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Quantum duality, unbounded operators, and inductive limits

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In this paper, we investigate the inductive limits of quantum normed (or operator) spaces. This construction allows us to treat the space of all noncommutative continuous functions over a quantum domain as a quantum (or local operator) space of all matrix continuous linear operators equipped with $\mathfrak{S}$-quantum topology. In particular, we classify all quantizations of the polynormed topologies compatible with the given duality proposing a noncommutative Arens–Mackey theorem. Further, the inductive limits of operator spaces are used to introduce locally compact and locally trace class unbounded operators on a quantum domain and prove the dual realization theorem for an abstract quantum space. © *2010 American Institute of Physics.* [doi:10.1063/1.3419771]

I. INTRODUCTION

The operator analogs of locally convex spaces have been started to develop in Ref. 14 by Effros and Webster. The central goal of this direction is to create a theory of quantum polynormed spaces or quantum spaces, which should reflect the "locally convex space chapters" of quantum functional analysis. This theory has been created as the basic language of quantum physics. To have a comprehensive mathematical model of quantum physics, it is necessary to consider the linear spaces of unbounded Hilbert space operators or "noncommutative variable spaces."[19] This is the mathematical background of Heisenberg's "matrix mechanics." The quantizations of a variable space have provided functional analysis with the new constructions, methods, and problems.[22] Being a new and modern branch of functional analysis, quantum functional analysis can be divided into the normed[13,20] and polynormed (or locally convex) topics,[14,25] as in the classical theory. The quantum (or local operator) spaces appear as the projective limits of quantum normed spaces. The known[21] representation theorem by Ruan asserts that each quantum normed space can be realized as an operator space (up to a matrix isometry) in the space $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space H, whereas quantum spaces lead to linear spaces of unbounded operators on H (see Ref. 6). These representation theorems are based on a quantum (or operator) version of the classical bipolar theorem, which asserts that the double operator polar of an absolutely matrix convex set is reduced to its weak closure. This result was proved in Ref. 14 (Proposition 4.1) by Effros and Webster. In a certain sense the bipolar theorem is equivalent to Ruan's representation theorem for quantum normed spaces (see Ref. 9). The bipolar theorem allows us also to describe a continuous matrix seminorm on a quantum space in terms of the matrix duality.[6] Furthermore, it provides a scale of possible quantizations of a polynormed space, namely, the scale of min and max quantizations. The Krein–Milman theorem for quantum spaces was proposed in Ref. 26 by Webster and Winkler.

In this paper, we investigate the inductive limits of quantum normed spaces. That is, the main technical machinery of the paper which allows to prove the main results, namely, classification of

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all quantum topologies compatible with the given duality, representation theorem for quantum spaces and the dual realization of a quantum dual space equipped with $\mathfrak{S}$-quantum topology. Note that the inductive limits of operator spaces have been successfully used in the quantum moment problems,[7] the quantum operator valued measures are treated as matrix contractive and matrix positive linear mappings between certain inductive limits of operator spaces.

The known (see, for instance, Ref. 23 (Sec. 4.3.2) and Ref. 18 (Sec. 10.4.5)) classical result by Arens and Mackey asserts that all polynormed (or locally convex) topologies $\mathfrak{s}$ compatible with the given duality (V,W) can be arranged into the Arens–Mackey scale $\sigma(V,W)\subseteq\mathfrak{s}\subseteq\tau(V,W)$ within the weak $\sigma(V,W)$ and Mackey $\tau(V,W)$ topologies. Moreover, all bounded sets with respect to the polynormed topologies compatible with the duality (V,W) are the same, thanks to the known (Ref. 18, Sec. 40.4.6) classical result by Mackey. These duality results play the fundamental role in the classical theory of locally convex spaces.

The duality theory for quantum spaces has been developed in Ref. 25 by Webster. There was proposed a $\mathfrak{S}$-quantum dual of a quantum space and proved that a quantum topology on V is compatible with the given duality (V,W) if and only if it is generated by matrix polars of weakly matrix compact sets on W. Let us recall that by *a quantum topology* $\mathfrak{f}$ on a linear space V we mean a polynormed topology in the space $M(V)$ of all finite matrices over V whose neighborhood filter base is generated by a family of absolutely matrix convex sets (see Sec. II D). Certainly, each quantum topology $\mathfrak{f}$ in $M(V)$ inherits a polynormed topology $\mathfrak{s}=\mathfrak{f}|V$ on the linear space V. We are saying that $\mathfrak{f}$ is *a quantization of* $\mathfrak{s}$. Actually, each polynormed topology $\mathfrak{s}$ in V admits a quantization. All these quantum topologies are placed within the min and max quantizations, that is, $\min\mathfrak{s}\subseteq\mathfrak{f}\subseteq\max\mathfrak{s}$ (see Sec. II E). It was proved in Ref. 9 that the weak topology $\sigma(V,W)$ admits precisely one quantization $\mathfrak{s}(V,W)$ called *the weak quantum topology*, that is, all quantum topologies in $M(V)$ reduced to the same $\sigma(V,W)$ are equivalent. In particular, $\min\sigma(V,W)=\max\sigma(V,W)=\mathfrak{s}(V,W)$. Furthermore, a noncommutative Mackey theorem proposed in Ref. 9 asserts that if $\mathfrak{s}$ runs within the classical Arens–Mackey scale $\sigma(V,W)\subseteq\mathfrak{s}\subseteq\tau(V,W)$, then all matrix bounded sets with respect to the quantum topologies $\mathfrak{s}(V,W)\subseteq\min\mathfrak{s}\subseteq\min\tau(V,W)$ are the same. Similar result for the max quantization is not true (see Ref. 9).

One of the central goal of the present paper is to classify all quantum topologies compatible with the given duality (V,W). We represent these quantum topologies as the elements of *a quantum scale on a concrete quantum space*. Namely, fix a set J. Assume that for each point $w\in J$ we have an "atomic" algebra M_{n_w} of all finite n_w-square complex matrices. If $I\subseteq J$ is a subset-then all atomic algebras M_{n_w}, $w\in I$, associate the operator space (von Neumann algebra) $M_I=\oplus_{w\in I}^{\infty}M_{n_w}$-direct sum of the full matrix algebras. Fix a family $J=\{J_\kappa\}$ of sets. The family J associates the quantum space (local von Neumann algebra)

$$\mathfrak{D}_J=\mathrm{op}\prod_\kappa M_{J_\kappa}\subseteq\{\text{unbounded operators}\},$$

which is the quantum (or operator) product of von Neumann algebras. The quantum space $\mathfrak{D}_J$ has a realization as unbounded operators.[6] The algebra $\mathfrak{D}_J$ possesses a family of so-called *divided quantum topologies*, which can easily be described in terms of divisors of J. Namely, a family $I=\{I_\alpha\}$ of sets is said to be *a divisor of* J if for each I_α there corresponds unique J_κ such that $I_\alpha\subseteq J_\kappa$, and $J=\cup I$. For instance, so are

$$A=\{\{w\}:w\in J\}\quad(\textit{atomic divisor})\quad\text{and}\quad J=\{J_\kappa\}\quad(\textit{top divisor})\text{ itself}.$$

Each divisor I of J generates the matrix seminorms

$$\pi_{I_\alpha}(a)=\sup\{\|a_w\|:w\in I_\alpha\},\quad a=(a_w)_{w\in J}\in M(\mathfrak{D}_J).$$

This family $\mathfrak{d}_I=\{\pi_{I_\alpha}\}$ of matrix seminorms defines, in turn, a quantum topology (denoted by $\mathfrak{d}_I$ too) on $\mathfrak{D}_J$ called *the divided quantum topology*. Put $\mathfrak{a}=\mathfrak{d}_A$ and $\mathfrak{t}=\mathfrak{d}_J$. They are so-called *atomic and top quantum topologies* (or *boundaries*) and $\mathfrak{a}\subseteq\mathfrak{d}_I\subseteq\mathfrak{t}$ for each divisor I of J. If $V\subseteq\mathfrak{D}_J$ is a linear subspace, then we have a scale of *divided quantum topologies* $\mathfrak{d}_I|M(V)$ in $M(V)$. Our first central result asserts that the quantizations of the classical Arens–Mackey scale can be realized as a quantum scale on a certain concrete quantum space.

Theorem 1: *(Noncommutative Arens–Mackey theorem) If (V,W) is a dual pair, then V can be identified with a subspace in a certain local von Neumann algebra $\mathfrak{D}_J$ such that the quantization of the classical Arens–Mackey scale $\sigma(V,W)\subseteq\mathfrak{s}\subseteq\tau(V,W)$ is precisely the quantum scale*

$$\mathfrak{a}|M(V)\subseteq\mathfrak{d}_J|M(V)\subseteq\mathfrak{t}|M(V).$$

In particular, the atomic quantum topology $\mathfrak{a}|M(V)$ represents the unique quantization $\mathfrak{s}(V,W)$ of the weak topology $\sigma(V,W)$.

In order to represent the elements of the quantum space $\mathfrak{D}_J$ as unbounded operators, we introduce *quantum domains* simplifying the construction used in Ref. 5. By *a quantum domain in a Hilbert space H* we mean an orthogonal family $\mathfrak{T}=\{N_\kappa\}$ of its closed subspaces whose sum $\mathcal{D}=\Sigma\mathfrak{T}=\Sigma_\kappa N_\kappa$ is dense in H. If all "nest" subspaces N_κ of a quantum domain $\mathfrak{T}$ are finite dimensional, then we say that it is *an atomic quantum domain*. The algebra of all noncommutative continuous functions[6] over a quantum domain $\mathfrak{T}$ is reduced to the unital multinormed C^*-algebra

$$C^*_{\mathfrak{T}}(\mathcal{D})=\{T\in L(\mathcal{D}):T|N_\kappa\in\mathcal{B}(N_\kappa)\text{ for all }\kappa\}$$

with the family $p_\kappa(T)=\|T|N_\kappa\|$ of C^*-seminorms, where $L(\mathcal{D})$ is the associative algebra of all linear transformations on $\mathcal{D}$. The elements of $C^*_{\mathfrak{T}}(\mathcal{D})$ are closable unbounded operators on H, and it possesses a canonical quantum space structure being a multinormed C^*-algebra. Note that $(\mathfrak{D}_J,\mathfrak{t})\subseteq C^*_{\mathfrak{T}}(\mathcal{D})$ is a quantum space inclusion with $\mathfrak{T}=\{N_\kappa\}$, $N_\kappa=\oplus_{w\in J_\kappa}\mathbb{C}^{n_w}$ and $H=\oplus_{w\in J}\mathbb{C}^{n_w}$. Moreover, $(\mathfrak{D}_J,\mathfrak{a})\subseteq C^*_{\mathfrak{A}}(\mathcal{D})$ with the atomic quantum domain $\mathfrak{A}=\{\mathbb{C}^{n_w}:w\in J\}$ in H.

The sum $\mathcal{D}$ of a quantum domain $\mathfrak{T}=\{N_\kappa\}$ can be quantized $\mathcal{D}_q=\mathrm{op}\oplus\mathfrak{T}_q=\mathrm{op}\oplus_\kappa N_{\kappa,q}$ being a quantum direct sum of the quantum normed spaces $N_{\kappa,q}$, where q indicates a quantization (see Sec. III A) over a certain class of normed spaces including $\mathfrak{T}$. The quantum direct sums have advantages to be handled in many technical results, for instance, to classify all matrix bounded sets in $\mathcal{D}_q$ we may prove Dieudonné–Schwartz type theorems on the matrix level $M(\mathcal{D}_q)=\oplus_\kappa M(N_{\kappa,q})$, which is a quantum direct sum of the quantum normed spaces. In Sec. III and IV, we convert the algebra of all noncommutative continuous functions on a quantum domain into the quantum space of all matrix continuous linear operators on a certain inductive limit (see Ref. 13, Theorem 3.4.1, for the normed case).

Theorem 2: *(Representation theorem for quantum spaces) If V is quantum space, then there is a quantum domain $\mathfrak{T}$ in a Hilbert space such that*

$$V\hookrightarrow\mathcal{MC}(\mathcal{D}_c)_\beta\quad(\textit{up to a topological matrix isomorphism}),$$

where $\mathcal{D}_c=\mathrm{op}\oplus\mathfrak{T}_c$ is the column quantization c of the sum $\mathcal{D}=\Sigma\mathfrak{T}$ and $\mathcal{MC}(\mathcal{D}_c)_\beta$ is the quantum space of all matrix continuous linear operators equipped with the strong quantum topology on the quantum space $\mathcal{D}_c$.

Recall that $\mathcal{B}(H)$ is an operator dual of the operator space $\mathcal{T}(H)$ of all trace class operators on H. A quantum space version of $\mathcal{T}(H)$ can be constructed using the inductive limit of operator spaces too. If $\mathfrak{T}=\{N_\kappa\}$ is a quantum domain in H, then we introduce the quantum spaces $\mathcal{K}_{\mathfrak{T}}(\mathcal{D})$ and $\mathcal{T}_{\mathfrak{T}}(\mathcal{D})$ over $\mathcal{D}$ of so-called *locally compact and locally trace class operators on $\mathcal{D}$*. They are $*$-ideals in $C^*_{\mathfrak{T}}(\mathcal{D})$. The quantum space $\mathcal{T}_{\mathfrak{T}}(\mathcal{D})$ of locally trace class operators on $\mathcal{D}$ is defined as the quantum direct sum $\mathcal{T}_{\mathfrak{T}}(\mathcal{D})=\mathrm{op}\oplus_\kappa\mathcal{T}(N_\kappa)$ of the operator subspaces $\mathcal{T}(N_\kappa)\subseteq\mathcal{T}(H)$. We prove that $\mathcal{T}_{\mathfrak{T}}(\mathcal{D})'_\beta=C^*_{\mathfrak{T}}(\mathcal{D})$ up to the canonical topological matrix isomorphism, where $\mathcal{T}_{\mathfrak{T}}(\mathcal{D})'_\beta$ is the strong quantum dual (see Ref. 14) of $\mathcal{T}_{\mathfrak{T}}(\mathcal{D})$. Moreover, $\mathcal{K}_{\mathfrak{T}}(\mathcal{D})'_\beta=\mathcal{T}_{\mathfrak{T}}(\mathcal{D})$ whenever $\mathfrak{T}$ is countable. The identification $C^*_{\mathfrak{T}}(\mathcal{D})=\mathcal{T}_{\mathfrak{T}}(\mathcal{D})'_\beta$ restricted to the local von Neumann algebra $\mathfrak{D}_J$ associates the quantum space isomorphism $\mathfrak{D}_J=(\mathcal{T}_J)'_\beta$, where $\mathcal{T}_J=\mathrm{op}\oplus_\kappa\oplus^1_{w\in J_\kappa}\mathcal{T}_{n_w}$ is the space of all trace class matrices in $\mathfrak{D}_J$. In particular, we have the weak* $\mathfrak{s}(\mathfrak{D}_J,\mathcal{T}_J)$ (see Theorem 1) and strong $\beta(\mathfrak{D}_J,\mathcal{T}_J)$ quantum topologies in $M(\mathfrak{D}_J)$, they are quantizations of the relevant weak* and strong dual topologies. We prove the following *dual realization theorem* for quantum spaces.

Theorem 3: *(Dual realization theorem) If $\mathfrak{a}$ and $\mathfrak{t}$ are the quantum boundaries in $\mathfrak{D}_J$, then*

$\mathfrak{a} \subseteq \mathfrak{s}(\mathfrak{D}_J, \mathcal{T}_J)$ and $\beta(\mathfrak{D}_J, \mathcal{T}_J) = \mathfrak{t}$. Moreover, if V is a complete quantum space, then its $\mathfrak{S}$-quantum dual $V'_{\mathfrak{S}}$ can be identified with a subspace in a certain $\mathfrak{D}_J$ such that

$$\mathfrak{s}(V', V) = \mathfrak{s}(\mathfrak{D}_J, \mathcal{T}_J)|M(V') = \mathfrak{a}|M(V') \quad \textit{and} \quad \mathfrak{S}(V', V) = \mathfrak{t}|M(V'),$$

where $\mathfrak{s}(V', V)$ is the quantum weak topology and $\mathfrak{S}(V', V)$ is the $\mathfrak{S}$-quantum topology in $M(V')$.*

The assertion effectively generalizes the dual realization theorem of an operator space proved by Blecher[2] (see also Refs. 3 and 13).

II. PRELIMINARY RESULTS

In this section we propose key notions and results of the quantum space theory. All central results of the paper[14] by Effros and Webster will be used in the present investigations. Therefore, we explore in detail basic concepts and tools of Ref. 14 to facilitate the reading of the present one.

A. The basic notations

The direct product of complex linear spaces E and F is denoted by $E \times F$ and we put E^k for the k-times product $E \times \cdots \times E$. If E is a linear space, then $\bar{E}$ denotes the *conjugate space* for E. So, $\bar{E} = E$ with its addition and the conjugate scalar multiplication $\lambda \bar{u} = \bar{\lambda} u$, $u \in E$, where $\bar{u}$ indicates the same u from E but in the conjugate space $\bar{E}$. If E is a normed space with the norm $\|\cdot\|$, then so is $\bar{E}$ with the norm $\|\bar{u}\| = \|u\|$, $u \in E$. If H is a Hilbert space with its inner product $\langle \cdot, \cdot \rangle$, then $\bar{H}$ turns out to be a Hilbert space with the inner product $\langle \bar{u}, \bar{v} \rangle = \langle v, u \rangle$, $u, v \in H$. The linear space of all linear transformations between linear spaces E and F is denoted by $L(E,F)$, and we write $L(E)$ instead of $L(E,E)$. The identity operator on E is denoted by I_E. It is the unit of the associative algebra $L(E)$. Take $T \in L(E)$. The n-fold inflation $T^{\oplus n} = T \oplus \cdots \oplus T \in L(E^n)$ of T is acting as $(x_i)_i \mapsto (Tx_i)_i$, $x_i \in E$, $1 \leq i \leq n$. If T leaves invariant a subspace $F \subseteq E$, then $T|F$ denotes the restriction of T onto F. If $A \subseteq E$ is a subset, then abc A denotes the absolutely convex hull of A in the linear space E. The unit set of a normed space V is denoted by ball V. If p is a gauge (or seminorm) on a linear space V, then the unit set $\{p \leq 1\}$ in V is denoted by ball p. The domain of an unbounded operator T on a Hilbert space H is denoted by dom(T). For unbounded operators T and S on H, we write $T \subseteq S$ if dom$(T) \subseteq$ dom(S) and $Tx = Sx$ for all $x \in$ dom(T). If T is a densely defined operator on H, then $T^\star$ denotes its dual operator, thus $\langle Tx, y \rangle = \langle x, T^\star y \rangle$ for all $x \in$ dom(T), $y \in$ dom$(T^\star)$, where $\langle \cdot, \cdot \rangle$ is the inner product in H. The C^*-algebra of all bounded linear operators on H is denoted by $\mathcal{B}(H)$, whose ideals comprising all finite-rank and compact operators are denoted by $\mathcal{F}(H)$ and $\mathcal{K}(H)$, respectively. The space of all trace class operators on H is denoted by $\mathcal{T}(H)$. The trace norm of an operator $A \in \mathcal{T}(H)$ is denoted by $\|A\|_t$, that is, if $|A| = (A^*A)^{1/2}$, then $\|A\|_t = \mathrm{tr}(|A|) = \Sigma_{e \in \Delta} \langle |A| e, e \rangle$ for a (Hilbert) basis Δ in H. Note that if $H = H_1 \oplus H_2$ is an orthogonal sum of Hilbert spaces, $A \in \mathcal{B}(H_1)$, $B \in \mathcal{B}(H_2)$, and $T = A \oplus B \in \mathcal{B}(H)$, then $T \in \mathcal{T}(H)$ if and only if $A \in \mathcal{T}(H_1)$ and $B \in \mathcal{T}(H_2)$. In this case, $\mathrm{tr}(T) = \mathrm{tr}(A) + \mathrm{tr}(B)$.

The linear space of all $m \times n$-matrices $x = [x_{ij}]$ over a linear space V is denoted by $M_{m,n}(V)$, and we set $M_{m,n} = M_{m,n}(\mathbb{C})$ and $M_m(V) = M_{m,m}(V)$. Further, $M(V)$ (M) denotes the linear space of all infinite (scalar) matrices $[x_{ij}]$, $x_{ij} \in V$, where all but finitely many of x_{ij} are zero. Each $M_{m,n}(V)$ is a subspace in $M(V)$ comprising those matrices $x = [x_{ij}]$ with $x_{ij} = 0$ whenever $i > m$ or $j > n$. Note that $M_{m,n}(L(E)) = L(E^n, E^m)$ up to the canonical identification. In particular, $M_n(L(E)) = L(E^n)$. If $E = H$ is a Hilbert space, then $M_n(\mathcal{B}(H)) = \mathcal{B}(H^n)$ is a normed space. In particular, $M_{m,n}$ is the space $M_{m,n}$ endowed with the operator norm $\|\cdot\|$ between the canonical Hilbert spaces $\mathbb{C}^n$ and $\mathbb{C}^m$. In particular, M is a normed space.

Now we introduce the main quantum operations, *the direct sum* and *M-bimodule structure* in the space $M(V)$ of all matrices over V, which plays a basic role in the theory of quantum spaces. Take $v \in M_{s,t}(V)$ and $w \in M_{m,n}(V)$. Their direct sum is defined as

$$v \oplus w = \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} \in M_{s+m,t+n}(V).$$

If $a \in M_{m,s}$, $v \in M_{s,t}(V)$ and $b \in M_{t,n}$, then we have the matrix product

$$avb = \left[\sum_{k,l} a_{ik} v_{kl} b_{lj} \right]_{i,j} \in M_{m,n}(V).$$

A finite sum like $\Sigma_s a_s v_s b_s$ is called *a matrix combination in* $M(V)$. A linear mapping $\varphi: V \to W$ has the canonical linear extensions $\varphi^{(n)}: M_n(V) \to M_n(W)$ $[\varphi^{(\infty)}: M(V) \to M(W)]$ over all matrix spaces defined as $\varphi^{(n)}([x_{ij}]) = [\varphi(x_{ij})]$ $[\varphi^{(\infty)} | M_n(V) = \varphi^{(n)}]$. One can easily verify that $\varphi^{(\infty)}$ preserves the quantum operations, that is,

$$\varphi^{(\infty)}(v \oplus w) = \varphi^{(\infty)}(v) \oplus \varphi^{(\infty)}(w) \quad \text{and} \quad \varphi^{(\infty)}(avb) = a\varphi^{(\infty)}(v)b. \tag{2.1}$$

By a *matrix set* $\mathfrak{B}$ *in the matrix space* $M(V)$ *over a linear space* V, we mean a collection $\mathfrak{B} = (\mathfrak{b}_n)$ of subsets $\mathfrak{b}_n \subseteq M_n(V)$, $n \in \mathbb{N}$. For matrix subsets $\mathfrak{B}$ and $\mathfrak{M}$ in $M(V)$, we write $\mathfrak{B} \subseteq \mathfrak{M}$ whenever $\mathfrak{b}_n \subseteq \mathfrak{m}_n$ for all n. In particular, all set-theoretic operations over all matrix sets can easily be defined. Each subset $\mathfrak{b} \subseteq V$ determines a matrix set $\mathfrak{b} = (\mathfrak{b}_n)$ with $\mathfrak{b}_1 = \mathfrak{b}$ and $\mathfrak{b}_n = \{0\}$ if $n > 1$. A matrix set $\mathfrak{B}$ in $M(V)$ is said to be *absolutely matrix convex* if $\mathfrak{b}_m \oplus \mathfrak{b}_n \subseteq \mathfrak{b}_{m+n}$ and $a\mathfrak{b}_m b \subseteq \mathfrak{b}_n$ for all contractions $a \in M_{n,m}$, $b \in M_{m,n}$, m, $n \in \mathbb{N}$. For brevity, we write

$$\mathfrak{B} \oplus \mathfrak{B} \subseteq \mathfrak{B} \quad \text{and} \quad a\mathfrak{B}b \subseteq \mathfrak{B}, \quad a,b \in \text{ball } M.$$

Remark 2.1: If $\mathfrak{B} = (\mathfrak{b}_n)$ *is an absolutely matrix convex set, then each* $\mathfrak{b}_n$ *is an absolutely convex set in* $M_n(V)$. *Indeed, take* $v, w \in \mathfrak{b}_n$, *and* $\lambda, \mu \in \mathbb{C} \backslash \{0\}$ *with* $|\lambda| + |\mu| \leq 1$. *Put*

$$a = [|\lambda|^{-1/2} \lambda I_n \quad |\mu|^{-1/2} \mu I_n] \in M_{n,2n}, \quad b = \begin{bmatrix} |\lambda|^{1/2} I_n \\ |\mu|^{1/2} I_n \end{bmatrix} \in M_{2n,n},$$

where I_n *is the identity matrix in* M_n. *Then* $\|a\|^2 = \|aa^*\| = |\lambda|^{-1} \lambda \bar{\lambda} + |\mu|^{-1} \mu \bar{\mu} = |\lambda| + |\mu| \leq 1$ *and* $\|b\|^2 = \|b^* b\| = |\lambda| + |\mu| \leq 1$. *It follows that* $\lambda v + \mu w = a(v \oplus w)b \in a(\mathfrak{b}_n \oplus \mathfrak{b}_n)b \subseteq a\mathfrak{b}_{2n} b \subseteq \mathfrak{b}_n$, *that is, any linear combination can be converted into the matrix one. In particular,* $\mathfrak{b}_n$ *is an absolutely convex set.*

Evidently, any intersection of absolutely matrix convex sets is absolutely matrix convex. *The absolutely matrix convex hull of a matrix set* $\mathfrak{B}$ is denoted by amc $\mathfrak{B}$.

The following nice result is due to Johnson, which has been proved in Ref. 14, Lemma 3.2. For the sake of a reader, we briefly sketch the proof.

