_{t,K_{\mathrm{r}}}\right\}$, where $\|f\|_{t,K_{\mathrm{r}}} = \max\{\|f_{J_{\mathrm{r}}}\|_t : J_{\mathrm{r}} \leq K_{\mathrm{r}}\}$ (see (2.1)), $f = \sum_{J_{\mathrm{r}}} f_{J_{\mathrm{r}}} e_{\mathrm{r}}^{J_{\mathrm{r}}} \in \mathcal{F}_{\mathfrak{g}}$. By assumption, there exists a (unique) continuous algebra homomorphism $\widehat{\tau} : \mathcal{F}_{\mathfrak{g}} \to \mathcal{L}(X)$ extending $\tau$. Thereby

$$\|\widehat{\tau}(f)\|_{\mathcal{L}(X)} \leq C \|f\|_{t,K_{\mathrm{r}}}$$

for a certain positive constant $C$ and some seminorm $\|\cdot\|_{t,K_{\mathrm{r}}}$. It follows that $\left\|\tau(e_{\mathrm{r}})^{I_{\mathrm{r}}}\right\|_{\mathcal{L}(X)} = \left\|\widehat{\tau}\left(e_{\mathrm{r}}^{I_{\mathrm{r}}}\right)\right\|_{\mathcal{L}(X)} = 0$ whenever $|I_{\mathrm{r}}| > |K_{\mathrm{r}}|$. But $\tau(e_{\mathrm{r}})$ generates $[\mathfrak{g}_T, \mathfrak{g}_T]$, therefore $\mathfrak{g}_T$ is a supernilpotent Lie subalgebra in $\mathcal{L}(X)$.

Conversely, let us assume that $\mathfrak{g}_T$ is a supernilpotent Lie subalgebra in $\mathcal{L}(X)$. Let $B$ be an associative subalgebra in $\mathcal{L}(X)$ generated by $[\mathfrak{g}_T, \mathfrak{g}_T]$, and let $\overline{\tau} : \mathcal{O}_{\mathfrak{g}} \to \mathcal{L}(X)$ be the continuous algebra homomorphism extending $\tau$. By Lemma 2.2, $B$ is finite dimensional and $B^k = 0$ for some $k$. Then $\overline{\tau}\left(e_{\mathrm{r}}^{J_{\mathrm{r}}}\right) = 0$ for all $J_{\mathrm{r}}$, $|J_{\mathrm{r}}| > k(n-m)$. We set

$$\widehat{\tau} : \mathcal{F}_{\mathfrak{g}} \to \mathcal{L}(X), \quad \widehat{\tau}\left(\sum_{J_{\mathrm{r}}} f_{J_{\mathrm{r}}} e_{\mathrm{r}}^{J_{\mathrm{r}}}\right) = \sum_{|J_{\mathrm{r}}| \leq k(n-m)} \overline{\tau}\left(f_{J_{\mathrm{r}}} e_{\mathrm{r}}^{J_{\mathrm{r}}}\right),$$

which is a continuous algebra homomorphism. Whence $X$ is a left $\mathcal{F}_{\mathfrak{g}}$-module. Finally, the equality of spectra follows from [12]. □

By using the main result Theorem 5.1 and Proposition 5.2, we derive the following assertion.

**Corollary 5.3.** *Let $T = (T_1, \ldots, T_m)$ be a family of bounded linear operators on a Banach space generating a supernilpotent Lie algebra in $\mathcal{L}(X)$. Then $X$ turns out to be a left Banach $\mathcal{F}_{\mathfrak{g}}$-module for a certain nilpotent Lie algebra $\mathfrak{g}$ and the resolvent set $\mathbb{C}^m \backslash \sigma(T)$ with respect to the Taylor spectrum $\sigma(T)$ consists of those $\lambda \in \mathbb{C}^m$ such that $\mathcal{F}_{\mathfrak{g}}(U) \perp_{\mathcal{F}_{\mathfrak{g}}} X$ for a certain small polydisk $U$ containing $\lambda$.*

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