Lemma 2.1: If $\mathfrak{M} = \text{amc } \mathfrak{B}$, *then* $\mathfrak{M} = (\mathfrak{m}_n)$ *is a matrix set in* $M(V)$ *with*

$$\mathfrak{m}_n = \left\{ \sum_s a_s v_s b_s : a_s \in M_{n,k_s}, v_s \in \mathfrak{b}_{k_s}, b_s \in M_{k_s,n}, \sum_s a_s a_s^* \leq 1, \sum_s b_s^* b_s \leq 1 \right\}.$$

Proof: First note that each indicated matrix combination $\Sigma_s a_s v_s b_s$ can be picked up into a "big" matrix. Namely,

$$\sum_s a_s v_s b_s = [\cdots \quad a_s \quad \cdots] \begin{bmatrix} \ddots & & \\ & v_s & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ b_s \\ \vdots \end{bmatrix} = avb$$

and $\|a\|^2 = \|aa^*\| = \Sigma_s a_s a_s^* \leq 1$, $\|b\|^2 = \|b^* b\| = \Sigma_s b_s^* b_s \leq 1$. Note that $v \in \oplus_s \mathfrak{b}_{k_s}$. Consequently, $\Sigma_s a_s v_s b_s \in \text{amc } \mathfrak{B}$. It remains to prove that $\mathfrak{M} = (\mathfrak{m}_n)$ is absolutely matrix convex. If $c \in M_{m,n}$ and $d \in M_{n,m}$ are contractions then $c(\Sigma_s a_s v_s b_s)d = \Sigma_s c a_s v_s b_s d$ and $\Sigma_s c a_s a_s^* c^* \leq cc^* \leq 1$, $\Sigma_s d^* b_s^* b_s d \leq d^* d \leq 1$. Finally, if $u = \Sigma_s a_s v_s b_s \in \mathfrak{m}_m$ and $v = \Sigma_s c_s w_s d_s \in \mathfrak{m}_n$, then $u \oplus v = \Sigma_s (a_s \oplus 0)(v_s \oplus 0)(b_s \oplus 0) + \Sigma_s (0 \oplus c_s)(0 \oplus w_s)(0 \oplus d_s) \in \mathfrak{m}_{m+n}$, for

$$\sum_s (a_s a_s^* \oplus 0) + \sum_s (0 \oplus c_s c_s^*) = \begin{bmatrix} \sum_s a_s a_s^* & 0 \\ 0 & \sum_s c_s c_s^* \end{bmatrix} \leqslant \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \leqslant 1$$

and $\Sigma_s(b_s^* b_s \oplus 0) + \Sigma_s(0 \oplus c_s^* c_s) \leqslant 1$. Whence $\mathfrak{M}$ is absolutely matrix convex. □

Let $\lambda = (\lambda_1, \ldots, \lambda_s) \in \Pi_{i=1}^s M_{n,k_i}$ and $\mu = (\mu_1, \ldots, \mu_s) \in \Pi_{i=1}^s M_{k_i,n}$ be tuples of (scalar) matrices. For a tuple $\varepsilon = (\varepsilon_i) \in \mathbb{R}_+^s$ of positive real numbers, we use the notations $\lambda_{\varepsilon,i} = \varepsilon_i^{-1/2}\lambda_i$, $\mu_{\varepsilon,i} = \varepsilon_i^{-1/2}\mu_i$, $1 \leqslant i \leqslant s$. If $\varepsilon, \delta \in \mathbb{R}_+^s$, then we write $\varepsilon \geqslant \delta$ whenever $\varepsilon_i \geqslant \delta_i$ for all i. Consider the following matrix:

$$A_{\lambda,\varepsilon,\mu} = \begin{bmatrix} \lambda_{\varepsilon,1} & 0 & \cdots & \lambda_{\varepsilon,s} & 0 \\ 0 & \mu_{\varepsilon,1}^* & & 0 & \mu_{\varepsilon,s}^* \end{bmatrix} \in M_{2n,2k}$$

associated with tuples λ, ε, and μ, where $k = \Sigma_{i=1}^s k_i$. If $\lambda' \in \Pi_{i=1}^s M_{m,l_i}$ and $\mu' \in \Pi_{i=1}^s M_{l_i,m}$ are similar tuples, then we may generate their direct sums as follows:

$$\lambda \oplus \lambda' = (\lambda_1 \oplus \lambda_1', \ldots, \lambda_s \oplus \lambda_s') \in \prod_{i=1}^s M_{n+m,k_i+l_i}, \quad \mu \oplus \mu' = (\mu_1 \oplus \mu_1', \ldots, \mu_s \oplus \mu_s')$$

$$\in \prod_{i=1}^s M_{k_i+l_i,n+m}.$$

Note that $A_{\lambda\oplus\lambda',\varepsilon,\mu\oplus\mu'} \in M_{2(n+m),2(k+l)}$, where $l = \Sigma_{i=1}^s l_i$. The following simple lemma will be used later.

Lemma 2.2: If $\varepsilon \geqslant \delta$ for some $\varepsilon, \delta \in \mathbb{R}_+^s$, then $\|A_{\lambda,\varepsilon,\mu}\| \leqslant \|A_{\lambda,\delta,\mu}\|$. Furthermore,

$$\|A_{\lambda\oplus\lambda',\varepsilon,\mu\oplus\mu'}\| = \max\{\|A_{\lambda,\varepsilon,\mu}\|, \|A_{\lambda',\varepsilon,\mu'}\|\}.$$

Proof: Indeed,

$$\begin{aligned}\|A_{\lambda,\varepsilon,\mu}\|^2 &= \|A_{\lambda,\varepsilon,\mu}A_{\lambda,\varepsilon,\mu}^*\| = \left\|\sum_i \begin{bmatrix} \lambda_{\varepsilon,i}\lambda_{\varepsilon,i}^* & 0 \\ 0 & \mu_{\varepsilon,i}^*\mu_{\varepsilon,i} \end{bmatrix}\right\| = \left\|\sum_i \varepsilon_i^{-1}\begin{bmatrix} \lambda_i\lambda_i^* & 0 \\ 0 & \mu_i^*\mu_i \end{bmatrix}\right\| \\ &\leqslant \left\|\sum_i \delta_i^{-1}\begin{bmatrix} \lambda_i\lambda_i^* & 0 \\ 0 & \mu_i^*\mu_i \end{bmatrix}\right\| = \left\|\sum_i \begin{bmatrix} \lambda_{\delta,i}\lambda_{\delta,i}^* & 0 \\ 0 & \mu_{\delta,i}^*\mu_{\delta,i} \end{bmatrix}\right\| = \|A_{\lambda,\delta,\mu}A_{\lambda,\delta,\mu}^*\| = \|A_{\lambda,\delta,\mu}\|^2,\end{aligned}$$

that is, $\|A_{\lambda,\varepsilon,\mu}\| \leqslant \|A_{\lambda,\delta,\mu}\|$, whenever $\varepsilon \geqslant \delta$.

In order to prove the equality $\|A_{\lambda\oplus\lambda',\varepsilon,\mu\oplus\mu'}\| = \max\{\|A_{\lambda,\varepsilon,\mu}\|, \|A_{\lambda',\varepsilon,\mu'}\|\}$ we use the known (Ref. 13, Sec. 2.1.5) fact that any permutation of the rows or columns of a matrix over an operator space does not affect its matrix norm. We have

$$\begin{aligned}\|A_{\lambda\oplus\lambda',\varepsilon,\mu\oplus\mu'}\| &= \left\|\begin{bmatrix} \lambda_{\varepsilon,1} & 0 & 0 & 0 & & \lambda_{\varepsilon,s} & 0 & 0 & 0 \\ 0 & \lambda_{\varepsilon,1}' & 0 & 0 & & 0 & \lambda_{\varepsilon,s}' & 0 & 0 \\ 0 & 0 & \mu_{\varepsilon,1}^* & 0 & \cdots & 0 & 0 & \mu_{\varepsilon,s}^* & 0 \\ 0 & 0 & 0 & \mu_{\varepsilon,1}'^* & & 0 & 0 & 0 & \mu_{\varepsilon,s}'^* \end{bmatrix}\right\| \\ &= \left\|\begin{bmatrix} \lambda_{\varepsilon,1} & 0 & 0 & 0 & & \lambda_{\varepsilon,s} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{\varepsilon,1}' & 0 & & 0 & 0 & \lambda_{\varepsilon,s}' & 0 \\ 0 & \mu_{\varepsilon,1}^* & 0 & 0 & \cdots & 0 & \mu_{\varepsilon,s}^* & 0 & 0 \\ 0 & 0 & 0 & \mu_{\varepsilon,1}'^* & & 0 & 0 & 0 & \mu_{\varepsilon,s}'^* \end{bmatrix}\right\|\end{aligned}$$

$$= \left\| \begin{bmatrix} \lambda_{\varepsilon,1} & 0 & 0 & 0 & & \lambda_{\varepsilon,s} & 0 & 0 & 0 \\ 0 & \mu_{\varepsilon,1}^{*} & 0 & 0 & \cdots & 0 & \mu_{\varepsilon,s}^{*} & 0 & 0 \\ 0 & 0 & \lambda'_{\varepsilon,1} & 0 & & 0 & 0 & \lambda'_{\varepsilon,s} & 0 \\ 0 & 0 & 0 & \mu'^{*}_{\varepsilon,1} & & 0 & 0 & 0 & \mu'^{*}_{\varepsilon,s} \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} \lambda_{\varepsilon,1} & 0 & \cdots & \lambda_{\varepsilon,s} & 0 & 0 & & & & \\ 0 & \mu_{\varepsilon,1}^{*} & & 0 & \mu_{\varepsilon,s}^{*} & & & & & \\ 0 & & & & & \lambda'_{\varepsilon,1} & 0 & \cdots & \lambda'_{\varepsilon,s} & 0 \\ & & & & & 0 & \mu'^{*}_{\varepsilon,s} & & 0 & \mu'^{*}_{\varepsilon,s} \end{bmatrix} \right\|$$

that is, $\|A_{\lambda\oplus\lambda',\varepsilon,\mu\oplus\mu'}\| = \|A_{\lambda,\varepsilon,\mu}\oplus A_{\lambda',\varepsilon,\mu'}\| = \max\{\|A_{\lambda,\varepsilon,\mu}\|,\|A_{\lambda',\varepsilon,\mu'}\|\}$. □

B. The inductive limits of normed spaces

Let $X=\Sigma_{\alpha\in\Lambda}X_\alpha$ be a linear space spanned by a family of linear subspaces $(X_\alpha)_{\alpha\in\Lambda}$ such that each X_α is a normed space with its norm $\|\cdot\|_\alpha$. The space X turns out to be a polynormed[16] (or locally convex) space equipped with the inductive topology, the finest topology on X such that all inclusions $X_\alpha\subseteq X$ are continuous. A neighborhood filter base of X is given by the family of convex hulls

$$U_\rho = \text{abc} \bigcup_{\alpha\in\Lambda} \{x_\alpha \in X_\alpha : \|x_\alpha\|_\alpha \leqslant \rho_\alpha\},$$

where $\rho=(\rho_\alpha)_{\alpha\in\Lambda}$ runs over all positive families of real numbers in $\mathbb{R}_+^\Lambda$. The Minkowski functional of the neighborhood U_ρ is denoted by σ_ρ, that is, $\sigma_\rho(x)=\inf\{t>0 : t^{-1}x\in U_\rho\}$, $x\in X$. The following lemma is due to Vasilescu.[24]

Lemma 2.3: For each $\rho=(\rho_\alpha)_{\alpha\in\Lambda}\in\mathbb{R}_+^\Lambda$ *and* $x\in X$, *we have*

$$\sigma_\rho(x) = \inf\left\{ \sum_{\alpha\in F} \rho_\alpha^{-1}\|x_\alpha\|_\alpha : x = \sum_{\alpha\in F} x_\alpha \right\},$$

where the greatest lower bound is taken over all finite expansions $x=\Sigma_{\alpha\in F}x_\alpha$ *with* $x_\alpha\in X_\alpha$.

Proof: First note that if $t^{-1}x\in U_\rho$, then $t^{-1}x=\Sigma_{\alpha\in F}s_\alpha x_\alpha$ with $\Sigma_{\alpha\in F}s_\alpha=1$, $s_\alpha\geqslant 0$ and $\|x_\alpha\|_\alpha \leqslant\rho_\alpha$, which, in turn, implies that $x=\Sigma_{\alpha\in F}t_\alpha x_\alpha$ and $\Sigma_{\alpha\in F}t_\alpha=t$. Conversely, if $x=\Sigma_{\alpha\in F}t_\alpha x_\alpha$ with $t_\alpha\geqslant 0$, $\|x_\alpha\|_\alpha\leqslant\rho_\alpha$, then $t^{-1}x=\Sigma_{\alpha\in F}s_\alpha x_\alpha\in U_\rho$ with $\Sigma_{\alpha\in F}s_\alpha=1$, $s_\alpha\geqslant 0$, where $t=\Sigma_{\alpha\in F}t_\alpha$ and $s_\alpha =t^{-1}t_\alpha$, $\alpha\in F$. We derive that

$$\sigma_\rho(x) = \inf\left\{ \sum_{\alpha\in F} t_\alpha : x = \sum_{\alpha\in F} t_\alpha x_\alpha, t_\alpha \geqslant 0, \|x_\alpha\|_\alpha \leqslant \rho_\alpha \right\}.$$

Now take the quantity $\Sigma_{\alpha\in F}\rho_\alpha^{-1}\|y_\alpha\|_\alpha$ for an expansion $x=\Sigma_{\alpha\in F}y_\alpha$. If $t_\alpha=\rho_\alpha^{-1}\|y_\alpha\|_\alpha$, $\alpha\in F$, then $x =\Sigma_{\alpha\in F}t_\alpha x_\alpha$ with $x_\alpha=t_\alpha^{-1}y_\alpha\in X_\alpha$ and $\|x_\alpha\|_\alpha=\rho_\alpha$. Whence

$$\left\{ \sum_{\alpha\in F} \rho_\alpha^{-1}\|x_\alpha\|_\alpha : x = \sum_{\alpha\in F} x_\alpha \right\} \subseteq \left\{ \sum_{\alpha\in F} t_\alpha : x = \sum_{\alpha\in F} t_\alpha x_\alpha, \|x_\alpha\|_\alpha \leqslant \rho_\alpha \right\},$$

which implies that $\sigma_\rho(x)\leqslant\inf\{\Sigma_{\alpha\in F}\rho_\alpha^{-1}\|x_\alpha\|_\alpha : x=\Sigma_{\alpha\in F}x_\alpha\}$. Take $\varepsilon>0$. Then $\sigma_\rho(x)+\varepsilon>\Sigma_{\alpha\in F}t_\alpha$ with $x=\Sigma_{\alpha\in F}t_\alpha x_\alpha$, $\|x_\alpha\|_\alpha\leqslant\rho_\alpha$. Put $y_\alpha=t_\alpha x_\alpha$. Then $x=\Sigma_{\alpha\in F}y_\alpha$ and $\Sigma_{\alpha\in F}\rho_\alpha^{-1}\|y_\alpha\|_\alpha=\Sigma_{\alpha\in F}\rho_\alpha^{-1}t_\alpha\|x_\alpha\|_\alpha \leqslant\Sigma_{\alpha\in F}t_\alpha<\sigma_\rho(x)+\varepsilon$. The rest is clear. □

Corollary 2.1: Let $X=\Sigma_{\alpha\in\Lambda}X_\alpha$ *and* $Y=\Sigma_{\alpha\in\Lambda}Y_\alpha$ *be inductive limits of normed spaces* X_α *and* Y_α, $\alpha\in\Lambda$, *respectively. The direct product topology in* $X\times Y$ *is precisely the inductive limit topology* $X\times Y=\Sigma_{\alpha\in\Lambda}X_\alpha\times Y_\alpha$.

Proof: First note that all canonical inclusions $X_\alpha\times Y_\alpha\subseteq X\times Y$ are continuous with respect to the direct product topology. Hence the inductive limit topology is finer than the direct product topology. Conversely, let (V,p) be a seminormed space and let $f:X\times Y\to V$ be a linear mapping such that all restrictions $f|_{X_\alpha \times Y_\alpha}: X_\alpha \times Y_\alpha \to V$ are continuous. Then $p(f(x,y)) \leq C_\alpha(\|x\|_\alpha + \|y\|_\alpha)$, $(x,y) \in X_\alpha \times Y_\alpha$, for some positive constants C_α, $\alpha \in \Lambda$. Put $\rho = (C_\alpha^{-1})_{\alpha\in\Lambda}$. Take $(x,y) \in X \times Y$ and $\varepsilon > 0$. Then $\Sigma_{\alpha\in F} C_\alpha \|x_\alpha\|_\alpha < \sigma_\rho(x) + \varepsilon$ and $\Sigma_{\alpha\in F} C_\alpha \|y_\alpha\|_\alpha < \sigma_\rho(y) + \varepsilon$ with $x = \Sigma_{\alpha\in F} x_\alpha$, $y = \Sigma_{\alpha \in F} y_\alpha$, $x_\alpha \in X_\alpha$, $y_\alpha \in Y_\alpha$, thanks to Lemma 2.3, where F is a finite subset in Λ. It follows that

$$p(f(x,y)) \leq \sum_{\alpha\in F} p(f(x_\alpha, y_\alpha)) \leq \sum_{\alpha \in F} C_\alpha(\|x_\alpha\|_\alpha + \|y_\alpha\|_\alpha) < \sigma_\rho(x) + \sigma_\rho(y) + 2\varepsilon.$$

Whence $p(f(x,y)) \leq \sigma_\rho(x) + \sigma_\rho(y)$, $(x,y) \in X \times Y$, which means that $f: X \times Y \to V$ is continuous with respect to the direct product topology. Consequently, $X \times Y = \Sigma_{\alpha\in\Lambda} X_\alpha \times Y_\alpha$ is the inductive limit of the normed subspaces $X_\alpha \times Y_\alpha$, $\alpha \in \Lambda$. □

C. The quantum duality

Let V and W be linear spaces. These spaces are said to be in duality if there is a pairing $\langle \cdot , \cdot \rangle : V \times W \to \mathbb{C}$ such that $\{\langle v, \cdot \rangle : v \in V\}$ and $\{\langle \cdot , w\rangle : w \in W\}$ are separating families of functionals on W and V, respectively. We briefly say that (V,W) is a dual pair. For instance, if V is a polynormed space, then the spaces V and V' are in the canonical duality $\langle x, f\rangle = f(x)$, where $V' = \mathcal{C}(V, \mathbb{C})$ is the space of all continuous linear functionals on V. The general case can also be reduced to the just considered example if we endow V and W with the relevant weak topologies $\sigma(V,W)$ and $\sigma(W,V)$, respectively. Namely, $(V, \sigma(V,W))' = W$ and $(W, \sigma(W,V))' = V$. A polynormed topology ς in V is said to be *compatible with the duality* (V,W) if $(V,\varsigma)' = W$. The least upper bound $\sup \varsigma$ of these topologies is called the Mackey topology and it is denoted by $\tau(V,W)$. The known (see, for instance, Ref. 18, Sec. 10.4.5) Arens–Mackey theorem asserts that all polynormed topologies compatible with the duality (V,W) are arranged within the weak topology $\sigma(V,W)$ and Mackey topology $\tau(V,W)$, that is, ς is compatible with the duality (V,W) if and only if $\sigma(V,W) \subseteq \varsigma \subseteq \tau(V,W)$.

The given pairing between V and W determines a *quantum* (or *matrix*) *pairing*

$$\langle\langle \cdot , \cdot \rangle\rangle : M_m(V) \times M_n(W) \to M_{mn}, \quad \langle\langle v, w\rangle\rangle = [\langle v_{ij}, w_{st}\rangle] = w^{(m)}(v) = v^{(n)}(w),$$

where $v = [v_{ij}] \in M_m(V)$, $w = [w_{st}] \in M_n(W)$, which are identified with the canonical linear mappings

$$v: W \to M_m, \quad v(y) = [\langle v_{ij}, y\rangle], \quad \text{and} \quad w: V \to M_n, \quad w(x) = [\langle x, w_{st}\rangle],$$

respectively. Each $M_m(V)$ $[M_n(W)]$ can be equipped with the polynormed topology induced from V^{n^2} (W^{n^2}). The polynormed spaces $M_n(V)$ and $M_n(W)$ are also in the canonical duality determined by *the scalar pairing*

$$\langle \cdot , \cdot \rangle : M_n(V) \times M_n(W) \to \mathbb{C}, \quad \langle v, w\rangle = \sum_{i,j} \langle v_{ij}, w_{ij}\rangle.$$

In particular, we have the relevant weak $\sigma(M_n(V), M_n(W))$ and Mackey $\tau(M_n(V), M_n(W))$ topologies, respectively. Moreover, $\sigma(M_n(V), M_n(W)) = \sigma(V,W)^{n^2}$ and $\tau(M_n(V), M_n(W)) = \tau(V,W)^{n^2}$ (see Ref. 23, Secs. 4.4.2 and 4.4.3), where $\sigma(V,W)^{n^2}$ and $\tau(V,W)^{n^2}$ are the relevant direct product topologies in V^{n^2}. In particular, if ς is a polynormed topology in V compatible with the duality (V,W), then the direct product ς^{n^2} is a polynormed topology in $M_n(V)$ compatible with the duality $(M_n(V), M_n(W))$. Indeed, since $\sigma(V,W) \subseteq \varsigma \subseteq \tau(V,W)$, we derive that

$$\sigma(M_n(V), M_n(W)) = \sigma(V,W)^{n^2} \subseteq \varsigma^{n^2} \subseteq \tau(V,W)^{n^2} = \tau(M_n(V), M_n(W)),$$

which, in turn, implies that ς^{n^2} is compatible with the duality $(M_n(V), M_n(W))$, thanks to Arens–Mackey theorem. Further, the bilinear mapping $\langle\langle \cdot , \cdot \rangle\rangle : V \times M_n(W) \to M_n$ determines all continuous linear mappings $\varphi: V \to M_n$, that is, $M_n(W) = \mathcal{C}(V, M_n)$ with respect to any polynormed topology in V compatible with the duality (V,W).

Given a matrix set $\mathfrak{B}$ in $M(V)$ let us introduce its *weak closure* $\mathfrak{B}^-$ as the matrix set $(\mathfrak{b}_n^-)$, where $\mathfrak{b}_n^-$ indicates $\sigma(M_n(V),M_n(W))$-closure of $\mathfrak{b}_n$. We say that $\mathfrak{B}$ is *a weakly closed matrix set in* $M(V)$ if and only if $\mathfrak{B}^-=\mathfrak{B}$. The *absolute matrix* (or *operator*) *polar* $\mathfrak{B}^{\odot}$ *in* $M(W)$ of a matrix set $\mathfrak{B}\subseteq M(V)$ is defined as the matrix set $(\mathfrak{b}_n^{\odot})$ with $\mathfrak{b}_n^{\odot}=\{w\in M_n(W):\|\langle\langle v,w\rangle\rangle\|\leq 1, v\in\mathfrak{b}_s, s\in\mathbb{N}\}$. We briefly write that

$$\mathfrak{B}^{\odot}=\{w\in M(W):\sup\|\langle\langle\mathfrak{B},w\rangle\rangle\|\leq 1\}.$$

Similarly, it is defined the absolute matrix polar $\mathfrak{M}^{\odot}\subseteq M(V)$ of a matrix set $\mathfrak{M}\subseteq M(W)$.

Lemma 2.4: Let (V,W) *be a dual pair and let* $\mathfrak{B}\subseteq M(V)$ *be a matrix set. Then* $\mathfrak{B}^{\odot}$ *is an absolutely matrix convex and weakly closed set in* $M(W)$, $(\mathrm{amc}\,\mathfrak{B})^{\odot}=\mathfrak{B}^{\odot}$, *and* $(\lambda\mathfrak{B})^{\odot}=\lambda^{-1}\mathfrak{B}^{\odot}$ *if* $\lambda\in\mathbb{C}\backslash\{0\}$. *Moreover*, $(\cup_{\alpha\in\Lambda}\mathfrak{B}\alpha)^{\odot}=\cap_{\alpha\in\Lambda}\mathfrak{B}_\alpha^{\odot}$ *for a family* $\{\mathfrak{B}_\alpha\}_{\alpha\in\Lambda}$ *of matrix sets in* $M(V)$.

Proof: First, let us prove that $\mathfrak{b}_n^{\odot}$ is weakly closed. Put $A_v=\{w\in M_n(W):\|\langle\langle v,w\rangle\rangle\|\leq 1\}$ whenever $v\in\mathfrak{b}_s$. The mapping $M_n(W)\to M_{ns}$, $w\mapsto\langle\langle v,w\rangle\rangle$, is weakly continuous. Indeed, each functional $M_n(W)\to\mathbb{C}$, $w\mapsto\langle v_{ij},w_{st}\rangle$ being a composition $M_n(W)\to W\to\mathbb{C}$, $w\mapsto w_{st}\mapsto\langle v_{ij},w_{st}\rangle$ of weakly continuous mappings, turns out to be weakly continuous. In particular, A_v is weakly closed. However, $\mathfrak{b}_n^{\odot}=\cap_s\cap_{v\in\mathfrak{b}_s}A_v$. Hence, $\mathfrak{b}_n^{\odot}$ is weakly closed. Further, if $w\in\mathfrak{b}_m^{\odot}$ and $w'\in\mathfrak{b}_n^{\odot}$, then

$$\|\langle\langle v,w\oplus w'\rangle\rangle\|=\|\langle\langle v,w\rangle\rangle\oplus\langle\langle v,w'\rangle\rangle\|=\max\{\|\langle\langle v,w\rangle\rangle\|,\|\langle\langle v,w'\rangle\rangle\|\}\leq 1$$

whenever $v\in\mathfrak{b}_s$. By its very definition, $w\oplus w'\in\mathfrak{b}_{m+n}^{\odot}$, that is, $\mathfrak{b}_m^{\odot}\oplus\mathfrak{b}_n^{\odot}\subseteq\mathfrak{b}_{m+n}^{\odot}$. If $a\in M_{n,m}$, $b\in M_{m,n}$ are contractions and $v\in\mathfrak{b}_s$, then

$$\|\langle\langle v,awb\rangle\rangle\|=\|a\otimes 1\langle\langle v,w\rangle\rangle b\otimes 1\|\leq\|a\|\|\langle\langle v,w\rangle\rangle\|\|b\|\leq 1,$$

that is, $a\mathfrak{b}_m^{\odot}b\subseteq\mathfrak{b}_n^{\odot}$. Consequently, $\mathfrak{B}^{\odot}$ is an absolutely matrix convex and weakly closed set in $M(W)$. The equalities $(\lambda\mathfrak{B})^{\odot}=\lambda^{-1}\mathfrak{B}^{\odot}$ and $(\cup_{\alpha\in\Lambda}\mathfrak{B}_\alpha)^{\odot}=\cap_{\alpha\in\Lambda}\mathfrak{B}_\alpha^{\odot}$ are directly derived from the definition of the absolute matrix polar. It remains to prove that $(\mathrm{amc}\,\mathfrak{B})^{\odot}=\mathfrak{B}^{\odot}$. Since $\mathfrak{B}\subseteq\mathrm{amc}\,\mathfrak{B}$, it follows that $(\mathrm{amc}\,\mathfrak{B})^{\odot}\subseteq\mathfrak{B}^{\odot}$. Take $w\in\mathfrak{b}_n^{\odot}$. Let us prove that $w\in\mathfrak{m}_n^{\odot}$, where $\mathfrak{M}=(\mathfrak{m}_n)=\mathrm{amc}\,\mathfrak{B}$. If $v\in\mathfrak{m}_r$, then $v=\Sigma_{i=1}^s a_iv_ib_i$ is a matrix combination for some $a_i\in M_{r,k_i}$, $v_i\in\mathfrak{b}_{k_i}$, $b_i\in M_{k_i,r}$ with $\Sigma_i a_ia_i^*\leq 1$, $\Sigma_i b_i^*b_i\leq 1$, thanks to Lemma 2.1. Then

$$v=a(v_1\oplus\cdots\oplus v_s)b,\quad\text{where } a=[a_1\quad\cdots\quad a_s]\in M_{r,k}\quad\text{and}\quad b=\begin{bmatrix}b_1\\ \vdots\\ b_s\end{bmatrix}\in M_{k,r},$$

where $k=\Sigma_i k_i$. It follows that

$$\begin{aligned}\|\langle\langle v,w\rangle\rangle\|&=\|\langle\langle a(v_1\oplus\cdots\oplus v_s)b,w\rangle\rangle\|=\|a\otimes 1\langle\langle v_1\oplus\cdots\oplus v_s,w\rangle\rangle b\otimes 1\|\leq\|a\|\|\langle\langle v_1,w\rangle\rangle\oplus\cdots\\ &\oplus\langle\langle v_s,w\rangle\rangle\|\|b\|=\|aa^*\|^{1/2}\max_i\|\langle\langle v_i,w\rangle\rangle\|\|b^*b\|^{1/2}\leq\left\|\sum_i a_ia_i^*\right\|^{1/2}\max_i\|\langle\langle v_i,w\rangle\rangle\|\\ &\times\left\|\sum_i b_i^*b_i\right\|^{1/2}\leq 1.\end{aligned}$$

Whence $w\in\mathfrak{m}_n^{\odot}$. So, $\mathfrak{b}_n^{\odot}=\mathfrak{m}_n^{\odot}$ for all n. □

Corollary 2.2: If $\mathfrak{B}=(\mathfrak{b}_n)$ *is a matrix set in* $M(V)$, *then* $\mathfrak{b}_1^{\odot}$ *coincides with the classical absolute polar of* $\mathfrak{b}_1$ *in* W, *that is*, $\mathfrak{b}_1^{\odot}=\mathfrak{b}_1^{\circ}=\{w\in W:|\langle v,w\rangle|\leq 1, v\in\mathfrak{b}_1\}$.

Proof: It can be assumed that $\mathfrak{B}$ is an absolutely matrix convex set thanks to Lemma 2.4. Without any doubt, $\mathfrak{b}_1^{\odot}\subseteq\mathfrak{b}_1^{\circ}$. Conversely, take $w\in\mathfrak{b}_1^{\circ}$. If $v=[v_{ij}]\in\mathfrak{b}_s$, then for all unit column vectors $\zeta,\eta\in\mathbb{C}^s$ we have

$$\langle\langle\langle v,w\rangle\rangle\zeta|\eta\rangle = \langle[\langle v_{ij},w\rangle]\zeta|\eta\rangle = \sum_{i,j}\langle v_{ij},w\rangle\zeta_j\bar{\eta}_i = \left\langle \sum_{i,j}\bar{\eta}_i v_{ij}\zeta_j, w\right\rangle = \langle\eta^* v\zeta,w\rangle.$$

However, $\eta^* v\zeta \in \eta^*\mathfrak{b}_s\zeta \subseteq \mathfrak{b}_1$. It follows that

$$\|\langle\langle v,w\rangle\rangle\| = \sup\{|\langle\langle\langle v,w\rangle\rangle\zeta|\eta\rangle| : \|\zeta\|,\|\eta\| \leq 1\} \leq \sup\{|\langle x,w\rangle| : x \in \mathfrak{b}_1\} \leq 1,$$

that is, $w \in \mathfrak{b}_1^{\odot}$. □

The classical bipolar theorem asserts that the double absolute polar $S^{\circ\circ}$ of a subset $S \subseteq V$ is the smallest weakly closed absolutely convex set containing S. The quantum version of this result was proved in Ref. 14 by Effros and Webster.

Theorem 2.1: *Let (V,W) be a dual pair and let $\mathfrak{B}$ be a matrix set in $M(V)$. Then $\mathfrak{B}^{\odot\odot}$ is the weak closure of* amc $\mathfrak{B}$.

D. The quantum spaces

Let $\mathfrak{p}=\{\mathfrak{B}\}$ be a filter base in $M(V)$ of absorbing, absolutely matrix convex sets such that $\cap\mathfrak{p}=\{0\}$, which defines a (Hausdorff) polynormed (or locally convex) topology in $M(V)$. We are saying that $(V,\mathfrak{p})$ is *a quantum space* (or *abstract local operator space*) *with its quantum topology* $\mathfrak{p}$. The terminology "*quantum space*" is due to Helemskii [Ref. 17, Sec. 1.7]. Note that the quantum topology $\mathfrak{p}$ in $M(V)$ inherits a polynormed topology $\mathfrak{s}=\mathfrak{p}|V$ in V, that is, V is a polynormed space. A matrix set $\mathfrak{M} \subseteq M(V)$ is said to be *a matrix bounded set* if it is bounded in the polynormed space $M(V)$ in the usual sense. A linear mapping $\varphi:(V,\mathfrak{p}) \to (Y,\mathfrak{q})$ between quantum spaces is said to be a *matrix continuous* if $\varphi^{(\infty)}:(M(V),\mathfrak{p}) \to (M(Y),\mathfrak{q})$ is a continuous linear mapping of the relevant polynormed spaces. If φ is invertible and φ^{-1} is matrix continuous too, then we say that φ is a *topological matrix isomorphism*.

Equivalently, a quantum topology $\mathfrak{p}=\{\mathfrak{B}\}$ in $M(V)$ can be defined in terms of the Minkowski functionals $p_{\mathfrak{B}}:M(V) \to [0,\infty]$ of the absolutely matrix convex neighborhoods $\mathfrak{B}$. In order to characterize the Minkowski functionals of absolutely matrix convex sets in $M(V)$ let us introduce a matrix gauge (matrix seminorm) on V see Refs. 14, 13, and 25 and Ref. 17, Sec. 1.7. A mapping $p:M(V) \to [0,\infty]$ is said to be *a matrix gauge* if it possesses the following properties:

$$\mathbf{M1}\quad p(v\oplus w) \leq \max\{p(v),p(w)\},\qquad \mathbf{M2}\quad p(avb) \leq \|a\|p(v)\|b\|$$

for all $v,w \in M(V)$, $a,b \in M$. Put $p^{(n)}=p|_{M_n(V)}$, $n \in \mathbb{N}$. Note that **M2** implies that

$$p^{(1)}(v_{ij}) \leq p^{(m)}(v) \leq \sum_{i,j=1}^{m} p^{(1)}(v_{ij}) \tag{2.2}$$

for any matrix $v=[v_{ij}] \in M_m(V)$. Indeed, $p^{(1)}(v_{ij})=p^{(1)}(\varepsilon_i v\varepsilon_j^*) \leq p^{(m)}(v)=p^{(m)}(\Sigma\varepsilon_i^* v_{ij}\varepsilon_j) \leq \Sigma p^{(m)} \times(\varepsilon_i^* v_{ij}\varepsilon_j) \leq \Sigma p^{(1)}(v_{ij})$, where ε_i are the canonical row matrices. Note also that

$$p^{(m+n)}(v\oplus 0) \leq p^{(m)}(v) = p^{(m)}\left([I_m \quad 0]\begin{bmatrix} v & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} I_m \\ 0\end{bmatrix}\right) \leq p^{(m+n)}(v\oplus 0),$$

which means that $(p^{(m)})$ is a compatible family of gauges on $M(V)$. If p and q are matrix gauges on V, then we write $p \leq q$ whenever $p^{(n)}(v) \leq q^{(n)}(v)$ for all $v \in M_n(V)$, $n \in \mathbb{N}$. It is a partial order structure over all matrix gauges on V. The following assertion indicated in Ref. 14 plays an important role.

Proposition 2.1: Let (V,W) be a dual pair. The correspondence $p \mapsto$ ball p *is a one-to-one mapping between the matrix gauges on V whose unit sets are weakly closed, and the matrix sets in $M(V)$ which are absolutely matrix convex and weakly closed.*

Proof: Assume that p is a matrix gauge with its weakly closed unit set $\mathfrak{B}=$ball p. Take $v,w \in \mathfrak{B}$. Then $p(v\oplus w) \leq \max\{p(v),p(w)\} \leq 1$, that is, $v\oplus w \in \mathfrak{B}$. Similarly, $p(awb) \leq \|a\|p(w)\|b\| \leq 1$ whenever $a,b \in$ ball M, and $w \in \mathfrak{B}$. Hence, $awb \in \mathfrak{B}$. Thus, $\mathfrak{B}$ is an absolutely matrix convex set.

Conversely, assume that $\mathfrak{B}$ is a weakly closed, absolutely matrix convex set in $M(V)$. According to Remark 2.1, $\mathfrak{B}$ is an absolutely convex set in $M(V)$, so its Minkowski functional $p(v) = \inf\{t>0 : t^{-1}v \in \mathfrak{B}\}$ is a gauge on $M(V)$. Let us prove that $p(v\oplus w) \leq \max\{p(v), p(w)\}$. One may assume that $p(w) \leq p(v) < \infty$. Take $\varepsilon > 0$ and $t>0$ with $p(w) \leq p(v) < t < p(v)+\varepsilon$. Since $\mathfrak{B}$ is absolutely convex, it follows that $t^{-1}w$, $t^{-1}v \in \mathfrak{B}$. Moreover, $t^{-1}(v\oplus w) \in \mathfrak{B}\oplus\mathfrak{B} \subseteq \mathfrak{B}$, which, in turn, implies that $p(v\oplus w) \leq t < p(v)+\varepsilon$. Consequently, $p(v\oplus w) \leq p(v) = \max\{p(v), p(w)\}$. Further, take $t>0$ with $p(w) < t < p(w)+\varepsilon$ and $t^{-1}w \in \mathfrak{B}$. If $a, b \in M$ are nontrivial matrices, then $t^{-1}awb \in a\mathfrak{B}b \subseteq \|a\|\|b\|(\|a\|^{-1}a)\mathfrak{B}(\|b\|^{-1}b) \subseteq \|a\|\|b\|\mathfrak{B}$. It follows that $p(awb) \leq \|a\|\|b\|t < \|a\|(p(w)+\varepsilon)\|b\|$. We derive that $p(awb) \leq \|a\|p(w)\|b\|$. Thus, we have both **M1** and **M2** properties, that is, p is a matrix gauge on V. Moreover, $\mathfrak{B} = \text{ball } p$. Indeed, if $p(w)=1$, then $w_\varepsilon = (1+\varepsilon)^{-1}w \in \mathfrak{B}$ for any $\varepsilon>0$, and $w = \lim_{\varepsilon\to 0} w_\varepsilon \in \mathfrak{B}^- = \mathfrak{B}$, for $\mathfrak{B}$ is weakly closed.

Finally, we have to prove that if p is a matrix gauge on V with its weakly closed unit set $\mathfrak{B}$, and $p_{\mathfrak{B}}$ is the Minkowski functional of $\mathfrak{B}$, then $p = p_{\mathfrak{B}}$. Note that $p(tv) = tp(v)$, $t>0$, $v \in M(V)$, thanks to **M2**. For $\varepsilon>0$ there corresponds t such that $0 < t < p_{\mathfrak{B}}(v)+\varepsilon$ and $t^{-1}v \in \mathfrak{B}$. Then $p(v) \leq t$, that is, $p \leq p_{\mathfrak{B}}$. Conversely, if $p(v)=0$, then $t^{-1}v \in \mathfrak{B}$ for all $t>0$, that is, $p_{\mathfrak{B}}(v)=0$. If $0 < p(v) < \infty$, then $p_{\mathfrak{B}}(v) \leq p(v)p_{\mathfrak{B}}(p(v)^{-1}v) \leq p(v)$, that is, $p \leq p_{\mathfrak{B}}$. The rest is clear. □

Now let $(V, \mathfrak{p})$ be a quantum space with its quantum topology $\mathfrak{p} = \{\mathfrak{B}\}$ in $M(V)$. It can be assumed that all $\mathfrak{B}$ from $\mathfrak{p}$ are closed. In particular, all $\mathfrak{B}$ are absorbing, weakly closed [with respect to the dual pair (V, V')] and absolutely matrix convex sets. Using Proposition 2.1, we derive that the quantum topology can be defined by means of a (separated and saturated) family of matrix seminorms $\mathfrak{p} = \{p\}$. In particular, each matrix space $M_n(V)$ turns into a polynormed space with its defining family of seminorms $\{p^{(n)}\}$, which is just the direct-product topology inherited by means of the canonical identifications $M_n(V) = V^{n^2}$ [see (2.2)]. Therefore, each $M_n(V)$ is a closed subspace in $M(V)$.

When we deal with a single matrix norm, then V is called *a quantum normed* (or *abstract operator*) *space*. The space $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space is an example of a quantum normed space. In this case $M_n(\mathcal{B}(H)) = \mathcal{B}(H^n)$ for all $n \in \mathbb{N}$, and $\|\cdot\| = (\|\cdot\|_{\mathcal{B}(H^n)})$ is a matrix norm on $\mathcal{B}(H)$. In particular, each subspace of $\mathcal{B}(H)$ is a quantum normed space called *an operator space*. A linear mapping $\varphi : (V, \|\cdot\|) \to (W, \|\cdot\|)$ between quantum normed spaces is said to be *a matrix isometry* if $\|\varphi^{(\infty)}(v)\| = \|v\|$ for all $v \in M(V)$.

Let (V, W) be a dual pair and let $\mathfrak{B}$ be a matrix set in $M(V)$. We define the mapping $q_{\mathfrak{B}} : M(W) \to [0, \infty]$ as

$$q_{\mathfrak{B}}(w) = \sup\|\langle\langle \mathfrak{B}, w\rangle\rangle\|, \quad w \in M(W). \tag{2.3}$$

It can easily be verified that $q_{\mathfrak{B}}$ is a matrix gauge on W called *the dual gauge of* $\mathfrak{B}$. If $\mathfrak{B} = \text{ball } p$ is the unit set of a matrix gauge p on V, then $q_{\mathfrak{B}}$ is called the *dual gauge of* p and it is denoted by $p^{\odot}$. Thus,

$$p^{\odot}(w) = \sup\|\langle\langle \text{ball } p, w\rangle\rangle\|$$

for all $w \in M(W)$.

Corollary 2.3: If $\mathfrak{B}$ *is a matrix set in* $M(V)$ *and* $q_{\mathfrak{B}}$ *its dual gauge, then* $q_{\mathfrak{B}}$ *is the Minkowski functional of the absolute matrix polar* $B^{\odot}$ *in* $M(W)$. *In particular*,

$$\text{ball } p^{\odot} = (\text{ball } p)^{\odot}.$$

Proof: By Lemma 2.4, the matrix polar $\mathfrak{B}^{\odot}$ is an absolutely matrix convex and weakly closed set in $M(W)$. Therefore, it suffices to prove that $\mathfrak{B}^{\odot} = \{q_{\mathfrak{B}} \leq 1\}$, thanks to Proposition 2.1. However, the latter equality directly follows from the definition of $\mathfrak{B}^{\odot}$. □

The space of all matrix continuous linear mappings $V \to Y$ between quantum spaces is denoted by $\mathcal{MC}(V, Y)$, whereas $\mathcal{C}(V, Y)$ denotes the space of all continuous linear mappings $V \to Y$ between the relevant polynormed spaces. If V and Y are quantum normed spaces, then $\mathcal{MC}(V, Y)$ is reduced to the quantum normed space $\mathcal{MB}(V, Y)$ of all matrix (or completely) bounded linear operators $T:V\to Y$ equipped with the matrix norm $\|T\|_{mb}=\sup\{\|T^{(n)}\|:n\in\mathbb{N}\}$. The notations $\mathcal{CB}(V,Y)$ and $\|T\|_{cb}$ can also be used (see Ref. 13) instead of $\mathcal{MB}(V,Y)$ and $\|T\|_{mb}$, respectively. It was proved in Ref. 14 (Lemma 5.2) that

$$\mathcal{MC}(V,M_n)=\mathcal{C}(V,M_n),\quad n\in\mathbb{N} \tag{2.4}$$

for a quantum space V. Namely, if $\varphi\in\mathcal{C}(V,M_n)$ with $\|\varphi^{(n)}(v)\|\leqslant cp^{(n)}(v)$, $v\in M_n(V)$, for some positive constant c and a continuous matrix seminorm p on V, then $\|\varphi^{(\infty)}(v)\|\leqslant cp(v)$, $v\in M(V)$. In particular, $V'=\mathcal{C}(V,\mathbb{C})=\mathcal{MC}(V,\mathbb{C})$. Furthermore,

$$M_n(V')=\mathcal{C}(V,M_n)=\mathcal{MC}(V,M_n), \tag{2.5}$$

where the first identification is given by the rule $[f_{ij}](v)=[f_{ij}(v)]$, $v\in V$, for each matrix $[f_{ij}]\in M_n(V')$.

Finally, let $V=\Sigma_{\alpha\in\Lambda}V_\alpha$ be a linear space, which is spanned by a family of linear subspaces $(V_\alpha)_{\alpha\in\Lambda}$ such that each V_α is a quantum normed space. Then V being an inductive limit of the quantum normed spaces V_α, $\alpha\in\Lambda$, turns out to be a quantum space. The relevant quantum topology on V is the finest quantum topology such that all inclusions $V_\alpha\subseteq V$ are matrix continuous, that is, $M(V)=\Sigma_{\alpha\in\Lambda}M(V_\alpha)$. In this case we write $V=\mathrm{op}\Sigma_{\alpha\in\Lambda}V_\alpha$ or $V=\mathrm{oplim}_{\rightarrow}\{V_\alpha\}$. If $V=\oplus_{\alpha\in\Lambda}V_\alpha$ is an algebraic direct sum of the quantum normed spaces, then we write $V=\mathrm{op}\oplus_{\alpha\in\Lambda}V_\alpha$ to indicate *the quantum* (or *local operator*) *direct sum of the quantum normed spaces,* thus $M(V)=\oplus_{\alpha\in\Lambda}M(V_\alpha)$. More detailed discussion of the inductive quantum topologies will be done in Sec. III.

E. The min and max quantizations

Now let $(V,\mathfrak{s})$ be a polynormed space. By *a quantization of V* we mean a quantum space structure $(V,\mathfrak{p})$ on V such that $\mathfrak{p}|V=\mathfrak{s}$. There is a scale of possible quantizations of a polynormed space $(V,\mathfrak{s})$, namely, the min and max quantizations (see Ref. 25), that we recall in this subsection.

Consider the dual pair (V,V') and let $\mathfrak{b}$ be a $\sigma(V,V')$-closed, absolutely convex set in the polynormed space $(V,\mathfrak{s})$. The set $\mathfrak{b}$ can be thought as a matrix set $\mathfrak{b}=(\mathfrak{b}_n)$ in $M(V)$ with $\mathfrak{b}_1=\mathfrak{b}$ and $\mathfrak{b}_n=\{0\}$ if $n>1$. Let us define (see Ref. 14, Sec. IV) the *minimal envelope* $\underline{\mathfrak{b}}\subseteq M(V)$ *of* $\mathfrak{b}$ by putting $\underline{\mathfrak{b}}=\mathfrak{b}^{\odot\odot}$. On the grounds of Bipolar Theorem 2.1, we conclude that $\underline{\mathfrak{b}}$ is the weak closure of the set amc $\mathfrak{b}$. Using Lemma 2.1, infer that

$$\mathrm{amc}\,\mathfrak{b}=\bigcup_{a,b\in\mathrm{ball}\ M}a(\mathfrak{b}\oplus\cdots\oplus\mathfrak{b})b.$$

The *maximal envelope* $\bar{\mathfrak{b}}\subseteq M(V)$ *of* $\mathfrak{b}$ is defined as the absolute matrix polar $(\mathfrak{b}^{\circ})^{\odot}$ [in $M(V)$] of the classical polar $\mathfrak{b}^{\circ}\subseteq V'$. As above, $\mathfrak{b}^{\circ}$ is considered to be a matrix set in $M(V')$.

Now assume that $\mathfrak{B}=(\mathfrak{b}_n)$ is a weakly closed, absolutely matrix convex set in $M(V)$ such that $\mathfrak{b}_1=\mathfrak{b}$. Then

$$\underline{\mathfrak{b}}\subseteq\mathfrak{B}\subseteq\bar{\mathfrak{b}}. \tag{2.6}$$

Indeed, $\mathfrak{b}\subseteq\mathfrak{B}$ as the matrix sets in $M(V)$ (see Remark 2.1), which, in turn, implies that $\mathfrak{b}^{\odot}\supseteq\mathfrak{B}^{\odot}$ in $M(V')$. Using bipolar Theorem 2.1, we derive that $\mathfrak{B}=\mathfrak{B}^{\odot\odot}\supseteq\mathfrak{b}^{\odot\odot}=\underline{\mathfrak{b}}$. Further, $\mathfrak{b}_1^{\odot}=\mathfrak{b}_1^{\circ}=\mathfrak{b}^{\circ}$ thanks to Corollary 2.2. It follows that $\mathfrak{b}^{\circ}\subseteq\mathfrak{B}^{\odot}$ as the matrix sets in $M(V')$. Using again bipolar Theorem 2.1, infer that $\mathfrak{B}=\mathfrak{B}^{\odot\odot}\subseteq(\mathfrak{b}^{\circ})^{\odot}=\bar{\mathfrak{b}}$.

Further, $\mathfrak{B}$ is the unit set of a certain matrix gauge p on V, thanks to Proposition 2.1. In particular, $\mathfrak{b}=\mathrm{ball}\ \pi$, where $\pi=p^{(1)}$ is a gauge on V or the Minkowski functional of $\mathfrak{b}$. Appealing again Proposition 2.1, we conclude that $\underline{\mathfrak{b}}$ and $\bar{\mathfrak{b}}$ are unit sets of the uniquely defined matrix gauges $\underline{\pi}$ and $\bar{\pi}$, respectively. For each $r\in\mathbb{N}$ we denote by $\mathcal{C}_\pi(V,M_r)$ the set of all continuous π-contractive linear mappings $w:V\to M_r$, that is,

$$\mathcal{C}_\pi(V,M_r)=\{w\in M_r(V'):\pi^\circ(w)=\sup\|\langle\langle\mathfrak{b},w\rangle\rangle\|\leq 1\}.$$

Thus, $\mathfrak{b}^\odot=(\mathcal{C}_\pi(V,M_r))\subseteq M(V')$. In particular, $\mathcal{C}_\pi(V,\mathbb{C})=\mathfrak{b}^\circ\subseteq V'$. Further, $\mathrm{ball}p^\odot=\mathfrak{B}^\odot=(\mathfrak{b}_r^\odot)\subseteq M(V')$ and $\mathfrak{b}_r^\odot=\mathcal{MB}_p(V,M_r)\subseteq M_r(V')$ is the set of all continuous matrix p-contractive linear mappings $w:V\to M_r$, that is,

$$\mathcal{MB}_p(V,M_r)=\{w\in M_r(V'):p^\odot(w)=\sup\|\langle\langle\mathfrak{B},w\rangle\rangle\|\leq 1\}$$

(see Sec. II D). The following assertion demonstrates the scale of all possible quantizations of the gauge π (see Ref. 14).

Proposition 2.2: Let p be a matrix gauge on V with its weakly closed unit set $\mathfrak{B}$, and $\pi=p^{(1)}$. Then

$$\overline{\pi}(v)=\sup\|\langle\langle v,\mathcal{C}_\pi(V,\mathbb{C})\rangle\rangle\|\leq p(v)=\sup_r\|\langle\langle v,\mathcal{MB}_p(V,M_r)\rangle\rangle\|\leq \underline{\pi}(v)=\sup_r\|\langle\langle v,\mathcal{C}_\pi(V,M_r)\rangle\rangle\|$$

for all $v\in M(V)$. In particular, if p and q are matrix gauges with their weakly closed unit sets in $M(V)$, then

$$p\leq q \quad \text{iff } \mathcal{MB}_p(V,M_r)\subseteq\mathcal{MB}_q(V,M_r),\quad r\in\mathbb{N}.$$

Proof: First note that $\underline{\mathfrak{b}}\subseteq\mathfrak{B}\subseteq\overline{\mathfrak{b}}$, thanks to (2.6). Then we have $\overline{\pi}\leq p\leq\underline{\pi}$ for the Minkowski functionals (Proposition 2.1). Take a matrix $v\in M(V)$. Since $\underline{\mathfrak{b}}=\mathfrak{b}^{\odot\odot}$, it follows that $\underline{\pi}(v)=\sup\|\langle\langle v,\mathfrak{b}^\odot\rangle\rangle\|$, thanks to Corollary 2.3. However, $\mathfrak{b}^\odot=((\mathfrak{b}^\odot)_r)$ is a matrix set with

$$(\mathfrak{b}^\odot)_r=\{w\in M_r(V'):\sup\|\langle\langle\mathfrak{b},w\rangle\rangle\|\leq 1\}=\{w\in M_r(V'):\pi^\circ(w)\leq 1\}=\mathcal{C}_\pi(V,M_r),$$

that is, $\underline{\pi}(v)=\sup\{\|\langle\langle v,\mathcal{C}_\pi(V,M_r)\rangle\rangle\|:r\in\mathbb{N}\}$. Similarly, using Corollary 2.3, we derive that

$$\overline{\pi}(v)=\sup\{\|\langle\langle v,(\mathfrak{b}^\circ)_r\rangle\rangle\|:r\in\mathbb{N}\}=\sup\|\langle\langle v,\mathfrak{b}^\circ\rangle\rangle\|=\sup\|\langle\langle v,\mathcal{C}_\pi(V,\mathbb{C})\rangle\rangle\|.$$

Finally, $\mathfrak{B}=\mathfrak{B}^{\odot\odot}$ due to bipolar Theorem 2.1. Using Corollary 2.3 again, we deduce that $p(v)=\sup\|\langle\langle v,\mathfrak{B}^\odot\rangle\rangle\|$. However,

$$\mathfrak{b}_r^\odot=\{w\in M_r(V'):\sup\|\langle\langle\mathfrak{B},w\rangle\rangle\|\leq 1\}=\{w\in M_r(V'):p^\odot(w)\leq 1\}=\mathcal{MB}_p(V,M_r),$$

that is, $p(v)=\sup\{\|\langle\langle v,\mathcal{MB}_p(V,M_r)\rangle\rangle\|:r\in\mathbb{N}\}$. Finally, observe that

$$\mathcal{C}_\pi(V,\mathbb{C})=\mathcal{C}_{p^{(1)}}(V,\mathbb{C})=\mathcal{MB}_p(V,\mathbb{C})\subseteq\mathcal{MB}_p(V,M_r)\subseteq\mathcal{C}_{p^{(1)}}(V,M_r)=\mathcal{C}_\pi(V,M_r)$$

for all r [see (2.5)]. □

A matrix seminorm p on a polynormed space V is said to be *continuous* if each $p^{(n)}:M_n(V)\to\mathbb{C}$ is a continuous mapping with respect to the direct product topology in $M_n(V)$.

Corollary 2.4: If p is a continuous matrix seminorm on a polynormed space V and $\pi=p^{(1)}$, then both $\underline{\pi}$ and $\overline{\pi}$ are continuous matrix seminorms on V such that $\overline{\pi}^{(1)}=\pi=\underline{\pi}^{(1)}$.

Proof: Since p is a continuous matrix seminorm on V and $\sigma(M_n(V),M_n(V'))=\sigma(V,V')^{n^2}$, it follows that its unit set $\mathfrak{B}$ is a weakly closed, absolutely matrix convex set in $M(V)$. By Proposition 2.2, $\underline{\pi}$ and $\overline{\pi}$ are matrix gauges and $\overline{\pi}\leq p\leq\underline{\pi}$. In particular, $\overline{\pi}$ is a continuous matrix seminorm. If $v=[v_{ij}]\in M_n(V)$, then

$$\underline{\pi}^{(n)}(v)=\sup_r\|\langle\langle v,\mathcal{C}_\pi(V,M_r)\rangle\rangle\|\leq\sum_{i,j=1}^n\sup_r\|\langle\langle v_{ij},\mathcal{C}_\pi(V,M_r)\rangle\rangle\|\leq\sum_{i,j=1}^n\pi(v_{ij})<\infty,$$

that is, $\underline{\pi}^{(n)}$ is a continuous seminorm on $M_n(V)$.

Finally, take $v\in V$ with $\pi(v)\neq 0$. By Hahn–Banach theorem, $|\langle v,w\rangle|=\pi(v)$ for a certain $w\in V'$ such that $|\langle x,w\rangle|\leq\pi(x)$, $x\in V$. The latter means that $w\in\mathcal{C}_\pi(V,\mathbb{C})$. In particular, $\pi(v)$

$=\bar{\pi}^{(1)}(v)$. If $w\in\mathcal{C}_\pi(V,M_r)$, then $\|\langle\langle x,w\rangle\rangle\|=\|w(x)\|\leqslant\pi(x)$ for all $x\in V$. By Proposition 2.2, $\underline{\pi}^{(1)}\times(v)=\sup_r\|\langle\langle v,\mathcal{C}_\pi(V,M_r)\rangle\rangle\|\leqslant\pi(v)$. Thus, $\bar{\pi}^{(1)}(v)=\pi(v)=\underline{\pi}^{(1)}(v)$. □

If π runs over all defining seminorms on V, we obtain the quantum spaces $(V,\{\bar{\pi}\})$ and $(V,\{\underline{\pi}\})$ denoted by min V and max V, respectively. If ς is the polynormed topology in V, then we use the notations $\min\varsigma$ and $\max\varsigma$ for the relevant quantum topologies in $M(V)$. Thus, $\min\varsigma\subseteq\mathfrak{p}\subseteq\max\varsigma$ for each quantization $\mathfrak{p}$ of ς. In particular, we have the sequence of matrix continuous linear mappings

$$\max V\to(V,\mathfrak{p})\to\min V,$$

where both arrows are the identity operators. If V is a normed space (or quantum normed space) with the norm $\pi=\|\cdot\|$, then we use the denotations $\|\cdot\|_{\min}=\bar{\pi}$ and $\|\cdot\|_{\max}=\underline{\pi}$. By Proposition 2.2, we conclude that

$$\|v\|_{\min}=\sup\|\langle\langle v,\text{ball }V^*\rangle\rangle\|\leqslant\|v\|\leqslant\|v\|_{\max}=\sup\{\|\langle\langle v,f\rangle\rangle\|:\|f:V\to M_r\|\leqslant 1,r\in\mathbb{N}\}\qquad(2.7)$$

(see Ref. 13, sec. III C).

Corollary 2.5: Let V be a quantum space and let E be a polynormed space. Then

$$\mathcal{MC}(V,\min E)=\mathcal{C}(V,E)\quad\text{and}\quad\mathcal{MC}(\max E,V)=\mathcal{C}(E,V).$$

In particular, if $\varphi:E\to F$ *is a continuous linear mapping between polynormed spaces E and F, and* $\pi\cdot\varphi\leqslant\sigma$ *for some continuous seminorms* π *and* σ *on F and E, respectively, then* $\bar{\pi}\cdot\varphi^{(\infty)}\leqslant\bar{\sigma}$ *and* $\underline{\pi}\cdot\varphi^{(\infty)}\leqslant\underline{\sigma}$, *which, in turn, implies that*

$$\mathcal{MC}(\min E,\min F)=\mathcal{C}(E,F)=\mathcal{MC}(\max E,\max F).$$

Proof: Take $\varphi\in\mathcal{C}(V,E)$ and let $\{\pi\}$ be a defining family of seminorms on E. For each π there corresponds a continuous matrix seminorm q on V such that $\pi\cdot\varphi\leqslant q^{(1)}$. If $w\in\mathcal{C}_\pi(E,\mathbb{C})$, then $|w(\varphi(v))|\leqslant\pi(\varphi(v))\leqslant q^{(1)}(v)$, $v\in V$. Since $w\cdot\varphi\in V'$, we conclude that $\|(w\cdot\varphi)^{(\infty)}(v)\|\leqslant q(v)$, $v\in M(V)$, thanks to (2.4). Using Proposition 2.2, we derive that

$$\bar{\pi}(\varphi^{(\infty)}(v))=\sup\|\langle\langle\varphi^{(\infty)}(v),\mathcal{C}_\pi(V,\mathbb{C})\rangle\rangle\|=\sup\{\|(w\cdot\varphi)^{(\infty)}(v)\|:w\in\mathcal{C}_\pi(V,\mathbb{C})\}\leqslant q(v)$$

for all $v\in M(V)$, that is, $\bar{\pi}\cdot\varphi^{(\infty)}\leqslant q$. It follows that $\varphi\in\mathcal{MC}(V,\min E)$.

Now take $\varphi\in\mathcal{C}(E,V)$ and let p be a continuous matrix seminorm on V. Then $p^{(1)}\cdot\varphi\leqslant\pi$ for a certain continuous seminorm π on E. Let us prove that $p\cdot\varphi^{(\infty)}\leqslant\underline{\pi}$. First note that if $w\in\mathcal{MB}_p(V,M_r)$, that is, $p^{\odot}(w)\leqslant 1$, then $w\cdot\varphi\in\mathcal{C}_\pi(E,M_r)$. Indeed,

$$\begin{aligned}\pi^\circ(w\cdot\varphi)&=\sup\{\|w(\varphi(v))\|:\pi(v)\leqslant 1\}\leqslant\sup\{\|\langle\langle\varphi(v),w\rangle\rangle\|:p^{(1)}(\varphi(v))\leqslant 1\}\leqslant\sup\{\|\langle\langle u,w\rangle\rangle\|:p^{(1)}(u)\\&\leqslant 1\}\leqslant\sup\{\|\langle\langle u,w\rangle\rangle\|:p(u)\leqslant 1,u\in M(V)\}=p^{\odot}(w)\leqslant 1.\end{aligned}$$

Further, take $x\in M(E)$. Using Proposition 2.2, we derive that

$$\begin{aligned}p(\varphi^{(\infty)}(x))&=\sup_r\|\langle\langle\varphi^{(\infty)}(x),\mathcal{MB}_p(V,M_r)\rangle\rangle\|=\sup_r\{\|(w\cdot\varphi)^{(\infty)}(x)\|:w\in\mathcal{MB}_p(V,M_r)\}\leqslant\sup_r\{\|u^{(\infty)}\\&\times(x)\|:u\in\mathcal{C}_\pi(E,M_r)\}=\underline{\pi}^{(\infty)}(x),\end{aligned}$$

that is, $p\cdot\varphi^{(\infty)}\leqslant\underline{\pi}$ and $\varphi\in\mathcal{MC}(\max E,V)$. In particular, $\mathcal{C}(E,F)=\mathcal{C}(\min E,F)=\mathcal{MC}(\min E,\min F)$ and $\mathcal{C}(E,F)=\mathcal{C}(E,\max F)=\mathcal{MC}(\max E,\max F)$ for polynormed spaces E and F.

Finally, assume that $\varphi:E\to F$ is a continuous linear mapping, and $\pi\cdot\varphi\leqslant\sigma$ for some continuous seminorms π and σ on F and E, respectively. Using Corollary 2.4, we derive that $\pi\cdot\varphi\leqslant\bar{\sigma}^{(1)}$, which, in turn, implies that $\bar{\pi}\cdot\varphi^{(\infty)}\leqslant\bar{\sigma}$. On the same grounds, $\underline{\pi}^{(1)}\cdot\varphi\leqslant\sigma$ implies that $\underline{\pi}\cdot\varphi^{(\infty)}\leqslant\underline{\sigma}$. □

Corollary 2.6: Let F be a polynormed space with its subspace $E\subseteq F$. *If* π *is a continuous seminorm on F, then*

$$\overline{\pi}|_{M(E)} = \overline{\pi|_E} \quad \textit{and} \quad \underline{\pi}|_{M(E)} \leqslant \underline{\pi|_E}.$$

Moreover, if there is a continuous projection $P:F\to F$ *onto* E, *and* $\pi|_E \cdot P \leqslant \sigma$ *for a certain continuous seminorm* σ *on* F, *then*

$$\underline{\pi}|_{M(E)} \leqslant \underline{\pi|_E} \leqslant \underline{\sigma}|_{M(E)}.$$

In particular, $\underline{\pi}|_{M(E)} = \underline{\pi|_E}$, *whenever* $\pi|_E \cdot P \leqslant \pi$.

Proof: Applying Corollary 2.5 to the canonical embedding $\varphi:E\to F$, we conclude that $\overline{\pi}|_{M(E)} \leqslant \pi|_E$ and $\underline{\pi}|_{M(E)} \leqslant \underline{\pi|_E}$. Furthermore, $\underline{\pi|_E} \cdot P^{(\infty)} \leqslant \underline{\sigma}$ whenever there is a projection $P:F\to E$ and $\pi|_E \cdot P \leqslant \sigma$ for a certain continuous seminorm σ on F. It follows that $\underline{\pi|_E} \cdot P^{(\infty)} \leqslant \underline{\sigma} \cdot P^{(\infty)}$ or $\underline{\pi|_E} \leqslant \underline{\sigma}|_{M(E)}$.

It remains to prove that $\overline{\pi}|_{M(E)} = \overline{\pi|_E}$. If $w \in \mathcal{C}_{\pi|_E}(E,\mathbb{C})$, then $w = u|_E$ for a certain $u \in \mathcal{C}_\pi(F,\mathbb{C})$, thanks to Hahn–Banach theorem. Using Proposition 2.2, infer that

$$\overline{\pi|_E}(v) = \sup\{\|w^{(\infty)}(v)\| : w \in \mathcal{C}_{\pi|_E}(E,\mathbb{C})\} \leqslant \sup\{\|u^{(\infty)}(v)\| : u \in \mathcal{C}_\pi(F,\mathbb{C})\} = \overline{\pi}|_{M(E)}(v)$$

for all $v \in M(E)$, that is, $\overline{\pi|_E} \leqslant \overline{\pi}|_{M(E)}$. □

Corollary 2.7: Let F *be polynormed space with its subspace* $E \subseteq F$. *Then* $\min E \subseteq \min F$ *as the quantum spaces. Moreover, if* E *is the range of a continuous projection on* F, *then* $\max E \subseteq \max F$ *as the quantum spaces. In particular, if* F *is a normed space and* E *is the range of a contractive projection on* F, *then* $\min E \subseteq \min F$ *and* $\max E \subseteq \max F$ *up to the canonical matrix isometries.*

Proof: It suffices to use Corollary 2.6. □

If V is a nuclear polynormed space, then it admits precisely one quantization. Namely, the following assertion was proved in Ref. 13, Theorem 7.3.

Theorem 2.2: *Let* V *be a nuclear polynormed space. Then* $\max V = \min V$, *that is, the matrix seminorms* $\{\underline{\pi}\}$ *and* $\{\overline{\pi}\}$ *on* $M(V)$ *are equivalent.*

Similar result for the weak topology was proved in Ref. 9.

Theorem 2.3: *Let* (V,W) *be a dual pair of linear spaces. Then the weak topology* $\sigma(V,W)$ *admits precisely one quantization called the weak quantum topology and denoted by* $\mathfrak{s}(V,W)$. *Thus,*

$$\max \sigma(V,W) = \min \sigma(V,W) = \mathfrak{s}(V,W).$$

The weak quantum topology $\mathfrak{s}(V,W)$ can be defined in terms of the explicitly written matrix seminorms. Namely, for each $w \in M(W)$ we put $p_w(v) = \|\langle\langle v,w\rangle\rangle\|$, $v \in M(V)$. As in the proof of Lemma 2.4, one can easily verify that p_w is a matrix seminorm. The family $\{p_w : w \in M(W)\}$ of matrix seminorms defines the weak quantum topology $\mathfrak{s}(V,W)$.

III. THE INDUCTIVE LIMITS OF QUANTUM NORMED SPACES

In this section we propose a family of matrix seminorms that determines the quantum topology of an inductive limit of quantum normed spaces.

A. The quantizations over a normed space class

We shall use various quantizations over all Hilbert spaces to realize a quantum domain as a quantum space. Therefore, it is convenient to postulate a quantization over a class of normed spaces.

Fix a certain class $\mathfrak{N}$ of normed spaces. We assume that if the direct sum $N \oplus K$ belongs to $\mathfrak{N}$ for some normed spaces $N,K \in \mathfrak{N}$, then it is equipped with a norm defining the direct product topology such that both canonical projections $N\oplus K \to N$ and $N\oplus K\to K$ and injections $N\to N\oplus K$ and $K\to N\oplus K$ are contractions. In particular, if $H = N\oplus K \in \mathfrak{N}$ is a Hilbert space for some $N,K\in\mathfrak{N}$, then $N\oplus K$ is the orthogonal sum of the Hilbert space N and K. Indeed, the projection $P \in \mathcal{B}(H)$ onto N along K is a contraction. Then P is the orthoprojection onto N and $K = N^\perp$ (Ref. 18, Sec. 6.2.10), that is, H is the Hilbert space sum of N and K.

By *a quantization q over the class* $\mathfrak{N}$ we mean a correspondence $N \mapsto N_q$ that converts each normed space N from $\mathfrak{N}$ into a quantum normed space N_q (with the same underline normed space N) such that all canonical injections and projections associated with a possible direct sum $N \oplus K$ in $\mathfrak{N}$ are matrix contractions, that is, we have a diagram of the canonical matrix contractions

$$\begin{tikzcd} N_q \arrow[dr] & & N_q \\ & (N \oplus K)_q \arrow[ur] \arrow[dr] & \\ K_q \arrow[ur] & & K_q \end{tikzcd}$$

In particular, all canonical injections into $(N \oplus K)_q$ are matrix isometries.

So are the min and max quantizations over all normed spaces as follows from Corollaries 2.7 and 2.6. The other examples can be delivered by the column quantization c (row quantization r) over the class of all Hilbert spaces, which converts each Hilbert space H into the column Hilbert operator space H_c (row Hilbert space H_r) (Ref. 13, Sec. III D). Let us recall these constructions briefly. If $\zeta = [\zeta_{ij}] \in M_{m,n}(H)$, then it can be written as the matrices

$$\zeta = [\zeta_1 \quad \cdots \quad \zeta_n] \quad \text{and} \quad \zeta = \begin{bmatrix} \zeta^{(1)} \\ \vdots \\ \zeta^{(m)} \end{bmatrix}$$

of columns and rows, respectively. So, $\zeta_i = (\zeta_{ki}) \in H^m$ and $\zeta^{(j)} = (\zeta_{jk}) \in H^n$ for all i, j. Consider the matrices $[\langle \zeta_j, \zeta_i \rangle_{H^m}] \in M_n$ and $[\langle \zeta^{(i)}, \zeta^{(j)} \rangle_{H^n}] \in M_m$. The matrix norms $\|\zeta\|_c$ and $\|\zeta\|_r$ are defined (see Ref. 13, Sec. III D) by the following ways:

$$\|\zeta\|_c = \|[\langle \zeta_j, \zeta_i \rangle_{H^m}]\|^{1/2} \quad \text{and} \quad \|\zeta\|_r = \|[\langle \zeta^{(i)}, \zeta^{(j)} \rangle_{H^n}]\|^{1/2}. \tag{3.1}$$

In particular, if $\zeta = \sum_{h=1}^{p} \alpha^{(h)} \otimes e_h \in M_n(H)$ for some $\alpha^{(h)} \in M_n$ and orthonormal vectors $\{e_h\} \subseteq H$, then (see Ref. 13, Sec. III D)

$$\|\zeta\|_c = \left\| \begin{bmatrix} \alpha^{(1)} \\ \vdots \\ \alpha^{(p)} \end{bmatrix} \right\| \quad \text{and} \quad \|\zeta\|_r = \|[\alpha^{(1)} \quad \cdots \quad \alpha^{(p)}]\|.$$

Now if $K \subseteq H$ is a Hilbert space inclusion, then $K_c \subseteq H_c$ and $K_r \subseteq H_r$ are quantum normed space inclusions as follows from (3.1). Further, $\mathcal{B}(H,K) = \mathcal{MB}(H_c, K_c)$ up to the natural matrix isometry, thanks to Ref. 13, Theorem 3.4.1. In particular, if $P \in \mathcal{B}(H,K)$ is the projection onto K, then $\|P\| \leqslant 1$, which, in turn, implies that $P \in \text{ball } \mathcal{MB}(H_c, K_c)$. Similarly, the conjugate mapping implements the matrix isometry $\mathcal{B}(K^*, H^*) = \mathcal{MB}(H_r, K_r)$, thanks to Ref. 13, Proposition 3.4.2. It follows that $\|P : H_r \to K_r\|_{mb} = \|P^*\| = \|P\| \leqslant 1$. Whence c and r are quantizations over all Hilbert spaces.

Now let $V = \oplus_{\iota \in \Xi} V_\iota$ be a polynormed direct sum of normed spaces $\{V_\iota : \iota \in \Xi\}$ from a certain class $\mathfrak{N}$. We denote the set of all finite subsets of Ξ by Λ, and assume that each finite direct sum $V_\alpha = \oplus_{\iota \in \alpha} V_\iota$ belongs to the class $\mathfrak{N}$, where $\alpha \in \Lambda$. It is well known (Ref. 23, Sec. 2.6) that $V = \sum_{\alpha \in \Lambda} V_\alpha = \cup_{\alpha \in \Lambda} V_\alpha$ is the inductive limit of the upward filtered family $\{V_\alpha : \alpha \in \Lambda\}$ of normed spaces, that is, $V = \lim_{\to} \{V_\alpha : \alpha \in \Lambda\}$. If q is a quantization over the class $\mathfrak{N}$, then we put

$$V_q = \operatorname{op} \bigoplus_{\iota \in \Xi} V_{\iota,q}$$

to indicate the quantum (or local operator) direct sum of the quantum normed spaces $V_{\iota,q}$, $\iota \in \Xi$. Recall that V_q is equipped with the finest quantum topology such that all inclusions $V_{\iota,q} \to V_q$ are matrix continuous.

Proposition 3.1: Let q be a quantization over a normed space class $\mathfrak{N}$. *If* $V = \oplus_{\iota \in \Xi} V_\iota$ *is a* *direct sum of normed spaces* $\{V_\iota : \iota \in \Xi\}$ *from the class* $\mathfrak{N}$ *and all* $V_\alpha \in \mathfrak{N}$, $\alpha \in \Lambda$, *then*

$$V_q = \mathrm{op} \bigoplus_{\iota \in \Xi} V_{\iota,q} = \underset{\rightarrow}{\mathrm{oplim}}\{V_{\alpha,q} : \alpha \in \Lambda\},$$

where Λ *is the set of all finite subset of* Ξ *and* $V_\alpha = \oplus_{\iota \in \alpha} V_\iota$.

Proof: Let $W = \mathrm{oplim}_{\rightarrow}\{V_{\alpha,q} : \alpha \in \Lambda\}$ be a quantum space. Since each canonical embedding $V_{\iota,q} = V_{\{\iota\},q} \to W$ is matrix continuous, it follow that the quantum topology in V_q is finer than the quantum topology of W. To prove the reverse statement it suffices to observe that any linear mapping $f : V_{\alpha,q} \to X$ into a quantum space X whose restrictions $f_\iota : V_{\iota,q} \to X$, $\iota \in \alpha$, are matrix continuous is automatically matrix continuous. If p is a continuous matrix seminorm on X, then $p(f_\iota^{(\infty)} v_\iota) \leq C_\iota \|v_\iota\|_{\iota,q}$, $v_\iota \in M(V_{\iota,q})$ for some positive real C_ι, $\iota \in \alpha$, where $\|\cdot\|_{\iota,q}$ is the matrix norm on $V_{\iota,q}$. Take $v \in M_n(V_{\alpha,q}) = \oplus_{\iota \in \alpha} M_n(V_{\iota,q})$. Then $v = \oplus_{\iota \in \alpha} P_\iota^{(n)}(v)$, where each $P_\iota : V_\alpha \to V_\alpha$ is the canonical (contractive) projection onto V_ι. By assumption, V_α can be equipped by any norm such that all canonical projections are contractions. Since q is a quantization over the class $\mathfrak{N}$, it follows that $V_{\iota,q} \subseteq V_{\alpha,q}$ up to a matrix isometry and $P_\iota \in \mathrm{ball}\, \mathcal{MB}(V_{\alpha,q})$, $\iota \in \alpha$. Then

$$\begin{aligned} p(f^{(\infty)} v) &\leq \sum_{\iota \in \alpha} p(f^{(\infty)} P_\iota^{(\infty)}(v)) = \sum_{\iota \in \alpha} p(f_\iota^{(\infty)} P_\iota^{(\infty)}(v)) \leq \sum_{\iota \in \alpha} C_\iota \|P_\iota^{(\infty)}(v)\|_{\iota,q} = \sum_{\iota \in \alpha} C_\iota \|P_\iota^{(\infty)}(v)\|_{\alpha,q} \\ &\leq C_\alpha \|v\|_{\alpha,q}, \end{aligned}$$

where $C_\alpha = \Sigma_{\iota \in \alpha} C_\iota$. Consequently, $f : V_{\alpha,q} \to X$ is matrix continuous. Thereby, $W = V_q$. □

Corollary 3.1: Let X *be a quantum normed space and let* $V = \oplus_{\iota \in \Xi} V_\iota \subseteq X$ *be a subspace spanned as a direct sum by some subspaces* $V_\iota \subseteq X$, $\iota \in \Xi$. *Then all spaces* $V_\alpha = \oplus_{\iota \in \alpha} V_\iota$, $\alpha \in \Lambda$, *being subspaces of* X *turn into the quantum normed spaces. If all canonical projections* $V_\alpha \to V_\beta$, $\beta \subseteq \alpha$, $\alpha, \beta \in \Lambda$, *are matrix contractions, then*

$$\mathrm{op} \bigoplus_{\iota \in \Xi} V_\iota = \underset{\rightarrow}{\mathrm{oplim}}\{V_\alpha : \alpha \in \Lambda\}.$$

Proof: Let $\mathfrak{N} = \{V_\alpha : \alpha \in \Lambda\}$ be a class of normed spaces. Since all these spaces are subspaces of a quantum normed space X and all canonical projections are matrix contractions, we have a quantization q over the class $\mathfrak{N}$ which assigns to each V_α the same space but as a quantum (normed) subspace of X denoted by $V_{\alpha,q}$. It remains to apply Proposition 3.1. □

B. The matrix seminorms on the quantum inductive limit

Let $V = \Sigma_{\alpha \in \Lambda} V_\alpha$ be a linear space which is spanned by a family of its linear subspaces $\{V_\alpha : \alpha \in \Lambda\}$ such that each V_α is a quantum normed space, whose matrix norm is denoted by $\|\cdot\|_\alpha$. Then V being an inductive limit of the quantum normed spaces V_α, $\alpha \in \Lambda$, turns out to be a quantum space. For each $\rho = (\rho_\alpha)_{\alpha \in \Lambda} \in \mathbb{R}_+^\Lambda$, we put

$$\mathfrak{B}_\rho = \mathrm{amc} \bigcup_{\alpha \in \Lambda} \rho_\alpha \, \mathrm{ball}\, M(V_\alpha),$$

which is a matrix set in $M(V)$.

Lemma 3.1: The family $\{\mathfrak{B}_\rho : \rho \in \mathbb{R}_+^\Lambda\}$ *is a neighborhood filter base for the inductive quantum topology in* $M(V)$. *Moreover*,

$$\mathfrak{B}_\rho = \left\{ v = \sum_{i=1}^s \lambda_i v_{\alpha_i} \mu_i : \sum_{i=1}^s \lambda_{\rho,i} \lambda_{\rho,i}^* \leq 1, v_{\alpha_i} \in \mathrm{ball}\, M(V_{\alpha_i}), \sum_{i=1}^s \mu_{\rho,i}^* \mu_{\rho,i} \leq 1 \right\},$$

where λ_i, $\mu_i \in M$, *and* $\lambda_{\rho,i} = \rho_{\alpha_i}^{-1/2} \lambda_i$, $\mu_{\rho,i} = \rho_{\alpha_i}^{-1/2} \mu_i$.

Proof: First note $\mathfrak{B}_\rho$ consists of matrix combinations $v = \Sigma_s a_s v_s b_s$ with $v_s \in \rho_{\alpha_s} \mathrm{ball}\, M(V_{\alpha_s})$ and $\Sigma_s a_s a_s^* \leq 1$, $\Sigma_s b_s^* b_s \leq 1$, thanks to Lemma 2.1. Then $v = \Sigma_i \Sigma_{s \in F_i} a_s v_s b_s$ with $v_s \in \rho_{\alpha_i} \mathrm{ball}\, M(V_{\alpha_i})$ whenever $s \in F_i$. If $w_{\alpha_i} = \oplus_{s \in F_i} v_s \in M(V_{\alpha_i})$ then $\|w_{\alpha_i}\|_{\alpha_i} = \max_{s \in F_i} \|v_s\|_{\alpha_i} \leq \rho_{\alpha_i}$. Moreover, $\Sigma_{s \in F_i} a_s v_s b_s = a_i w_{\alpha_i} b_i$ with

$$a_i=[\cdots \quad a_s \quad \cdots], \quad b_i=\begin{bmatrix}\vdots\\ b_s\\ \vdots\end{bmatrix}, \quad s\in F_i.$$

It follows that $v=\Sigma_i a_i w_{\alpha_i} b_i$, $w_{\alpha_i}\in \rho_{\alpha_i}\mathrm{ball}\, M(V_{\alpha_i})$ and $\Sigma_i a_i a_i^*=\Sigma_i\Sigma_{s\in F_i} a_s a_s^*\leqslant 1$, $\Sigma_i b_i^* b_i=\Sigma_i\Sigma_{s\in F_i} b_s^* b_s\leqslant 1$. It remains to put $v_{\alpha_i}=\rho_{\alpha_i}^{-1} w_{\alpha_i}$, $\lambda_i=\rho_{\alpha_i}^{1/2} a_i$ and $\mu_i=\rho_{\alpha_i}^{1/2} b_i$. Conversely, each indicated sum belongs to $\mathfrak{B}_\rho$, for the latter set is absolutely matrix convex.

Further, note that $\mathfrak{B}_\rho$ is an absorbing set in $M(V)$. Indeed, if $x=\Sigma_{i=1}^s x_{\alpha_i}\in M(V)$ with $x_{\alpha_i}\in M(V_{\alpha_i})\backslash\{0\}$, and $\lambda_i=\mu_i=s^{-1/2}\delta^{1/2}\|x_{\alpha_i}\|_{\alpha_i}^{1/2}$, then $s^{-1}\delta x=\Sigma_{i=1}^s\lambda_i\|x_{\alpha_i}\|_{\alpha_i}^{-1}x_{\alpha_i}\mu_i\in\mathfrak{B}_\rho$ whenever $0<\delta\leqslant\min_i\rho_{\alpha_i}\|x_{\alpha_i}\|_{\alpha_i}^{-1}$. Thus $\{\mathfrak{B}_\rho:\rho\in\mathbb{R}_+^\Lambda\}$ is a neighborhood filter base of a certain quantum topology in $M(V)$, say $\mathfrak{t}$.

Since $\rho_\alpha\mathrm{ball}\, M(V_\alpha)\subseteq\mathfrak{B}_\rho\cap M(V_\alpha)$ for all α, it follows that all inclusions $V_\alpha\subseteq(V,\mathfrak{t})$ are matrix continuous. In particular, the inductive quantum topology in $M(V)$ is finer than $\mathfrak{t}$. Conversely, if $\mathfrak{A}$ is an absolutely matrix convex neighborhood in $M(V)$ with respect to the inductive quantum topology then $\delta_\alpha\mathrm{ball}\, M(V_\alpha)\subseteq\mathfrak{A}\cap M(V_\alpha)$ for some positive δ_α, $\alpha\in\Lambda$. Hence, $\cup_\alpha\delta_\alpha\mathrm{ball}\, M(V_\alpha)\subseteq\mathfrak{A}$, which, in turn, implies that $\mathfrak{B}_\rho\subseteq\mathfrak{A}$ by its very definition. Thus, $\mathfrak{t}$ is the inductive quantum topology. □

Remark 3.1: Note that the polynormed sum $M(V)=\Sigma_{\alpha\in\Lambda}M(V_\alpha)$ *has a neighborhood filter base given by the family of absolutely convex hulls* $\mathrm{abc}\cup_{\alpha\in\Lambda}\rho_\alpha\mathrm{ball}\, M(V_\alpha)$, $\rho=(\rho_\alpha)_{\alpha\in\Lambda}\in\mathbb{R}_+^\Lambda$. *In particular, the inductive quantum topology is coarser than the classical inductive polynormed topology in* $M(V)$ *(see Remark 2.1).*

Corollary 3.2: Let q be a quantization over a normed space class $\mathfrak{N}$. *If* $V=\oplus_{\iota\in\Xi}V_\iota$ *is a direct sum of normed spaces* $\{V_\iota:\iota\in\Xi\}$ *from the class* $\mathfrak{N}$ *and all* $V_\alpha\in\mathfrak{N}$, $\alpha\in\Lambda$, *then any matrix bounded set in the quantum space* V_q *is contained in a certain* $M(V_{\alpha,q})$ *and it is matrix bounded there.*

Proof: Let $\mathfrak{M}$ be a matrix bounded set in $M(V_q)$. Note that $M(V_q)=\oplus_{\iota\in\Xi}M(V_{\iota,q})$ is an algebraic direct sum equipped with the inductive quantum topology. Let us prove that there is a finite subset $\alpha\subseteq\Xi$ such that $Q_\kappa^{(\infty)}(\mathfrak{M})=0$ for all $\kappa\notin\alpha$, and $Q_\iota^{(\infty)}(\mathfrak{M})$ is matrix bounded in $M(V_{\iota,q})$ for all $\iota\in\alpha$, where $Q_\iota:V\to V$ is the canonical projection onto V_ι. If that is not the case then there is a sequence $\{v^{(n)}\}$ in $\mathfrak{M}$ such that $v_{\kappa_n}^{(n)}=Q_{\kappa_n}^{(\infty)}(v^{(n)})\neq 0$, $n\in\mathbb{N}$ for infinitely many different indices $\{\kappa_n\}$ in Ξ. Take $\rho\in\mathbb{R}_+^\Xi$ with $\rho_{\kappa_n}<\|n^{-1}v_{\kappa_n}^{(n)}\|_{\kappa_n}$, $n\in\mathbb{N}$. If $n^{-1}v^{(n)}\in\mathfrak{B}_\rho$ for some n, then $n^{-1}v^{(n)}=\Sigma_\iota a_\iota v_\iota b_\iota$ with $v_\iota\in\rho_\iota\mathrm{ball}\, M(V_\iota)$ and $\Sigma_\iota a_\iota a_\iota^*\leqslant 1$, $\Sigma_\iota b_\iota^* b_\iota\leqslant 1$, thanks to Lemma 3.1. Since we deal with the direct sum, we conclude that $n^{-1}v_{\kappa_n}^{(n)}=a_{\kappa_n}v_{\kappa_n}b_{\kappa_n}$. It follows that

$$\|n^{-1}v_{\kappa_n}^{(n)}\|_{\kappa_n}\leqslant\|a_{\kappa_n}\|\|v_{\kappa_n}\|_{\kappa_n}\|b_{\kappa_n}\|\leqslant\rho_{\kappa_n}\|a_{\kappa_n}a_{\kappa_n}^*\|^{1/2}\|b_{\kappa_n}^*b_{\kappa_n}\|^{1/2}\leqslant\rho_{\kappa_n}\left\|\sum_\iota a_\iota a_\iota^*\right\|^{1/2}\left\|\sum_\iota b_\iota^* b_\iota\right\|^{1/2},$$

$$\leqslant\rho_{\kappa_n},$$

a contradiction. Hence, $\mathfrak{B}\subseteq M(V_{\alpha,q})$ for some $\alpha\in\Lambda$. Finally, taking into account that $\mathfrak{B}_\rho\cap M(V_\iota)=\rho_\iota\mathrm{ball}\, M(V_\iota)$, $\iota\in\Xi$, we conclude that all projections Q_ι are matrix continuous. In particular, $Q_\iota^{(\infty)}(\mathfrak{M})$ is matrix bounded in $M(V_{\iota,q})$ for all $\iota\in\alpha$. □

The Minkowski functional p_ρ of the neighborhood $\mathfrak{B}_\rho$ (see Proposition 2.1) is a matrix seminorm on $M(V)$, and the family $\{p_\rho:\rho\in\mathbb{R}_+^\Lambda\}$ is a defining family of matrix seminorms on V.

Each expansion $v=\Sigma_{i=1}^s\lambda_i v_{\alpha_i}\mu_i\in M_n(V)$ indicated in Lemma 3.1 associates the matrix tuples

$$\lambda=(\lambda_1,\ldots,\lambda_s)\in\prod_{i=1}^s M_{n,k_i},\quad \mu=(\mu_1,\ldots,\mu_s)\in\prod_{i=1}^s M_{k_i,n}$$

and the following matrix (see Sec. II C)

$$A_{\lambda,\rho,\mu}=\begin{bmatrix}\lambda_{\rho,1} & 0 & \cdots & \lambda_{\rho,s} & 0\\ 0 & \mu^*_{\rho,1} & & 0 & \mu^*_{\rho,s}\end{bmatrix}\in M_{2n,2k}, \tag{3.2}$$

where $\lambda_{\rho,i}=\rho_{\alpha_i}^{-1/2}\lambda_i$, $\mu_{\rho,i}=\rho_{\alpha_i}^{-1/2}\mu_i$, and $k=\Sigma_{i=1}^s k_i$.

Proposition 3.2: If $\rho\in\mathbb{R}_+^\Lambda$ *and* $v\in M(V)$, *then*

$$p_\rho(v)^{1/2}=\inf\left\{\|A_{\lambda,\rho,\mu}\| : v=\sum_{i=1}^s \lambda_i v_{\alpha_i}\mu_i,\quad \lambda_i,\mu_i\in M,\quad v_{\alpha_i}\in \text{ball } M_{k_i}(V_{\alpha_i})\right\}.$$

Proof: By its very definition $p_\rho(v)=\inf\{t>0 : t^{-1}v\in\mathfrak{B}_\rho\}$. By Lemma 3.1, we have $p_\rho(v)=\inf\{t>0 : v=\Sigma_{i=1}^s\lambda_i v_{\alpha_i}\mu_i, \Sigma_{i=1}^s\lambda_{\rho,i}\lambda^*_{\rho,i}\leq t, \|v_{\alpha_i}\|_{\alpha_i}\leq 1, \Sigma_{i=1}^s\mu^*_{\rho,i}\mu_{\rho,i}\leq t\}$. Therefore, $p_\rho(v)=\inf\{\Delta_{\lambda,v,\mu} : v=\Sigma_{i=1}^s\lambda_i v_{\alpha_i}\mu_i, \|v_{\alpha_i}\|_{\alpha_i}\leq 1\}$, where $\Delta_{\lambda,v,\mu}=\max\{\|\Sigma_{i=1}^s\lambda_{\rho,i}\lambda^*_{\rho,i}\|, \|\Sigma_{i=1}^s\mu^*_{\rho,i}\mu_{\rho,i}\|\}$. However,

$$\Delta_{\lambda,v,\mu}=\left\|\sum_{i=1}^s\begin{bmatrix}\lambda_{\rho,i}\lambda^*_{\rho,i} & 0\\ 0 & \mu^*_{\rho,i}\mu_{\rho,i}\end{bmatrix}\right\|=\left\|\sum_{i=1}^s\begin{bmatrix}\lambda_{\rho,i} & 0\\ 0 & \mu^*_{\rho,i}\end{bmatrix}\begin{bmatrix}\lambda_{\rho,i} & 0\\ 0 & \mu^*_{\rho,i}\end{bmatrix}^*\right\|=\left\|\begin{bmatrix}\cdots & \lambda_{\rho,i} & 0 & \cdots\\ & 0 & \mu^*_{\rho,i} & \end{bmatrix}\right\|^2,$$

that is, $\Delta_{\lambda,v,\mu}=\|A_{\lambda,\rho,\mu}\|^2$. The rest is clear. □

Corollary 3.3: If $\rho\in\mathbb{R}_+^\Lambda$, *then* $p_\rho^{(1)}=\sigma_\rho$ *(see Sec. II B). In particular, for each n the family* $\{p_\rho^{(n)} : \rho\in\mathbb{R}_+^\Lambda\}$ *on* $M_n(V)$ *determines the inductive polynormed topology* $M_n(V)=\Sigma_{\alpha\in\Lambda}M_n(V_\alpha)$.

Proof: Take $v\in V$. By Lemma 2.3, $\sigma_\rho(v)=\inf_F\{\Sigma_{\alpha\in F}\rho_\alpha^{-1}\|v_\alpha\|_\alpha : v=\Sigma_{\alpha\in F}v_\alpha\}$. Calculate the norm of the matrix (3.2) associated with the expansion $v=\Sigma_{i=1}^s\|v_{\alpha_i}\|^{1/2}(\|v_{\alpha_i}\|^{-1}v_{\alpha_i})\|v_{\alpha_i}\|^{1/2}$, $v_{\alpha_i}\in V_{\alpha_i}\setminus\{0\}$. In this case $\lambda_{\rho,i}=\mu_{\rho,i}=\|v_{\alpha_i}\|^{1/2}\rho_{\alpha_i}^{-1/2}$ for all i. Then

$$\|A_{\lambda,\rho,\mu}\|^2=\left\|\begin{bmatrix}\lambda_{\rho,1} & 0 & \cdots & \lambda_{\rho,s} & 0\\ 0 & \mu^*_{\rho,1} & & 0 & \mu^*_{\rho,s}\end{bmatrix}\right\|^2=\left\|\begin{bmatrix}\cdots & \|v_{\alpha_i}\|^{1/2}\rho_{\alpha_i}^{-1/2} & 0 & \cdots\\ & 0 & \|v_{\alpha_i}\|^{1/2}\rho_{\alpha_i}^{-1/2} & \end{bmatrix}\right\|^2$$
$$=\left\|\begin{bmatrix}\sum_{i=1}^s\rho_{\alpha_i}^{-1}\|v_{\alpha_i}\| & 0\\ 0 & \sum_{i=1}^s\rho_{\alpha_i}^{-1}\|v_{\alpha_i}\|\end{bmatrix}\right\|=\sum_{i=1}^s\rho_{\alpha_i}^{-1}\|v_{\alpha_i}\|.$$

In particular, $p_\rho^{(1)}(v)\leq\sigma_\rho(v)$, thanks to Proposition 3.2. Conversely, take $\varepsilon>0$ and an expansion $v=\Sigma_{i=1}^s\lambda_i v_{\alpha_i}\mu_i$ with $\lambda_i\in M_{1,k_i}$, $v_{\alpha_i}\in\text{ball}M_{k_i}(V_{\alpha_i})$ and $\mu_i\in M_{k_i,1}$ such that $\|A_{\lambda,\rho,\mu}\|^2<p_\rho^{(1)}(v)+\varepsilon$. Then

$$\sigma_\rho(v)\leq\sum_{i=1}^s\rho_{\alpha_i}^{-1}\|\lambda_i v_{\alpha_i}\mu_i\|_{\alpha_i}\leq\sum_{i=1}^s\rho_{\alpha_i}^{-1}\|\lambda_i\|\|v_{\alpha_i}\|_{\alpha_i}\|\mu_i\|\leq\sum_{i=1}^s\|\lambda_{\rho,i}\|\|\mu_{\rho,i}\|\leq\frac{1}{2}\sum_{i=1}^s\|\lambda_{\rho,i}\|^2+\|\mu_{\rho,i}\|^2$$
$$\leq\max\left\{\sum_{i=1}^s\|\lambda_{\rho,i}\|^2,\sum_{i=1}^s\|\mu_{\rho,i}\|^2\right\}=\max\left\{\sum_{i=1}^s\lambda_{\rho,i}\lambda^*_{\rho,i},\sum_{i=1}^s\mu^*_{\rho,i}\mu_{\rho,i}\right\}$$
$$=\left\|\begin{bmatrix}\sum_{i=1}^s\lambda_{\rho,i}\lambda^*_{\rho,i} & 0\\ 0 & \sum_{i=1}^s\mu^*_{\rho,i}\mu_{\rho,i}\end{bmatrix}\right\|=\left\|\begin{bmatrix}\lambda_{\rho,1} & 0 & \cdots & \lambda_{\rho,s} & 0\\ 0 & \mu^*_{\rho,1} & & 0 & \mu^*_{\rho,s}\end{bmatrix}\right\|^2=\|A_{\lambda,\rho,\mu}\|^2<p_\rho^{(1)}(v)+\varepsilon.$$

Consequently, $\sigma_\rho(v)\leq p_\rho^{(1)}(v)$.

As we have observed in Sec. II D, the polynormed topology on $M_n(V)$ associated with $\{p_\rho^{(n)} : \rho\in\mathbb{R}_+^\Lambda\}$ is just the direct product topology in V^{n^2}, where V is equipped with the defining family $\{p_\rho^{(1)} : \rho\in\mathbb{R}_+^\Lambda\}$ of seminorms [see (2.2)]. As we have just proved the latter family determines the original inductive polynormed topology generated by $\{\sigma_\rho : \rho \in \mathbb{R}_+^\Lambda\}$. It remains to use Corollary 2.1. □

It can independently be proved that the seminorms given as in Proposition 3.2 are matrix seminorms on V. Namely, the following assertion is valid.

Proposition 3.3: Let $\rho \in \mathbb{R}_+^\Lambda$ and for each $v \in M(V)$ define

$$q_\rho(v) = \inf\left\{ \|A_{\lambda,\rho,\mu}\|^2 : v = \sum_{i=1}^{s} \lambda_i v_{\alpha_i} \mu_i, \quad \lambda_i, \mu_i \in M, \quad v_{\alpha_i} \in \text{ball } M(V_{\alpha_i}) \right\}.$$

Then q_ρ is a matrix seminorm on V.

Proof: First note that $q_\rho(rv) = rq_\rho(v)$ for each positive real r and $v \in M(V)$. Take an expansion $rv = \Sigma_{i=1}^s \lambda_i v_{\alpha_i} \mu_i$ with $\lambda_i, \mu_i \in M$, $v_{\alpha_i} \in \text{ball } M(V_{\alpha_i})$. Then $v = \Sigma_{i=1}^s \lambda_i' v_{\alpha_i} \mu_i'$ with $\lambda_i' = r^{-1/2}\lambda_i$, $\mu_i' = r^{-1/2}\mu_i$, and $\lambda_{\rho,i} = \rho_{\alpha_i}^{-1/2}\lambda_i = r^{1/2}\lambda_{\rho,i}'$, $\mu_{\rho,i} = \rho_{\alpha_i}^{-1/2}\mu_i = r^{1/2}\mu_{\rho,i}'$ for all i. Moreover,

$$A_{\lambda,\rho,\mu} = r^{1/2}\begin{bmatrix} \lambda_{\rho,1}' & 0 & \cdots & \lambda_{\rho,s}' & 0 \\ 0 & \mu_{\rho,1}'^{*} & & 0 & \mu_{\rho,s}'^{*} \end{bmatrix} = r^{1/2} A_{\lambda',\rho,\mu'}.$$

In particular, $q_\rho(rv) = \inf\{\|r^{1/2}A_{\lambda',\rho,\mu'}\|^2 : v = \Sigma_{i=1}^s \lambda_i' v_{\alpha_i} \mu_i'\} = rq_\rho(v)$.

Now take nonzero matrices $a \in M_{m,n}$, $v \in M_n(V)$ and $b \in M_{n,m}$ and prove that $q_\rho(avb) \le \|a\| q_\rho(v) \|b\|$. Since $q_\rho(avb) = \|a\| q_\rho(\|a\|^{-1} av \|b\|^{-1} b)\|b\|$, one may assume that $\|a\| = \|b\| = 1$. Take an expansion $v = \Sigma_{i=1}^s \lambda_i v_{\alpha_i} \mu_i$ with $\|A_{\lambda,\rho,\mu}\|^2 < q_\rho^{(n)}(v) + \varepsilon$. Then $avb = \Sigma_{i=1}^s \lambda_i' v_{\alpha_i} \mu_i'$ with $\lambda_i' = a\lambda_i$ and $\mu_i' = \mu_i b$. It follows that

$$q_\rho(avb) \le \left\| \begin{bmatrix} \lambda_{\rho,1}' & 0 & \cdots & \lambda_{\rho,s}' & 0 \\ 0 & \mu_{\rho,1}'^{*} & & 0 & \mu_{\rho,s}'^{*} \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} a & 0 \\ 0 & b^* \end{bmatrix} \begin{bmatrix} \lambda_{\rho,1} & 0 & \cdots & \lambda_{\rho,s} & 0 \\ 0 & \mu_{\rho,1}^{*} & & 0 & \mu_{\rho,s}^{*} \end{bmatrix} \right\|^2$$
$$\le \max\{\|a\|, \|b\|\}^2 \|A_{\lambda,\rho,\mu}\|^2 \le \|A_{\lambda,\rho,\mu}\|^2 < q_\rho(v) + \varepsilon,$$

that is, $q_\rho(avb) \le q_\rho(v)$. So, we have the property **M2** (see Sec. II D).

It remains to prove the property **M1**, that is, $q_\rho(v \oplus w) \le \max\{q_\rho(v), q_\rho(w)\}$. Take expansions $v = \Sigma_{i=1}^s \lambda_i v_{\alpha_i} \mu_i$, $v_{\alpha_i} \in \text{ball} M_{k_i}(V_{\alpha_i})$, and $w = \Sigma_{i=1}^s \lambda_i' w_{\alpha_i} \mu_i'$, $w_{\alpha_i} \in \text{ball} M_{l_i}(V_{\alpha_i})$ such that $\|A_{\lambda,\rho,\mu}\|^2 < q_\rho(v) + \varepsilon$ and $\|A_{\lambda',\rho,\mu'}\|^2 < q_\rho(w) + \varepsilon$. Then $v \oplus w = \Sigma_{i=1}^s \lambda_i'' u_{\alpha_i} \mu_i''$ is the same type expansion with

$$\lambda_i'' = \begin{bmatrix} \lambda_i & 0 \\ 0 & \lambda_i' \end{bmatrix} \in M, \quad u_{\alpha_i} = \begin{bmatrix} v_{\alpha_i} & 0 \\ 0 & w_{\alpha_i}' \end{bmatrix} \in \text{ball } M(V_{\alpha_i}), \quad \mu_i'' = \begin{bmatrix} \mu_i & 0 \\ 0 & \mu_i' \end{bmatrix} \in M.$$

Hence the matrix (3.2) associated with the latter expansion of $v \oplus w$ is just $A_{\lambda\oplus\lambda',\rho,\mu\oplus\mu'}$ (see Sec. II A). Using Lemma 2.2, infer that $q_\rho(v \oplus w) \le \|A_{\lambda\oplus\lambda',\rho,\mu\oplus\mu'}\|^2 = \max\{\|A_{\lambda,\rho,\mu}\|^2, \|A_{\lambda',\rho,\mu'}\|^2\} < \max\{q_\rho(v), q_\rho(w)\} + \varepsilon$, that is, $q_\rho(v \oplus w) \le \max\{q_\rho(v), q_\rho(w)\}$. In particular (see Ref. 13, Sec. 2.3.6), all $q_\rho^{(n)}$ are seminorms on $M_n(V)$, respectively. □

C. The linear space $\mathcal{MC}_\Lambda(V,W)$

Now assume that $V = \Sigma_{\alpha\in\Lambda} V_\alpha$ and $W = \Sigma_{\alpha\in\Lambda} W_\alpha$ are inductive limits of the quantum normed spaces V_α and W_α, respectively. We introduce the following subspace

$$\mathcal{MC}_\Lambda(V,W) = \{T \in \mathcal{MC}(V,W) : T(V_\alpha) \subseteq W_\alpha, \quad \alpha \in \Lambda\}$$

in the space $\mathcal{MC}(V,W)$ of all matrix continuous linear mappings acting from V into W. By Proposition 3.2, we have the family of matrix seminorms $\{p_\rho : \rho \in \mathbb{R}_+^\Lambda\}$ on V defining its quantum topology. For the same family on W, we use the notation $\{q_\rho : \rho \in \mathbb{R}_+^\Lambda\}$.

Proposition 3.4: Let $T : V \to W$ be a linear mapping such that $T(V_\alpha) \subseteq W_\alpha$ and $T|V_\alpha \in \mathcal{MB}(V_\alpha, W_\alpha)$ for all $\alpha \in \Lambda$. Then $T \in \mathcal{MC}_\Lambda(V,W)$.

Proof: Fix $\rho \in \mathbb{R}_+^\Lambda$. Consider $\theta = (\theta_\alpha)_{\alpha\in\Lambda} \in \mathbb{R}_+^\Lambda$ with $\theta_\alpha = \rho_\alpha \| T|V_\alpha\|_{mb}^{-1}$ if $T|V_\alpha \ne 0$, and θ_α is arbitrary otherwise. Take $v \in M_n(V)$ and consider its expansion $v = \Sigma_{i=1}^s \lambda_i v_{\alpha_i} \mu_i$ for some λ_i

$\in M_{n,k_i}$, $v_{\alpha_i}\in \mathrm{ball} M_{k_i}(V_{\alpha_i})$, $\mu_i\in M_{k_i,n}$, $1\leq i\leq s$. Then $T^{(n)}(v)=\Sigma_{i=1}^s\lambda_i T^{(k_i)}(v_{\alpha_i})\mu_i=\Sigma_{i=1}^s\lambda_i' w_{\alpha_i}\mu_i'$ with $\lambda_i'=\lambda_i\delta_{\alpha_i}^{-1/2}$, $\mu_i'=\delta_{\alpha_i}^{-1/2}\mu_i$ and $w_{\alpha_i}=\delta_{\alpha_i}T^{(k_i)}(v_{\alpha_i})$ if $T^{(k_i)}(v_{\alpha_i})\neq 0$, where $\delta_{\alpha_i}=\|T^{(k_i)}(v_{\alpha_i})\|_{\alpha_i}^{-1}$. If $T^{(k_i)}\times(v_{\alpha_i})=0$, then we put $w_{\alpha_i}=0$ and $\lambda_i'=\lambda_i\delta_{\alpha_i}^{-1/2}$, $\mu_i'=\mu_i\delta_{\alpha_i}^{-1/2}$ with $\delta_{\alpha_i}\geq\rho_{\alpha_i}^{-1}\theta_{\alpha_i}$. Note that $(\rho_{\alpha_i}\delta_{\alpha_i})^{-1}=\rho_{\alpha_i}^{-1}\|T^{(k_i)}(v_{\alpha_i})\|_{\alpha_i}\leq\rho_{\alpha_i}^{-1}\|T^{(k_i)}|M_{k_i}(V_{\alpha_i})\|\leq\rho_{\alpha_i}^{-1}\|T|V_{\alpha_i}\|_{mb}\leq\theta_{\alpha_i}^{-1}$ if $T^{(k_i)}(v_{\alpha_i})\neq 0$. Consequently, $\rho_{\alpha_i}\delta_{\alpha_i}\geq\theta_{\alpha_i}$ for all i, $1\leq i\leq s$. Consider the tuple $\varepsilon\in\mathbb{R}_+^s$ with $\varepsilon_i=\rho_{\alpha_i}\delta_{\alpha_i}$ for all i. Using Proposition 3.2 and Lemma 2.2, we deduce that $q_\rho^{(n)}(T^{(n)}(v))\leq\|A_{\lambda',\rho,\mu'}\|^2=\|A_{\lambda,\varepsilon,\mu}\|^2\leq\|A_{\lambda,\theta,\mu}\|^2$. Using again Proposition 3.2, we conclude that $q_\rho^{(n)}(T^{(n)}(v))\leq p_\theta^{(n)}(v)$, that is, T is matrix continuous (see also Proposition 3.3). Hence, $T\in\mathcal{MC}_\Lambda(V,W)$. □

D. The direct quantum families

Let $V=\Sigma_{\alpha\in\Lambda}V_\alpha$ be an inductive limit of quantum normed spaces V_α, $\alpha\in\Lambda$. Put $\alpha\leq\beta$ whenever $V_\alpha\subseteq V_\beta$, $\alpha,\beta\in\Lambda$. In particular, Λ is a partially ordered set. We say that $\{V_\alpha:\alpha\in\Lambda\}$ is *a direct quantum family* if Λ is an upward filtered and $V_\alpha\subseteq V_\beta$ is a quantum normed space (or matrix isometric) inclusion whenever $\alpha\leq\beta$.

Theorem 3.1: *Let $V=\Sigma_{\alpha\in\Lambda}V_\alpha$ be an inductive limit of a direct quantum family $\{V_\alpha:\alpha\in\Lambda\}$. Then the inductive quantum topology restricted to each V_α is reduced to the original quantum normed topology. Moreover*,

$$\max V=\sum_{\alpha\in\Lambda}\max V_\alpha=V_{\max}.$$

Proof: Fix $\alpha\in\Lambda$. If $v\in M(V_\alpha)\backslash\{0\}$, then $v=\lambda v_\alpha\mu$ with $\lambda=\mu=\|v\|_\alpha^{1/2}$ and $v_\alpha=\|v\|_\alpha^{-1}v$. By Proposition 3.2,

$$p_\rho(v)^{1/2}\leq\left\|\begin{bmatrix}\lambda_\rho & 0\\ 0 & \mu_\rho^*\end{bmatrix}\right\|=\left\|\begin{bmatrix}\rho_\alpha^{-1/2}\lambda & 0\\ 0 & \rho_\alpha^{-1/2}\mu^*\end{bmatrix}\right\|=\rho_\alpha^{-1/2}\|v\|_\alpha^{1/2},$$

that is,

$$p_\rho|_{M(V_\alpha)}\leq\rho_\alpha^{-1}\|\cdot\|_\alpha. \tag{3.3}$$

So, the original quantum normed topology on V_α is finer than the inductive quantum topology inherited from V. Conversely, take $v\in M(V_\alpha)\backslash\{0\}$ and its expansion $v=\Sigma_{i=1}^s\lambda_i v_{\alpha_i}\mu_i$ with $\lambda_i,\mu_i\in M$, $v_{\alpha_i}\in\mathrm{ball}\ M(V_{\alpha_i})$, $1\leq i\leq s$. By assumption, all $V_{\alpha_i}\subseteq V_\beta$ for a certain $\beta\in\Lambda$. In particular, $v=\Sigma_{i=1}^s\lambda_{\rho,i}\rho_{\alpha_i}v_{\alpha_i}\mu_{\rho,i}\in M(V_\beta)$ and

$$v=\lambda_\rho\begin{bmatrix}\rho_{\alpha_1}v_{\alpha_1} & & 0\\ & \ddots & \\ 0 & & \rho_{\alpha_s}v_{\alpha_s}\end{bmatrix}\mu_\rho\quad\text{with}\quad\lambda_\rho=[\lambda_{\rho,1}\quad\cdots\quad\lambda_{\rho,s}],\quad\mu=\begin{bmatrix}\mu_{\rho,1}\\ \vdots\\ \mu_{\rho,s}\end{bmatrix}. \tag{3.4}$$

Furthermore,

$$\begin{aligned}\|v\|_\alpha&=\|v\|_\beta\leq\|\lambda_\rho\|\max\{\rho_{\alpha_i}\|v_{\alpha_i}\|_\beta\}\|\mu_\rho\|=\|\lambda_\rho\|\max\{\rho_{\alpha_i}\|v_{\alpha_i}\|_{\alpha_i}\}\|\mu_\rho\|\leq\max\{\rho_{\alpha_i}\}\|\lambda_\rho\|\|\mu_\rho\|\\ &\leq\max\{\rho_{\alpha_i}\}\max\{\|\lambda_\rho\|^2,\|\mu_\rho\|^2\}=\max\{\rho_{\alpha_i}\}\max\{\|\lambda_\rho\lambda_\rho^*\|,\|\mu_\rho^*\mu_\rho\|\}=\max\{\rho_{\alpha_i}\}\left\|\begin{bmatrix}\lambda_\rho\lambda_\rho^* & 0\\ 0 & \mu_\rho^*\mu_\rho\end{bmatrix}\right\|\\ &=\max\{\rho_{\alpha_i}\}\left\|\sum_i\begin{bmatrix}\lambda_{\rho,i}\lambda_{\rho,i}^* & 0\\ 0 & \mu_{\rho,i}^*\mu_{\rho,i}\end{bmatrix}\right\|=\max\{\rho_{\alpha_i}\}\|A_{\lambda,\rho,\mu}\|^2.\end{aligned}$$

Hence, if ρ is a bounded family, then $\|v\|_\alpha\leq\sup(\rho)p_\rho(v)$, thanks to Proposition 3.2, that is, $\|\cdot\|_\alpha\leq\sup(\rho)\ p_\rho|_{M(V_\alpha)}$. Thus,

$$\sup(\rho)^{-1}\|\cdot\|_\alpha \leq p_\rho|_{M(V_\alpha)} \leq \rho_\alpha^{-1}\|\cdot\|_\alpha$$

whenever ρ is bounded. It follows that the inductive quantum topology on each V_α is finer than the original quantum normed topology, therefore they coincide.

Now consider the quantum normed spaces max V_α, $\alpha\in\Lambda$, and let $\mathcal{C}_{\sigma_\rho}(V,M_r)$ be the set of all continuous σ_ρ-contractive linear mappings $w:V\to M_r$ $(w\in M_r(V'))$, that is, $\sigma_\rho^\circ(w)=\sup\|\langle\langle \mathrm{ball}\sigma_\rho,w\rangle\rangle\|\leq 1$ or $\|\langle\langle x,w\rangle\rangle\|\leq\sigma_\rho(x)$ for all $x\in V$ (see Sec. II E). Consider also the set $\mathcal{MB}_{p_\rho}(V,M_r)=\mathrm{ball}p_\rho^\odot$, where $p_\rho^\odot(w)=\sup\|\langle\langle \mathrm{ball}p_\rho,w\rangle\rangle\|$, $w\in\mathcal{C}(V,M_r)$ [see (2.3)]. Since $p_\rho^{(1)}=\sigma_\rho$ (see Corollary 3.3), it follows that $\mathrm{ball}\sigma_\rho\subseteq\mathrm{ball}p_\rho$ and $\sigma_\rho^\circ\leq p_{\rho,r}^\odot$, which, in turn, implies that $\mathcal{MB}_{p_\rho}(V,M_r)\subseteq\mathcal{C}_{\sigma_\rho}(V,M_r)$. Let us prove that $\mathcal{MB}_{p_\rho}(V,M_r)=\mathcal{C}_{\sigma_\rho}(V,M_r)$. Take $w\in\mathcal{C}_{\sigma_\rho}(V,M_r)$. Using (3.3) and Corollary 3.3, infer that $\|\rho_\alpha w(x)\|\leq\rho_\alpha\sigma_\rho(x)=\rho_\alpha p_\rho^{(1)}(x)\leq\|x\|_\alpha=\|x\|_{\max}$ for all $x\in V_\alpha$. Then $\rho_\alpha w|V_\alpha\in\mathrm{ball}\ \mathcal{B}(V_\alpha,M_r)=\mathrm{ball}\ \mathcal{MB}(\max V_\alpha,M_r)$, thanks to Corollary 2.5 [see also Ref. 13, (3.3.9)], that is, $\|\langle\langle\rho_\alpha v,w\rangle\rangle\|\leq\|v\|_{\max}$ for all $v\in M(\max V_\alpha)$. Take $v\in M(V)$ and its expansion as in (3.4). Then

$$\langle\langle v,w\rangle\rangle=\lambda_\rho\otimes 1\begin{bmatrix}\langle\langle\rho_{\alpha_1}v_{\alpha_1},w\rangle\rangle & & 0\\ & \ddots & \\ 0 & & \langle\langle\rho_{\alpha_s}v_{\alpha_s},w\rangle\rangle\end{bmatrix}\mu_\rho\otimes 1,$$

and as above we have

$$\|\langle\langle v,w\rangle\rangle\|\leq\|\lambda_\rho\otimes 1\|\max_i\{\|\langle\langle\rho_{\alpha_i}v_{\alpha_i},w\rangle\rangle\|\}\|\mu_\rho\otimes 1\|\leq\|\lambda_\rho\|\max_i\{\|v_{\alpha_i}\|_{\max}\}\|\mu_\rho\|\leq\|\lambda_\rho\|\|\mu_\rho\|\leq\|A_{\lambda,\rho,\mu}\|^2,$$

where $A_{\lambda,\rho,\mu}$ is the matrix (3.2) associated with the indicated expansion of v. Using Proposition 3.2, we derive that $\|\langle\langle v,w\rangle\rangle\|\leq p_\rho(v)$ or $p_\rho^\odot(w)\leq 1$, that is, $w\in\mathcal{MB}_{p_\rho}(V,M_r)$. Consequently, $\mathcal{C}_{\sigma_\rho}(V,M_r)=\mathcal{MB}_{p_\rho}(V,M_r)$. Using Proposition 2.2, we deduce that

$$p_\rho(v)=\sup_r\|\langle\langle v,\mathcal{MB}_{p_\rho}(V,M_r)\rangle\rangle\|=\sup_r\|\langle\langle v,\mathcal{C}_\rho(V,M_r)\rangle\rangle\|=\underline{\sigma_\rho}(v)$$

for all $v\in M(V)$, that is, $p_\rho=\underline{\sigma_\rho}$. It follows that $V=\Sigma_{\alpha\in\Lambda}\max V_\alpha=\max V$. □

Remark 3.2: Note that if $\rho=1$, *then* $p_\rho|_{M(V_\alpha)}=\|\cdot\|_\alpha$, $\alpha\in\Lambda$. *Similar argument used in the proof for the min quantization fails. The same equality for the min quantization takes place when we deal with the nuclear quantum spaces (see Theorem 2.2).*

Theorem 3.2: *Let* $V=\Sigma_{\alpha\in\Lambda}V_\alpha$ *and* $W=\Sigma_{\alpha\in\Lambda}W_\alpha$ *be inductive limits of quantum normed spaces. If* $\{W_\alpha:\alpha\in\Lambda\}$ *is a direct quantum family, then*

$$T\in\mathcal{MC}_\Lambda(V,W)\Leftrightarrow T|V_\alpha\in\mathcal{MB}(V_\alpha,W_\alpha)\quad \textit{for all } \alpha\in\Lambda.$$

Proof: If $T|V_\alpha\in\mathcal{MB}(V_\alpha,W_\alpha)$ for all $\alpha\in\Lambda$, then $T\in\mathcal{MC}_\Lambda(V,W)$, thanks to Proposition 3.4. Conversely, assume that $T\in\mathcal{MC}_\Lambda(V,W)$. By Theorem 3.1, the inductive quantum topology restricted to each W_α is reduced to the original normed one. However $T(V_\alpha)\subseteq W_\alpha$, therefore the mapping $T|V_\alpha:V_\alpha\to W_\alpha$ as a superposition $V_\alpha\hookrightarrow V\xrightarrow{T}W$ of matrix continuous linear mappings is matrix continuous. Consequently, $T|V_\alpha\in\mathcal{MB}(V_\alpha,W_\alpha)$ for all $\alpha\in\Lambda$. □

Corollary 3.4: If $V=\Sigma_{\alpha\in\Lambda}V_\alpha$ *is an inductive limit of quantum normed spaces and* W *is a quantum normed space, then*

$$T\in\mathcal{MC}(V,W)\Leftrightarrow T|V_\alpha\in\mathcal{MB}(V_\alpha,W)\quad \textit{for all } \alpha\in\Lambda.$$

Proof: It suffices to put $W_\alpha=W$, $\alpha\in\Lambda$, in Theorem 3.2. Evidently, $\{W_\alpha:\alpha\in\Lambda\}$ is a direct quantum family, $\mathcal{MC}(V,W)=\mathcal{MC}_\Lambda(V,W)$ and $W=\Sigma_{\alpha\in\Lambda}W_\alpha$, thanks to Theorem 3.1. Appealing Theorem 3.2, we derive the result. □

E. Application to the quantum moment problem

Finally, we propose an extension theorem which is used in the quantum moment problems.[24,7,8]

Theorem 3.3: *Let $V=\Sigma_{\alpha\in\Lambda}V_\alpha$ be an inductive limit of quantum normed spaces, $X\subseteq V$ a linear subspace, $S:X\to\mathcal{B}(H)$ a linear mapping and let $X_\alpha=X\cap V_\alpha$, $S_\alpha=S|X_\alpha$, $\alpha\in\Lambda$. Suppose each $S_\alpha:X_\alpha\to\mathcal{B}(H)$ is a nontrivial matrix bounded linear mapping. The mapping $S:X\to\mathcal{B}(H)$ has a matrix continuous linear extension $T:V\to\mathcal{B}(H)$ such that $\|T|V_\alpha\|_{mb}=\|S_\alpha\|_{mb}$, $\alpha\in\Lambda$, if and only if*

$$\|S^{(\infty)}(x)\|\leq p_\rho(x),\quad x\in M(X),$$

where p_ρ is the matrix seminorm on $M(V)$ with $\rho=(\rho_\alpha)_{\alpha\in\Lambda}$, $\rho_\alpha=\|S_\alpha\|_{mb}^{-1}$, $\alpha\in\Lambda$.

Proof: First, assume that $\|S^{(\infty)}(x)\|\leq p_\rho(x)$, $x\in M(X)$, that is, $\|S^{(n)}(x)\|\leq p_\rho^{(n)}(x)$, $x\in M_n(X)$, $n\in\mathbb{N}$. By Hahn–Banach theorem,[6] we have a linear extension $T:V\to\mathcal{B}(H)$ of S such that $\|T^{(n)}\times(v)\|\leq p_\rho^{(n)}(v)$, $v\in M_n(V)$, $n\in\mathbb{N}$. If $v\in M_n(V_\alpha)$, then $\|T^{(n)}(v)\|\leq p_\rho^{(n)}(v)\leq\rho_\alpha^{-1}\|v\|_\alpha$ [see (3.3)]. It follows that $\|T|V_\alpha\|_{mb}\leq\rho_\alpha^{-1}=\|S_\alpha\|_{mb}=\|T|X_\alpha\|_{mb}\leq\|T|V_\alpha\|_{mb}$, that is, $\|T|V_\alpha\|_{mb}=\|S_\alpha\|_{mb}$ for all $\alpha\in\Lambda$. So, we have a matrix continuous linear extension $T:V\to\mathcal{B}(H)$ such that $\|T|V_\alpha\|_{mb}=\|S_\alpha\|_{mb}$, $\alpha\in\Lambda$.

Conversely, assume that the linear mapping $S:X\to\mathcal{B}(H)$ has an extension $T\in\mathcal{MC}(V,\mathcal{B}(H))$ such that $\|T|V_\alpha\|_{mb}=\|S_\alpha\|_{mb}$ for all $\alpha\in\Lambda$. Using Proposition 3.4 (see to the proof) and Corollary 3.4, infer that $\|T^{(\infty)}(v)\|\leq p_\theta(v)$, $v\in M(V)$, where $\theta=(\theta_\alpha)_{\alpha\in\Lambda}$, $\theta_\alpha=\|T|V_\alpha\|_{mb}^{-1}=\|S_\alpha\|_{mb}^{-1}$, that is, $\theta=\rho$. In particular, $\|S^{(\infty)}(x)\|\leq p_\rho(x)$ for all $x\in M(X)$. □

IV. QUANTUM DOMAINS

In this section we endow the quantum domains in a Hilbert space with a quantum space structures, which will allow us to treat the space of all noncommutative continuous functions over a quantum domain as a quantum space of matrix continuous linear operators on a certain quantum space equipped with a $\mathfrak{S}$-quantum topology. As the main result of this section we prove that each quantum domain which admits a gradation is a quantum reflexive space in the sense that its second strong quantum dual is reduced to itself up to the topological matrix isomorphism.

First we introduce $\mathfrak{S}$-quantum topology in its general setting.

A. The $\mathfrak{S}$-quantum topology in $\mathcal{MC}(V,W)$

Let V and W be quantum spaces with their defining families $\{p_\iota:\iota\in\Xi\}$ and $\{q_\kappa:\kappa\in\Omega\}$ of matrix seminorms, respectively. We introduce $\mathfrak{S}$-quantum topology (see Ref. 14, Sec. 9) in the space $\mathcal{MC}(V,W)$ of all matrix continuous linear mappings $V\to W$. Fix a family $\mathfrak{S}$ of *matrix bounded* sets $\mathfrak{B}$ such that *the matrix hull of the union* $\cup\mathfrak{S}$ is dense in $M(V)$, that is, all matrix combinations $\Sigma_i a_i v_i b_i$ with $a_i,b_i\in M$, $v_i\in\mathfrak{B}_i$, $\mathfrak{B}_i\in\mathfrak{S}$, generate a dense subspace in the polynormed space $M(V)$. In this case we briefly say that $\mathfrak{S}$ is *a matrix total family of matrix bounded sets in* $M(V)$.

The following simple assertion will be useful.

Lemma 4.1: For each $n\in\mathbb{N}$ we have $M_n(\mathcal{MC}(V,W))=\mathcal{MC}(V,M_n(W))$ up to the canonical linear isomorphism.

Proof: First note that we have a canonical linear isomorphism $M_n(L(V,W))\to L(V,M_n(W))$ which converts each matrix $[T_{st}]\in M_n(L(V,W))$ into a linear mapping $T:V\to M_n(W)$, $Tv=[T_{st}v]_{s,t}$. It remains to prove that its restriction to the subspace $M_n(\mathcal{MC}(V,W))$ implements the required isomorphism onto $\mathcal{MC}(V,M_n(W))$. Take $[T_{st}]\in M_n(\mathcal{MC}(V,W))$. Then for each $\kappa\in\Omega$ there corresponds positive constants C_{st} and $\iota_{st}\in\Xi$ such that $q_\kappa T_{st}^{(\infty)}\leq C_{st}p_{\iota_{st}}$. Taking into account that any permutation of rows and columns of a matrix over a quantum space does not affect the matrix seminorms [Ref. 13, 2.1.5], and applying (2.2) to the quantum space $M_m(W)$, we derive that

$$q_\kappa^{(mn)}(T^{(m)}(v)) = q_\kappa^{(mn)}([T(v_{kl})]_{k,l}) = q_\kappa^{(mn)}([[T_{st}(v_{kl})]_{s,t}]_{k,l}) = q_\kappa^{(mn)}([[T_{st}(v_{kl})]_{k,l}]_{s,t}) = q_\kappa^{(mn)}([T_{st}^{(m)}(v)]_{s,t})$$
$$\leqslant \sum_{s,t=1}^{n} q_\kappa^{(m)}(T_{st}^{(m)}(v)) \leqslant \sum_{s,t=1}^{n} C_{st} p_{\iota_{st}}^{(m)}(v) \leqslant C \max\{p_{\iota_{st}}^{(m)}(v): 1 \leqslant s,t \leqslant n\},$$

where $v=[v_{kl}] \in M_m(V)$, $m \in \mathbb{N}$, $C=\Sigma_{s,t=1}^{n} C_{st}$, that is, the linear mapping $T: V \to M_n(W)$ is matrix continuous. Conversely, take $T \in \mathcal{MC}(V, M_n(W))$, which defines a matrix $[T_{st}] \in M_n(L(V,W))$. Then each $T_{st}: V \to W$ is matrix continuous. In order to prove it, first note that $T_{st}=P_{st}T$, where $P_{st}: M_n(W) \to W$, $P_{st}[w_{kl}]=w_{st}$. However, P_{st} is matrix continuous. Indeed, $P_{st}w=\varepsilon_s w \varepsilon_t^*$, $w \in M_n(W)$, where $\varepsilon_s \in M_{1,n}$ are standard scalar row matrices. It follows that $P_{st}^{(m)}w=(1_m \otimes \varepsilon_s)w(1_m \otimes \varepsilon_t^*)$ for all $w \in M_m(M_n(W))$, $m \in \mathbb{N}$. Fix $\kappa \in \Omega$. Then

$$q_\kappa^{(m)}(P_{st}^{(m)}w) \leqslant \|1_m \otimes \epsilon_s\| q_\kappa^{(mn)}(w) \|1_m \otimes \epsilon_t^*\| = q_\kappa^{(mn)}(w)$$

for all $w \in M_m(M_n(W))$, $m \in \mathbb{N}$, which means that P_{st} is matrix continuous. Consequently, $T_{st} \in \mathcal{MC}(V,W)$. □

Take a matrix bounded set $\mathfrak{B}=(\mathfrak{b}_n) \in \mathfrak{S}$ and fix an index $\kappa \in \Omega$. For any $T=[T_{ij}] \in M_n(\mathcal{MC}(V,W))$, we put

$$p_{\kappa,\mathfrak{B}}^{(n)}(T) = \sup\{q_\kappa^{(rn)}(T^{(r)}(v)): v \in \mathfrak{b}_r, r \in \mathbb{N}\}, \tag{4.1}$$

that is, $p_{\kappa,\mathfrak{B}}(T)=\sup q_\kappa(T^{(\infty)}(\mathfrak{B}))$, where T is identified with $T \in \mathcal{MC}(V, M_n(W))$, thanks to Lemma 4.1. Note that $q_\kappa T^{(\infty)} \leqslant C \max_{\iota \in \alpha} p_\iota$ for a certain positive C and a finite subset $\alpha \subseteq \Xi$. However, $\mathfrak{B}$ is matrix bounded in $M(V)$, so $\max_{\iota \in \alpha} \sup p_\iota(\mathfrak{B})=C' < \infty$. Hence, $p_{\kappa,\mathfrak{B}}(T) < \infty$.

Lemma 4.2: Each $p_{\kappa,\mathfrak{B}}$ is a matrix seminorm on $\mathcal{MC}(V,W)$.

Proof: Take $T=[T_{ij}] \in M_m(\mathcal{MC}(V,W))$, $a \in M_{n,m}$ and $b \in M_{m,n}$. On the grounds of Lemma 4.1, we have $(aTb)^{(r)}(v)=(a \otimes 1_r)T^{(r)}(v)(b \otimes 1_r)$ for all $v \in M_r(V)$. Then

$$p_{\kappa,\mathfrak{B}}^{(n)}(aTb) = \sup\{q_\kappa^{(rn)}((a \otimes 1_r)T^{(r)}(v)(b \otimes 1_r)): v \in \mathfrak{b}_r, r \in \mathbb{N}\} \leqslant \sup\{\|a \otimes 1_r\| q_\kappa^{(rm)}(T^{(r)}(v)) \|b \otimes 1_r\|: v \in \mathfrak{b}_r, r \in \mathbb{N}\} \leqslant \|a\| p_{\kappa,\mathfrak{B}}^{(m)}(T) \|b\|.$$

Further, if $T=[T_{ij}] \in M_m(\mathcal{MC}(V,W))$ and $S=[S_{ij}] \in M_n(\mathcal{MC}(V,W))$, then $(T \oplus S)^{(r)}(v)=T^{(r)}(v) \oplus S^{(r)}(v)$, $v \in M_r(V)$, and

$$p_{\kappa,\mathfrak{B}}^{(m+n)}(T \oplus S) = \sup\{q_\kappa^{(rm+rn)}(T^{(r)}(v) \oplus S^{(r)}(v)): v \in \mathfrak{b}_r, r \in \mathbb{N}\} \leqslant \sup\{\max\{q_\kappa^{(rm)}(T^{(r)}(v)), q_\kappa^{(rn)}(S^{(r)}(v))\}: v \in \mathfrak{b}_r, r \in \mathbb{N}\} \leqslant \max\{p_{\kappa,\mathfrak{B}}^{(m)}(T), p_{\kappa,\mathfrak{B}}^{(n)}(S)\}.$$

Thus, we have both **M1** and **M2** properties, that is, $p_{\kappa,\mathfrak{B}}$ is a matrix seminorm. □

The quantum topology on $\mathcal{MC}(V,W)$ determined by the family $\{p_{\kappa\mathfrak{B}}: \kappa \in \Omega, \mathfrak{B} \in \mathfrak{S}\}$ of matrix seminorms (see Lemma 4.2) is called $\mathfrak{S}$-*quantum (or* $\mathfrak{S}$-*matrix) topology* (Ref. 14, Sec. 9) in $\mathcal{MC}(V,W)$ (for the usual $\mathfrak{S}$-topology in $\mathcal{C}(V,W)$ (see Ref. 23, Sec. III C). The quantum space $\mathcal{MC}(V,W)$ equipped with the $\mathfrak{S}$-quantum topology is denoted by $\mathcal{MC}(V,W)_{\mathfrak{S}}$. If $\mathfrak{S}$ is *a fundamental system of matrix bounded sets in* $M(V)$, that is, each matrix bounded set in $M(V)$ is contained in a certain set from $\mathfrak{S}$, then the relevant $\mathfrak{S}$-quantum topology in $\mathcal{MC}(V,W)$ is called *the strong quantum (or matrix) convergence topology,* and the quantum space $\mathcal{MC}(V,W)$ equipped with this topology is denoted by $\mathcal{MC}(V,W)_\beta$. Thus, $M_n(\mathcal{MC}(V,W)_{\mathfrak{S}})=(\mathcal{MC}(V,M_n(W)), \{p_{\kappa,B}^{(n)}\})$, $n \in \mathbb{N}$, up to the identification from Lemma 4.1.

Remark 4.1: Note that $\mathfrak{S}$-quantum topology on $\mathcal{MC}(V,W)$ is Hausdorff. Indeed, it suffices to prove that $\{p_{\kappa,B}^{(1)}\}$ determines a Hausdorff polynormed topology on $\mathcal{MC}(V,W)$ [see (2.2)]. *If $p_{\kappa,B}^{(1)}(T)=0$, $\kappa \in \Omega$, $\mathfrak{B} \in \mathfrak{S}$, for a certain $T \in \mathcal{MC}(V,W)$, then $T^{(\infty)}(\mathfrak{B})=\{0\}$ for all $\mathfrak{B} \in \mathfrak{S}$. Hence, $T^{(\infty)}(\cup \mathfrak{S})=\{0\}$. Take a matrix combination $\Sigma_{i=1}^{s} a_i v_i b_i$ with $a_i, b_i \in M$, $v_i \in \cup \mathfrak{S}$. Then $T^{(\infty)}(\Sigma_i a_i v_i b_i)=\Sigma_i a_i T^{(\infty)}(v_i) b_i=0$. Taking into account that $T^{(\infty)}: M(V) \to M(W)$ is a continuous linear mapping and the matrix hull of $\cup \mathfrak{S}$ is dense in $M(V)$, we conclude that $T^{(\infty)}=0$ or $T=0$.*

B. The quantum topology in $\mathcal{MC}_\Lambda(V,W)$

Let again $V=\Sigma_{\alpha\in\Lambda}V_\alpha$ and $W=\Sigma_{\alpha\in\Lambda}W_\alpha$ be inductive limits of the quantum normed spaces and let $\mathfrak{S}$ be a family in $M(V)$ of the unit sets ball $M(V_\alpha)$, $\alpha\in\Lambda$. Since $M(V)=\Sigma_{\alpha\in\Lambda}M(V_\alpha)$ possesses the inductive quantum topology, it follows that $\mathfrak{S}$ is a matrix total family of matrix bounded sets in $M(V)$ (see Lemma 3.1). Note that $\sup p_\rho$ (ball $M(V_\alpha))\leqslant\rho_\alpha^{-1}$, $\rho\in\mathbb{R}_+^\Lambda$, $\alpha\in\Lambda$ [see (3.3)]. Consider the subspace $\mathcal{MC}_\Lambda(V,W)\subseteq\mathcal{MC}(V,W)$. By Lemma 4.1, $M_n(\mathcal{MC}_\Lambda(V,W))=\mathcal{MC}_\Lambda(V,M_n(W))$ up to the canonical linear isomorphism. Fix an index $\alpha\in\Lambda$ and a family $\rho\in\mathbb{R}_+^\Lambda$. For any $T=[T_{ij}]\in M_n(\mathcal{MC}_\Lambda(V,W))$, we put

$$p_{\rho,\alpha}(T)=\sup q_\rho(T^{(\infty)}(\text{ball }M(V_\alpha))) \tag{4.2}$$

[see (4.1)]. By Lemma 4.2, $p_{\rho,\alpha}$ is a matrix seminorm on $\mathcal{MC}_\Lambda(V,W)$, and $\{p_{\rho,\alpha}:\rho\in\mathbb{R}_+^\Lambda,\alpha\in\Lambda\}$ determines the $\mathfrak{S}$-quantum topology in the quantum space $\mathcal{MC}_\Lambda(V,W)_{\mathfrak{S}}$.

Finally, $\mathfrak{S}=\{\mathfrak{B}_\alpha:\alpha\in\Lambda\}$ is a fundamental system of matrix bounded sets in the quantum inductive limit $M(V)=\Sigma_{\alpha\in\Lambda}M(V_\alpha)$ whenever all V_α are complete and Λ is countable (see Ref. 23, Sec. 2.6.5) for the classical case), or we deal with the direct sum (see Corollary 3.2). In this case, $\mathcal{MC}_\Lambda(V,W)_{\mathfrak{S}}=\mathcal{MC}_\Lambda(V,W)_\beta$ (see Sec. IV A).

C. The $\mathfrak{S}$-quantum topology on V' and quantum bornological spaces

Now consider the particular case of the quantum space $\mathcal{MC}(V,W)$ when $W=\mathbb{C}$. As we have confirmed above in this case $\mathcal{MC}(V,\mathbb{C})=\mathcal{C}(V,\mathbb{C})=V'$ and $\{q_\kappa:\kappa\in\Omega\}$ consists of a single matrix norm on $\mathbb{C}$. For each $\mathfrak{B}\in\mathfrak{S}$, we write $p_{\mathfrak{B}}$ instead of $p_{\kappa,\mathfrak{B}}$. Take $f\in M_n(V')=\mathcal{MC}(V,M_n)$. Using (4.1) and (2.3), we derive that

$$p_{\mathfrak{B}}(f)=\sup\|f^{(\infty)}(\mathfrak{B})\|=\sup\|\langle\langle\mathfrak{B},f\rangle\rangle\|=q_{\mathfrak{B}}(f),$$

where $q_{\mathfrak{B}}$ is the dual gauge of $\mathfrak{B}$. By Corollary 2.3, we conclude that $p_{\mathfrak{B}}$ is just the Minkowski functional of the absolute matrix polar $\mathfrak{B}^\odot\subseteq M(V')$. It follows that the $\mathfrak{S}$-quantum topology in $V'_{\mathfrak{S}}$ has a neighborhood filter base of absolute matrix polars $\{\mathfrak{B}^\odot:\mathfrak{B}\in\mathfrak{S}\}$. Furthermore, by Lemma 2.4, it can be assumed that $\mathfrak{B}$ is an absolutely matrix convex, that is, $\mathfrak{B}=\text{amc}\,\mathfrak{B}$. Then $\mathfrak{B}^\odot=(\mathfrak{B}^\odot)^{\odot\odot}=(\mathfrak{B}^{\odot\odot})^\odot=(\overline{\text{amc }\mathfrak{B}})^\odot=\bar{B}^\odot$ by virtue of the bipolar Theorem 2.1, where $\bar{\mathfrak{B}}$ denotes the weak closure. Consequently, it can be assumed that all $\mathfrak{B}$ from the family $\mathfrak{S}$ are weakly closed absolutely matrix convex sets. In this case

$$p_{\mathfrak{B}}=q_{\mathfrak{B}}=\gamma_{\mathfrak{B}}^\odot,$$

where $\gamma_{\mathfrak{B}}$ is the Minkowski functional of $\mathfrak{B}$ [see (2.3)]. Thus, the $\mathfrak{S}$-quantum topology on V' is just the quantum topology generated by the dual seminorms $\{\gamma_{\mathfrak{B}}^\odot:\mathfrak{B}\in S\}$.

Let us note observe that a net $\{f_\theta\}$ in $M(V')$ converges to a "function" $f\in M(V')$ in the $\mathfrak{S}$-quantum topology if the matrix-valued functions $v\mapsto\langle\langle v,f_\theta\rangle\rangle$ converge uniformly to the function $v\mapsto\langle\langle v,f\rangle\rangle$ over all matrix bounded sets from $\mathfrak{S}$. If $\mathfrak{S}$ is a fundamental system of matrix bounded sets in $M(V)$ then we write V'_β instead of $V'_{\mathfrak{S}}$ and it is called *the strong quantum dual of V*. Using Corollary 2.2, we conclude that the strong quantum dual topology in $M(V'_\beta)$ associates the (classical) strong dual topology in V'.

Finally, let V be a quantum space. By a *quantum bornivorous on V* we mean an absolutely matrix convex set $\mathfrak{P}$ in $M(V)$ which absorbs each matrix bounded set in $M(V)$, that is, if $\mathfrak{B}$ is a matrix bounded set, then $\lambda\mathfrak{B}\subseteq\mathfrak{P}$ for a certain $\lambda>0$. Any matrix set from the neighborhood filter base of the quantum topology in $M(V)$ is obviously a quantum bornivorous. If each quantum bornivorous is a neighborhood of the origin in $M(V)$, then we say that V is a *quantum bornological space* [Ref. 14 (Sec. 8) and Ref. 10]. Evidently, each quantum normed space is a quantum bornological space. Another example is delivered by the following assertion.

Proposition 4.1: Let $V=\Sigma_{\alpha\in\Lambda}V_\alpha$ be a quantum inductive limit of the quantum normed spaces. Then V is a quantum bornological space.

Proof: Take a quantum bornivorous $\mathfrak{P}$ in $M(V)$. Since each embedding $V_\alpha\hookrightarrow V$ is matrix continuous, it follows that ball $M(V_\alpha)$ is matrix bounded in $M(V)$. Hence, ρ_α ball $M(V_\alpha)\subseteq\mathfrak{P}$ for some $\rho_\alpha>0$, $\alpha\in\Lambda$. Taking into account that $\mathfrak{P}$ is an absolutely matrix convex set, we conclude that $\mathfrak{B}_\rho=\text{amc}\cup_\alpha\rho_\alpha$ ball $M(V_\alpha)\subseteq\mathfrak{P}$, $\rho=(\rho_\alpha)_{\alpha\in\Lambda}$ (see Sec. III B). It follows that $\mathfrak{P}$ belongs to the filter generated by $\{\mathfrak{B}_\rho\}$, that is, V is a quantum bornological space. □

The following assertion was proved in Ref. 14 (Proposition 9.1) by Effros and Webster.

Proposition 4.2: If V is a quantum bornological space, then the canonical embedding $V\hookrightarrow(V'_\beta)'_\beta$ of V into its second strong quantum dual is a topological matrix isomorphism.

D. The conjugate space of a quantum domain

Let H be a Hilbert space with its inner product $\langle\cdot,\cdot\rangle$. By *a (quantum) domain $\mathcal{E}$ in H* Ref. 6 and 7 we mean an upward filtered family $\mathcal{E}=\{H_\alpha:\alpha\in\Lambda\}$ of closed subspaces in H such that their union $\mathcal{D}=\cup\mathcal{E}$ is dense in H. Obviously, $\mathcal{D}$ is a dense subspace in H called *the union space of the domain $\mathcal{E}$*. Being an inductive limit of the Hilbert spaces, the union space $\mathcal{D}$ turns out to be a polynormed space equipped with the inductive topology.

A family $\mathcal{N}=\{N_\iota:\iota\in\Xi\}$ of closed subspaces in H is said to be*a gradation* if they are pairwise orthogonal and their algebraic orthogonal sum $\mathcal{D}=\Sigma_{\iota\in\Xi}N_\iota$ is dense in H. Each gradation $\mathcal{N}=\{N_\iota:\iota\in\Xi\}$ automatically defines a domain $\mathcal{E}=\{H_\alpha:\alpha\in\Lambda\}$, where Λ is the set of all finite subsets of Ξ and $H_\alpha=\oplus_{\iota\in\alpha}N_\iota$ for a finite subset $\alpha\subseteq\Xi$. Furthermore, $\mathcal{D}=\Sigma_{\iota\in\Xi}N_\iota$ is the union space of the domain $\mathcal{E}$. We say that $\mathcal{E}$ is *a (quantum) domain with the gradation $\mathcal{N}$*. Further, $\mathcal{N}=\{N_\iota:\iota\in\Xi\}$ is said to be *a finite rank gradation* if $\mathcal{N}$ is a gradation with finite dimensional nest subspaces N_ι. In this case $\mathcal{E}$ is called *an atomic domain with the gradation $\mathcal{N}$*. Note that the inductive topology in the union space $\mathcal{D}=\Sigma_{\iota\in\Xi}N_\iota$ of an atomic domain is just the finest polynormed topology, for each linear mapping $\mathcal{D}\to X$ into a polynormed space X is continuous.

Lemma 4.3: Let H be a Hilbert space and let $\mathcal{N}=\{N_\iota:\iota\in\Xi\}$ be a family of its closed subspaces. Then $\mathcal{N}$ is a gradation if and only if $H=\bigoplus_{\iota\in\Xi}N_\iota$ is the Hilbert space sum.

Proof: Assume $\mathcal{N}$ is a gradation in H and $\mathcal{D}=\Sigma_{\iota\in\Xi}N_\iota$ is the relevant algebraic orthogonal sum. Then $K=\bigoplus_{\iota\in\Xi}N_\iota$ is a closed subspace in H (see, for instance, [Ref. 18, 6.3.3]). However, $\mathcal{D}\subseteq K$ and it is dense in H, therefore $K=H$. Conversely, assume that $H=\oplus_{\iota\in\Xi}N_\iota$ and $\mathcal{D}=\Sigma_{\iota\in\Xi}N_\iota$. For each $x\in H$ we have $x=\Sigma_{\iota\in\Xi}x_\iota$ with $\Sigma_{\iota\in\Xi}\|x_\iota\|^2=\|x\|^2$. It follows that $x=\lim_\alpha s_\alpha$, where $\alpha\subseteq\Xi$ is a finite subset and $s_\alpha=\Sigma_{\iota\in\alpha}x_\iota\in\mathcal{D}$. Thus, $\mathcal{D}$ is dense in H, that is, $\mathcal{N}$ is a gradation in H. □

Let $\mathcal{E}=\{H_\alpha:\alpha\in\Lambda\}$ be a domain in H with the union polynormed space $\mathcal{D}$. The family of the unit balls $\{\text{ball } H_\alpha:\alpha\in\Lambda\}$ is a total family of bounded sets in $\mathcal{D}$ denoted by $\mathfrak{S}$. In particular, the dual space $\mathcal{D}'$ turns out to be a polynormed space $\mathcal{D}'_{\mathfrak{S}}$ equipped with the $\mathfrak{S}$-topology. If $\mathcal{E}$ admits a gradation $\mathcal{N}=\{N_\iota:\iota\in\Xi\}$, that is, $H_\alpha=\oplus_{\iota\in\alpha}N_\iota$ for a finite subset $\alpha\subseteq\Xi$ (see Lemma 4.3), then $\mathcal{D}'_{\mathfrak{S}}=\mathcal{D}'_\beta$, that is, the $\mathfrak{S}$-topology is precisely the strong uniform convergence topology in the dual space $\mathcal{D}'$. Indeed, in this case $\{\text{ball } H_\alpha:\alpha\in\Lambda\}$ is a fundamental system of bounded sets in the inductive limit $\mathcal{D}$ (see Ref. 23, Sec. 3.6.3).

The projections in $\mathcal{B}(H)$ over all subspaces H_α are denoted by P_α, respectively. If $\alpha\leqslant\beta$, that is, $H_\alpha\subseteq H_\beta$ for some $\alpha,\beta\in\Lambda$, then we have a projection

$$P_{\alpha\beta}:H_\beta\to H_\alpha,\quad P_{\alpha\beta}(x_\beta)=P_\alpha(x_\beta),$$

onto H_α. Obviously, $P_{\alpha\beta}P_{\beta\gamma}=P_{\alpha\gamma}$ whenever $\alpha\leqslant\beta\leqslant\gamma$. So, we have the inverse system $\{\overline{H_\alpha},P_{\alpha\beta}\}$ of the conjugate Hilbert spaces. Its inverse limit $\lim_{\leftarrow}\{\overline{H_\alpha},P_{\alpha\beta}\}$ is called *a conjugate space of the domain $\mathcal{E}$* and it is denoted by $\mathcal{D}^-$. So, $\mathcal{D}^-$ consists of all compatible families $y^-=(\overline{y_\alpha})_{\alpha\in\Lambda}$ in $\Pi_{\alpha\in\Lambda}\overline{H_\alpha}$. The space $\mathcal{D}^-$ turns out to be a polynormed space with the family of seminorms

$$\|y^-\|_\alpha=\|\overline{y_\alpha}\|_{\overline{H_\alpha}}=\|y_\alpha\|,\quad y^-\in\mathcal{D}^-,\quad\alpha\in\Lambda.$$

Note that $\bar{H}$ can be identified with a subspace in $\mathcal{D}^-$. Indeed, take $\bar{y}\in\bar{H}$. Put $\overline{y_\alpha}=P_\alpha(\bar{y})$, $\alpha\in\Lambda$. If $\alpha\leqslant\beta$, then $P_{\alpha\beta}\overline{y_\beta}=P_{\alpha\beta}P_\beta\bar{y}=P_\alpha P_\beta\bar{y}=P_\alpha\bar{y}=\overline{y_\alpha}$, that is, the family $y^-=(\overline{y_\alpha})_{\alpha\in\Lambda}$ is compatible. Therefore $y^-\in\mathcal{D}^-$. Hence, $\bar{\mathcal{D}}\subseteq\bar{H}\subseteq\mathcal{D}^-$ up to the identification $\bar{y}=y^-$.

Lemma 4.4: The subspace $\bar{\mathcal{D}}$ is dense in $\mathcal{D}^-$.

Proof: Take $y^-=(\overline{y_\alpha})_{\alpha\in\Lambda}\in\mathcal{D}^-$ and consider the net $\mathfrak{y}=\{y_\alpha^-:\alpha\in\Lambda\}$ in $\bar{\mathcal{D}}$. Let us prove that $y^-=\lim\mathfrak{y}$ in $\mathcal{D}^-$. Fix $\alpha\in\Lambda$. Then

$$\lim_\beta\|y^--y_\beta^-\|_\alpha=\lim_{\beta,\beta\geqslant\alpha}\|y^--y_\beta^-\|_\alpha=\lim_{\beta,\beta\geqslant\alpha}\|\overline{y_\alpha}-P_\alpha\overline{y_\beta}\|_{\overline{H_\alpha}}=\lim_{\beta,\beta\geqslant\alpha}\|y_\alpha-y_\alpha\|_{H_\alpha}=0.$$

Whence $\bar{\mathcal{D}}$ is dense in $\mathcal{D}^-$.

Lemma 4.5: The linear mapping $\mathcal{D}^-\to\mathcal{D}'_{\mathfrak{S}}$, $y^-=(\overline{y_\alpha})_{\alpha\in\Lambda}\mapsto f_y$, $f_y|H_\alpha=\langle\cdot,y_\alpha\rangle$, implements a topological isomorphism of the conjugate space $\mathcal{D}^-$ of the domain $\mathcal{E}$ onto the dual space $\mathcal{D}'_{\mathfrak{S}}$.

Proof: Let $f:\mathcal{D}\to\mathbb{C}$ be a linear functional. We put $f_\alpha=f|_{H_\alpha}$, $\alpha\in\Lambda$. Since $\mathcal{D}$ is equipped with the inductive topology, it follows that f is continuous iff each f_α is bounded, that is, $f_\alpha\in H_\alpha^*$ for all $\alpha\in\Lambda$. Thus, $f_\alpha=\langle\cdot,y_\alpha\rangle$ for the uniquely determined vectors $y_\alpha\in H_\alpha$, $\alpha\in\Lambda$. If $\alpha\leqslant\beta$ then $\langle x,y_\alpha\rangle=f_\alpha(x)=f(x)=f_\beta(x)=\langle x,y_\beta\rangle$ for all $x\in H_\alpha$, that is, $y_\beta=y_\alpha+z_{\alpha\beta}$ for a certain $z_{\alpha\beta}\in H_\alpha^\perp\cap H_\beta$. Then $P_{\alpha\beta}y_\beta=P_\alpha(y_\alpha+z_{\alpha\beta})=y_\alpha$. Hence,

$$y_f^-=(\overline{y_\alpha})_{\alpha\in\Lambda}\in\mathcal{D}^-.$$

Conversely, take $y^-\in\mathcal{D}^-$. Put $f_y(x)=\langle x,y_\alpha\rangle$ whenever $x\in H_\alpha$. If $x\in H_\alpha\cap H_\beta$, then $H_\alpha\cup H_\beta\subseteq H_\gamma$ for a certain $\gamma\in\Lambda$, and

$$\langle x,y_\alpha\rangle=\langle x,P_\alpha y_\gamma\rangle=\langle P_\alpha x,y_\gamma\rangle=\langle x,y_\gamma\rangle=\langle P_\beta x,y_\gamma\rangle=\langle x,P_\beta y_\gamma\rangle=\langle x,y_\beta\rangle.$$

Consequently, $f_y:\mathcal{D}\to\mathbb{C}$ is a well defined linear functional. Since $f_y|_{H_\alpha}=\langle\cdot,y_\alpha\rangle$ for all α, it follows that $f_y\in\mathcal{D}'$. Thus, the assignments $f\mapsto y_f^-$ and $y^-\mapsto f_y$ implement the required isomorphism. Note that $y_{\lambda f}^-=(\bar{\lambda}y_\alpha)_{\alpha\in\Lambda}=(\lambda\overline{y_\alpha})_{\alpha\in\Lambda}=\lambda y_f^-$ (in the conjugate space $\mathcal{D}^-$) and $f_{\lambda y}^-=\lambda f_y$ for all $\lambda\in\mathbb{C}$, $f\in\mathcal{D}'$ and $y^-\in\mathcal{D}^-$.

Finally, by its very definition, the Minkowski functionals q_α, $\alpha\in\Lambda$, of the polars from the family $\mathfrak{S}$ is a defining family of seminorms of the dual space $\mathcal{D}'_{\mathfrak{S}}$ [Ref. 23, Chap. 3]. However,

$$q_\alpha(f)=\sup|f(\text{ball } H_\alpha)|=\sup|\langle\text{ball } H_\alpha,y_\alpha\rangle|=\|y_\alpha\|=\|y_f^-\|_\alpha$$

for all $f\in\mathcal{D}'$ and $\alpha\in\Lambda$. We conclude that the mapping $\mathcal{D}^-\to\mathcal{D}'_{\mathfrak{S}}$, $y^-\mapsto f_y$ is a topological isomorphism. □

Take $x\in H_\alpha$ and consider the linear functional

$$x'':\mathcal{D}^-\to\mathbb{C},\quad x''((\overline{y_\alpha})_{\alpha\in\Lambda})=\langle x,y_\alpha\rangle.$$

If $x\in H_\alpha\cap H_\beta$, then $H_\alpha\cup H_\beta\subseteq H_\gamma$ for a certain γ, and

$$x''((\overline{y_\alpha})_{\alpha\in\Lambda})=\langle x,P_{\alpha\gamma}y_\gamma\rangle=\langle x,P_\alpha y_\gamma\rangle=\langle P_\alpha x,y_\gamma\rangle=\langle x,y_\gamma\rangle=\langle P_\beta x,y_\gamma\rangle=\langle x,y_\beta\rangle,$$

that is, the functional x'' is well defined. Since

$$|x''((\overline{y_\alpha})_{\alpha\in\Lambda})|\leqslant\|y_\alpha\|_{H_\alpha}\|x\|_{H_\alpha}=\|(\overline{y_\alpha})_{\alpha\in\Lambda}\|_\alpha\|x\|,$$

we conclude that $x''\in\mathcal{C}(\mathcal{D}^-,\mathbb{C})$. However, $\mathcal{D}^-=\mathcal{D}'_{\mathfrak{S}}$ (up to the topological isomorphism), thanks to Lemma 4.5. Therefore x'' can be thought as a continuous linear functional on $\mathcal{D}'_{\mathfrak{S}}$, that is, $x''\in(\mathcal{D}'_{\mathfrak{S}})$ and

$$x''(f)=x''(y_f^-)=\langle x,y_\alpha\rangle=f(x)$$

(see the proof of Lemma 4.5). Hence, the correspondence $x\mapsto x''$ determines a linear embedding $\mathcal{D}\hookrightarrow C(\mathcal{D}^-,\mathbb{C})$, which is reduced to the canonical embedding $\mathcal{D}\hookrightarrow(\mathcal{D}'_{\mathfrak{S}})$.

Proposition 4.3: Let $\mathcal{E}$ be a domain with the union space $\mathcal{D}$. Then $\mathcal{D}=(\mathcal{D}'_{\mathfrak{S}})'$. Moreover, if $\mathcal{E}$ admits a gradation, then the polynormed space $\mathcal{D}$ is reflexive, that is, $\mathcal{D}=(\mathcal{D}'_\beta)'_\beta$.

Proof: Let $f\in\mathcal{C}(\mathcal{D}^-,\mathbb{C})$ be a continuous linear functional. Being an inverse limit of the Hilbert spaces, $\mathcal{D}^{-}$ is a (closed) subspace in the polynormed space $\Pi_{\alpha\in\Lambda}\overline{H_\alpha}$ with its defining family of seminorms $\|\bar{y}\|_\alpha=\|y_\alpha\|$, $\bar{y}=(\overline{y_\alpha})_{\alpha\in\Lambda}$, $\alpha\in\Lambda$. By Hahn–Banach theorem, f has a continuous linear extension $\hat{f}:\Pi_{\alpha\in\Lambda}\overline{H_\alpha}\to\mathbb{C}$. There are a positive constant C and a finite subset $F\subseteq\Lambda$ such that

$$|\hat{f}(\bar{y})| \leq C\max\{\|\bar{y}\|_\alpha:\alpha\in F\},\quad \bar{y}\in\prod_{\alpha\in\Lambda}\overline{H_\alpha}.$$

Consider the Hilbert space $H_F=\oplus_{\alpha\in F}H_\alpha$ and the canonical projection $P_F:\Pi_{\alpha\in\Lambda}\overline{H_\alpha}\to\overline{H_F}$, $P_F((\overline{y_\alpha})_{\alpha\in\Lambda})=(\overline{y_\alpha})_{\alpha\in F}$. Then $|\hat{f}(\bar{y})|\leq C\|P_F\bar{y}\|$ for all $\bar{y}\in\Pi_{\alpha\in\Lambda}\overline{H_\alpha}$. It follows that (see, for instance, Ref. 18, Sec. 7.4.12) we have a well defined bounded linear functional $g_F\in\overline{H_F}^*$ such that $\hat{f}=g_FP_F$. However, $g_F=\langle\cdot,\bar{z}\rangle_{\overline{H_F}}$ for a certain $\bar{z}=(\overline{x_\alpha})_{\alpha\in F}\in\overline{H_F}$, that is,

$$g_F((\overline{y_\alpha})_{\alpha\in F})=\sum_{\alpha\in F}\langle\overline{y_\alpha},\overline{x_\alpha}\rangle_{\overline{H_\alpha}}=\sum_{\alpha\in F}\langle x_\alpha,y_\alpha\rangle_{H_\alpha}=\sum_{\alpha\in F}\langle x_\alpha,y_\alpha\rangle.$$

In particular, $f(y^-)=g_FP_F(y^-)=g_F((\overline{y_\alpha})_{\alpha\in F})=\Sigma_{\alpha\in F}\langle x_\alpha,y_\alpha\rangle=\Sigma_{\alpha\in F}x''_\alpha(y^-)$ for all $y^-=(\overline{y_\alpha})_{\alpha\in\Lambda}\in\mathcal{D}^-$. Finally, $\{x_\alpha:\alpha\in F\}\subseteq H_\gamma$ for a certain $\gamma\in\Lambda$. In particular, $x=\Sigma_{\alpha\in F}x_\alpha\in H_\gamma$ and $x''(y^-)=\langle x,y_\gamma\rangle=\Sigma_{\alpha\in F}\langle x_\alpha,y_\gamma\rangle=\Sigma_{\alpha\in F}x''_\alpha(y^-)=f(y^-)$ for all $y^-\in\mathcal{D}^-$. Consequently, $f=x''\in\mathcal{D}$. Therefore, $\mathcal{D}=C(\mathcal{D}^-,\mathbb{C})$.

In particular, $\mathcal{D}=(\mathcal{D}'_{\mathfrak{S}})'$, thanks to Lemma 4.5. Finally, assume that $\mathcal{E}$ admits a gradation. As we have pointed out above $\mathcal{D}'_{\mathfrak{S}}=\mathcal{D}'_\beta$. It follows that $\mathcal{D}$ is semireflexive, that is, $\mathcal{D}=(\mathcal{D}'_\beta)'$. However, $\mathcal{D}$ being an inductive limit of the barreled spaces H_α, $\alpha\in\Lambda$, turns out to be a barreled polynormed space [Ref. 23, Sec. 2.7.2]. It remains to note that each semireflexive barreled polynormed space is reflexive [Ref. 12, Sec. 8.4.5] (see also Ref. 23, Sec. 4.5.6). □

E. The conjugate space as a quantum space

Assume that $\mathcal{E}=\{H_\alpha:\alpha\in\Lambda\}$ is a quantum domain in a Hilbert space H with its union space $\mathcal{D}$. Consider the class

$$\mathfrak{N}=\mathcal{E}\cup\{H_\alpha^\perp\cap H_\beta:\alpha\leq\beta\}$$

of Hilbert spaces and let q be a quantization over the class $\mathfrak{N}$ (see Sec. III A). Then we have a direct quantum family $\mathcal{E}_q=\{H_{\alpha,q}:\alpha\in\Lambda\}$ of quantum normed spaces (see Sec. III B). Indeed, if $\alpha\leq\beta$ then $H_\beta=H_\alpha\oplus(H_\alpha^\perp\cap H_\beta)$ and therefore the embedding $H_{\alpha,q}\subseteq H_{\beta,q}$ is a matrix isometry. In particular, $\mathcal{D}$ being an inductive limit of the direct quantum family $\mathcal{E}_q$ turns out to be a quantum space, that is,

$$\mathcal{D}_q=\underset{\rightarrow}{\mathrm{oplim}}\{H_{\alpha,q}:\alpha\in\Lambda\}.$$

Note that

$$M_n(\mathcal{D}'_q)=\mathcal{C}(\mathcal{D}_q,M_n)=\mathcal{MC}(\mathcal{D}_q,M_n)=\underset{\leftarrow}{\lim}\{\mathcal{MB}(H_{\alpha,q},M_n),P^{(n)}_{\alpha\beta}\},$$

where $P^{(n)}_{\alpha\beta}:\mathcal{MB}(H_{\beta,q},M_n)\to\mathcal{MB}(H_{\alpha,q},M_n)$, $P^{(n)}_{\alpha\beta}(T)=T|H_\alpha$, are the connecting mappings of the inverse system $\{\mathcal{MB}(H_{\alpha,q},M_n),P^{(n)}_{\alpha\beta}\}$ (see Corollary 3.4). Further, if $\mathcal{E}$ admits a gradation $\mathcal{N}=\{N_\iota:\iota\in\Xi\}$, that is, $H_\alpha=\oplus_{\iota\in\alpha}N_\iota$ for a finite subset $\alpha\subseteq\Xi$, then $\mathfrak{N}=\mathcal{E}=\{H_\alpha:\alpha\in\Lambda\}$, where Λ is the set of all finite subsets in Ξ. Using Proposition 3.1, we conclude that

$$\mathcal{D}_q=\underset{\rightarrow}{\mathrm{oplim}}\{H_{\alpha,q}:\alpha\in\Lambda\}=\mathrm{op}\bigoplus_{\iota\in\Xi}N_{\iota,q}. \tag{4.3}$$

Furthermore, in this case $\mathfrak{S}=\{\text{ball } M(H_{\alpha,q}):\alpha\in\Lambda\}$ is a fundamental system of matrix bounded sets in $\mathcal{D}_q$ (see Corollary 3.2). In particular,

$$(\mathcal{D}_q)'_{\mathfrak{S}} = (\mathcal{D}_q)'_\beta. \tag{4.4}$$

Now consider the conjugate space $\mathcal{D}^- = \lim_{\leftarrow}\{\overline{H_\alpha}, P_{\alpha\beta}\}$ of the domain $\mathcal{E}$ introduced in the previous subsection. Since all connecting morphisms $P_{\alpha\beta}: \overline{H_\beta} \to \overline{H_\alpha}$ of the inverse system $\{\overline{H_\alpha}, P_{\alpha\beta}\}$ are projections, it follows that $\{\overline{H_{\alpha,q}}, P_{\alpha\beta}\}$ is an inverse system of the quantum normed spaces with the matrix contractive morphisms $P_{\alpha\beta}: \overline{H_{\beta,q}} \to \overline{H_{\alpha,q}}$ (see Sec. III A). In particular, we have the quantum space $\mathcal{D}^{-q} = \operatorname{oplim}_{\leftarrow}\{\overline{H_{\alpha,q}}, P_{\alpha\beta}\}$.

The conjugate space $\mathcal{D}^-$ can also be equipped with another, more natural, quantum space structure using the quantum (or operator) duals $\underline{H_\alpha^*}$, $\alpha \in \Lambda$. Since the conjugate space $\overline{H_\alpha}$ is identified with H_α^* up to the canonical isometry $\theta_\alpha: \overline{H_\alpha} \to H_\alpha^*$, $\overline{v_\alpha} \mapsto \langle \cdot , v_\alpha\rangle$, it follows that $\overline{H_\alpha}$ is a quantum normed space denoted by $\overline{H_\alpha}^q$ equipped with the dual matrix norm denoted by $\|\cdot\|^q$ from the quantum dual $H_{\alpha,q}^*$.

Lemma 4.6: The family $\{\overline{H_\alpha}^q, P_{\alpha\beta}\}$ is an inverse system of the quantum normed spaces with the connecting matrix contractive morphisms. In particular, the conjugate space $\mathcal{D}^-$ turns out to be a quantum space

$$\mathcal{D}_q^- = \operatorname*{oplim}_{\leftarrow}\{\overline{H_\alpha}^q, P_{\alpha\beta}\}$$

with its defining family $\{\|\cdot\|_\alpha^q : \alpha \in \Lambda\}$ of matrix seminorms.

Proof: First note that if $\overline{y_\alpha} \in M_n(\overline{H_\alpha}^q)$, then (see Sec. IV C, see also Ref. 13, Sec. III B) $\|\overline{y_\alpha}\|^q = \sup\|\langle\langle \text{ball } M(\underline{H_{\alpha,q}}), y_\alpha\rangle\rangle\|$, where $\langle\langle x, y_\alpha\rangle\rangle = [\langle x_{ij}, y_\alpha^{kl}\rangle]$ if $x = [x_{ij}]$, $y_\alpha = [y_\alpha^{kl}]$. We have to prove that the projection $P_{\alpha\beta}: \overline{H_\beta}^q \to \overline{H_\alpha}^q$ is a matrix contraction. If $\alpha \leqslant \beta$, then the following diagram

$$\begin{array}{ccc} M_n(\overline{H_\beta}^q) & \xrightarrow{\theta_\beta^{(n)}} & M_n(H_{\beta,q}^*) = \mathcal{MB}(H_{\beta,q}, M_n) \\ P_{\alpha\beta}^{(n)}\downarrow & & \downarrow \\ M_n(\overline{H_\alpha}^q) & \xrightarrow{\theta_\alpha^{(n)}} & M_n(H_{\alpha,q}^*) = \mathcal{MB}(H_{\alpha,q}, M_n) \end{array}$$

commutes, where the second vertical arrow is the restriction mapping $T \mapsto T|H_\alpha$. Indeed, just observe that $\langle x_\alpha, y_\beta\rangle = \langle x_\alpha, P_{\alpha\beta} y_\beta\rangle$ whenever $x_\alpha \in H_\alpha$ and $y_\beta \in H_\beta$. Since the embedding $H_{\alpha,q} \to H_{\beta,q}$ is an isometrical inclusion, it follows that the restriction mapping $H_{\beta,q}^* \to H_{\alpha,q}^*$ is a matrix contraction (Ref. 13, Sec. 3.2.2). In particular, $P_{\alpha\beta}: \overline{H_\beta}^q \to \overline{H_\alpha}^q$ is a matrix contraction. Therefore, $\mathcal{D}^-$ turns out to be a quantum space $\mathcal{D}_q^- = \operatorname{oplim}_{\leftarrow}\{\overline{H_\alpha}^q, P_{\alpha\beta}\}$ with its defining family $\{\|\cdot\|_\alpha^q : \alpha \in \Lambda\}$ of matrix seminorms. If $y^- = [y_{ij}^-] \in M_n(\mathcal{D}_q^-) = \lim_{\leftarrow}\{M_n(\overline{H_\alpha}^q), P_{\alpha\beta}^{(n)}\}$, then

$$y^- = [y_{ij}^-] = [(\overline{y_\alpha^{ij}})] = ([\overline{y_\alpha^{ij}}]) = (\overline{y_\alpha}), \quad \overline{y_\alpha} = [\overline{y_\alpha^{ij}}] \in M_n(\overline{H_\alpha}^q),$$

and $\|y^-\|_\alpha^q = \|\overline{y_\alpha}\|^q$, $\alpha \in \Lambda$. □

Lemma 4.7: The linear mapping

$$\mathcal{D}_q^- \to (\mathcal{D}_q)'_{\mathfrak{S}}, \quad y^- = (\overline{y_\alpha})_{\alpha \in \Lambda} \mapsto f_y, \quad f_y|H_\alpha = \langle \cdot , y_\alpha\rangle,$$

is a topological matrix isomorphism.

Proof: By Lemma 4.5, the just indicated mapping is a topological isomorphism of the relevant polynormed spaces. By its very definition, the quantum dual $(\mathcal{D}_q)'_{\mathfrak{S}}$ has a defining family of the dual matrix seminorms $\|\cdot\|_\alpha^\odot$, $\alpha \in \Lambda$ (see Sec. IV C). Take $y^- = [y_{ij}^-] \in M_n(\mathcal{D}_q^-)$ and consider the matrix $f_y = [f_{y_{ij}}] \in M_n((\mathcal{D}_q)'_{\mathfrak{S}})$. Note that $f_{y_{ij}}|H_\gamma = \langle \cdot , y_\gamma^{ij}\rangle$, $\gamma \in \Lambda$, for all i, j. If $x = [x_{ij}] \in M_r(H_\alpha)$, then

$$\langle\langle x, f_y\rangle\rangle = [f_{y_{kl}}(x_{ij})] = [\langle x_{ij}, y_\alpha^{kl}\rangle] = \langle\langle [x_{ij}], [y_\alpha^{kl}]\rangle\rangle = \langle\langle x, y_\alpha\rangle\rangle$$

(see Lemma 4.6 and its proof). It follows that

$$\|f_y\|_\alpha^{\odot} = \sup\|\langle\langle \text{ball } M(H_\alpha), f_y\rangle\rangle\| = \sup\|\langle\langle \text{ball } M(H_\alpha), y_\alpha\rangle\rangle\| = \|\overline{y_\alpha}\|^q = \|\bar{y}\|_\alpha^q,$$

that is, $\|f_y\|_\alpha^{\odot} = \|\bar{y}\|_\alpha^q$ for all $\alpha \in \Lambda$. The rest is clear. □

Theorem 4.1: *Let $\mathcal{D}$ be the union space of a quantum domain $\mathcal{E}$ which admits a gradation. Then $\mathcal{D}_q$ is a quantum reflexive space, that is, $\mathcal{D}_q = ((\mathcal{D}_q)'_\beta)'_\beta$.* In particular, $\mathcal{D}_q = (\mathcal{D}_q^-)'_\beta$.

Proof: Since $\mathcal{D}_q = \text{op}\oplus_{\iota\in\Xi} N_{\iota,q}$ [see (4.3)] it follows that $\mathcal{D}_q$ is a quantum bornological space, thanks to Proposition 4.1. Using Proposition 4.2, we infer that the canonical embedding $\mathcal{D}_q \hookrightarrow ((\mathcal{D}_q)'_\beta)'_\beta$ is a matrix homeomorphic injection. However, $\mathcal{D} = (\mathcal{D}'_\beta)'_\beta$ as the polynormed spaces, thanks to Proposition 4.3. Whence $\mathcal{D}_q = ((\mathcal{D}_q)'_\beta)'_\beta$. Using (4.4) and Lemma 4.7, we conclude that $\mathcal{D}_q = (\mathcal{D}_q^-)'_\beta$. □

Corollary 4.1: Let $\mathcal{N} = \{N_\iota : \iota \in \Xi\}$, $\mathcal{M} = \{M_\kappa : \kappa \in \Phi\}$ be finite rank gradations in H with the same union space $\mathcal{D} = \Sigma_{\iota\in\Xi} N_\iota = \Sigma_{\kappa\in\Phi} M_\kappa$. Then

$$\mathcal{D}_{q_1} = \text{op} \bigoplus_{\iota\in\Xi} N_{\iota,q_1} = \text{op} \bigoplus_{\kappa\in\Phi} M_{\kappa,q_2} = \mathcal{D}_{q_2} = \max \mathcal{D}$$

for all quantizations q_1 and q_2.

Proof: For finite subsets $\alpha \subseteq \Xi$ and $\theta \subseteq \Phi$, we set $H_\alpha = \oplus_{\iota\in\alpha} N_\iota$ and $K_\theta = \oplus_{\kappa\in\theta} M_\kappa$. Then $\mathcal{D}_{q_1} = \text{oplim}_{\rightarrow}\{H_{\alpha,q_1}\} = \text{op}\oplus_{\iota\in\Xi} N_{\iota,q_1}$ and $\mathcal{D}_{q_2} = \text{oplim}_{\rightarrow}\{K_{\theta,q_2}\} = \text{op}\oplus_{\kappa\in\Phi} M_{\kappa,q_2}$, thanks to (4.3). Using (4.4) and Lemma 4.7, we conclude that

$$(\mathcal{D}_{q_1})'_\beta = \mathcal{D}_{q_1}^- = \underset{\leftarrow}{\text{oplim}}\{\overline{H_\alpha}^{q_1}\} \quad \text{and} \quad (\mathcal{D}_{q_2})'_\beta = \mathcal{D}_{q_2}^- = \underset{\leftarrow}{\text{oplim}}\{\overline{K_\theta}^{q_2}\}.$$

Since all spaces $\overline{H_\alpha}$ and $\overline{K_\theta}$ are finite dimensional, it follows that $\mathcal{D}^-$ is a nuclear polynormed space (Ref. 23, Sec. 3.7.4). By Theorem 2.2, $\mathcal{D}^-$ admits precisely one quantization. In particular, $\mathcal{D}_{q_1}^- = \mathcal{D}_{q_2}^-$. Using Theorem 4.1, we derive that $\mathcal{D}_{q_1} = (\mathcal{D}_{q_1}^-)'_\beta = (\mathcal{D}_{q_2}^-)'_\beta = \mathcal{D}_{q_2}$. In particular, putting $q = \max$ and using Theorem 3.1, we derive that $\mathcal{D}_{\max} = \text{oplim}_{\rightarrow}\{H_{\alpha,\max}\} = \max \mathcal{D}$. □

V. NONCOMMUTATIVE CONTINUOUS FUNCTIONS

In this section we investigate the spaces of noncommutative continuous functions on quantum domains.

A. The quantizations of $\mathcal{C}_\Lambda(\mathcal{D},\Delta)$

Let $\mathcal{E} = \{H_\alpha : \alpha \in \Lambda\}$ and $\mathcal{S} = \{K_\alpha : \alpha \in \Lambda\}$ be quantum domains in Hilbert spaces H and K with their union spaces $\mathcal{D}$ and Δ, respectively. The linear space of all *noncommutative continuous Δ-valued functions on $\mathcal{D}$* is defined as

$$\mathcal{C}_\Lambda(\mathcal{D},\Delta) = \{T \in L(\mathcal{D},\Delta) : T(H_\alpha) \subseteq K_\alpha, T|H_\alpha \in \mathcal{B}(H_\alpha,K_\alpha), \alpha \in \Lambda\}.$$

Note that $\mathcal{C}_\Lambda(\mathcal{D},\Delta)$ is a subspace in the space $L(\mathcal{D},\Delta)$ of all linear transformations acting from $\mathcal{D}$ into Δ. If $\mathcal{E} = S$ then we write $C_{\mathcal{E}}(\mathcal{D})$ (Ref. 6) instead of $\mathcal{C}_\Lambda(\mathcal{D},\mathcal{D})$. Obviously, $C_{\mathcal{E}}(\mathcal{D})$ is a unital subalgebra in the algebra $L(\mathcal{D})$ of all linear transformations on $\mathcal{D}$. Note that

$$M_n(\mathcal{C}_\Lambda(\mathcal{D},\Delta)) = \mathcal{C}_\Lambda(\mathcal{D}^n,\Delta^n) = \{T \in L(\mathcal{D}^n,\Delta^n) : T(H_\alpha^n) \subseteq K_\alpha^n, T|H_\alpha^n \in \mathcal{B}(H_\alpha^n,K_\alpha^n), \alpha \in \Lambda\}.$$

In particular, $M_n(C_{\mathcal{E}}(\mathcal{D})) = C_{\mathcal{E}^n}(\mathcal{D}^n)$ with $\mathcal{E}^n = \{H_\alpha^n : \alpha \in \Lambda\}$. The seminorms

$$\|T\|_\alpha^{(n)} = \|T|H_\alpha^n\|, \quad T \in M_n(\mathcal{C}_\Lambda(\mathcal{D},\Delta)), \quad n \in \mathbb{N},$$

define the matrix seminorm $\|\cdot\|_\alpha$ on $\mathcal{C}_\Lambda(\mathcal{D},\Delta)$ (see Ref. 6). Hence, $\mathcal{C}_\Lambda(\mathcal{D},\Delta)$ is a quantum space whose quantum topology is determined by the family $\{\|\cdot\|_\alpha : \alpha \in \Lambda\}$ of matrix seminorms. If $\mathcal{E}$ and $\mathcal{S}$ admit gradations $\mathcal{N} = \{N_\iota : \iota \in \Xi\}$ and $\mathcal{N}' = \{N'_\iota : \iota \in \Xi\}$, respectively, then $\mathcal{C}_\Lambda(\mathcal{D},\Delta) = \mathcal{C}_\Xi(\mathcal{D},\Delta)$, where

$$\mathcal{C}_\Xi(\mathcal{D},\Delta) = \{T \in L(\mathcal{D},\Delta) : T(N_\iota) \subseteq N_\iota, T|N_\iota \in \mathcal{B}(N_\iota,N'_\iota), \iota \in \Xi\}.$$

If $\mathcal{E}=\mathcal{S}$, we obtain (see Ref. 6 for the general case) *the* $*$*-algebra*

$$C^{*}_{\Xi}(\mathcal{D})=\mathcal{C}_{\Xi}(\mathcal{D},\mathcal{D})=\{T\in L(\mathcal{D}):T|N_{\iota}\in\mathcal{B}(N_{\iota}),\iota\in\Xi\},\tag{5.1}$$

of all noncommutative continuous functions on $\mathcal{D}$. Actually, $C^{*}_{\Xi}(\mathcal{D})$ possesses the natural involution as follows from the following assertion (see Ref. 6).

Proposition 5.1: Each unbounded operator $T\in C^{*}_{\Xi}(\mathcal{D})$ *has an unbounded dual* $T^{\bigstar}$ *such that* $\mathcal{D}\subseteq\mathrm{dom}(T^{\bigstar})$, $T^{\bigstar}(\mathcal{D})\subseteq\mathcal{D}$, *and* $T^{*}=T^{\bigstar}|\mathcal{D}\in C^{*}_{\Xi}(\mathcal{D})$. *The correspondence* $T\mapsto T^{*}$ *is an involution on* $C^{*}_{\Xi}(\mathcal{D})$, *thereby* $C^{*}_{\Xi}(\mathcal{D})$ *is a unital multinormed* C^{*}*-algebra. In particular,* $C^{*}_{\Xi}(\mathcal{D})$ *consists of closable unbounded operators.*

Proof: Take $T\in C^{*}_{\Xi}(\mathcal{D})$ and consider $S\in L(\mathcal{D})$, $S(\Sigma_{\iota}x_{\iota})=\Sigma_{\iota}S_{\iota}x_{\iota}$ with $S_{\iota}=(T|N_{\iota})^{*}\in\mathcal{B}(N_{\iota})$, $\iota\in\Xi$. Evidently, $S\in C^{*}_{\Xi}(\mathcal{D})$ [see (5.1)]. Using Lemma 4.3, we derive that $\langle Tx,y\rangle=\langle\Sigma_{\iota}Tx_{\iota},\Sigma_{\iota}y_{\iota}\rangle=\Sigma_{\iota}\langle Tx_{\iota},y_{\iota}\rangle=\Sigma_{\iota}\langle x_{\iota},S_{\iota}y_{\iota}\rangle=\langle x,Sy\rangle$ for all $x,y\in\mathcal{D}$. Consequently, we have an unbounded dual $T^{\bigstar}$ of T such that $S=T^{*}=T^{\bigstar}|\mathcal{D}\in C^{*}_{\Xi}(\mathcal{D})$.

Finally, take $T\in C^{*}_{\Xi}(\mathcal{D})$, and assume that $\lim x_n=0$ and $\lim Tx_n=z\in H$ for a certain sequence $\{x_n\}$ in $\mathcal{D}$. If $y\in\mathcal{D}$, then $\langle z,y\rangle=\lim\langle Tx_n,y\rangle=\lim\langle x_n,T^{*}y\rangle=0$, that is, $z\perp\mathcal{D}$. Being $\mathcal{D}$ a dense subspace, infer that $z=0$. Whence T admits the closure.□

The following assertion plays a key role in further investigations.

Theorem 5.1: *The space* $\mathcal{C}_{\Lambda}(\mathcal{D},\Delta)$ *is exactly the space of all matrix continuous linear mappings* $T:\mathcal{D}_c\to\Delta_c$ *such that* $T(H_\alpha)\subseteq K_\alpha$ *for all* $\alpha\in\Lambda$, *that is,*

$$\mathcal{C}_{\Lambda}(\mathcal{D},\Delta)=\mathcal{MC}_{\Lambda}(\mathcal{D}_c,\Delta_c),$$

where c is the column quantization over all Hilbert spaces.

Proof: Take $T\in\mathcal{MC}_{\Lambda}(\mathcal{D}_c,\Delta_c)$. Then $T(H_\alpha)\subseteq K_\alpha$ and $T|H_\alpha\in\mathcal{MB}(H_{\alpha,c},K_{\alpha,c})$, $\alpha\in\Lambda$, by virtue of Theorem 3.2. In particular, $T|H_\alpha\in\mathcal{B}(H_\alpha,K_\alpha)$ for all $\alpha\in\Lambda$. Therefore, $T\in\mathcal{C}_{\Lambda}(\mathcal{D},\Delta)$. Conversely, take $T\in\mathcal{C}_{\Lambda}(\mathcal{D},\Delta)$. Then $T(H_\alpha)\subseteq K_\alpha$ and $T|H_\alpha\in\mathcal{B}(H_\alpha,K_\alpha)$ for all $\alpha\in\Lambda$. However, $\mathcal{B}(H_\alpha,K_\alpha)=\mathcal{MB}(H_{\alpha,c},K_{\alpha,c})$ up to the canonical matrix isometry (see Ref. 13, Sec. 3.4.1). It follows that $T\in\mathcal{MC}_{\Lambda}(\mathcal{D}_c,\Delta_c)$, thanks to Proposition 3.4. □

Let q_1 and q_2 be quantizations over Hilbert space classes. Let us introduce the following quantum space:

$$\mathcal{C}_{q_1,\Lambda,q_2}(\mathcal{D},\Delta)=\{T\in L(\mathcal{D},\Delta):T(H_\alpha)\subseteq K_\alpha,T|H_\alpha\in\mathcal{MB}(H_{\alpha,q_1},K_{\alpha,q_2}),\alpha\in\Lambda\}$$

with the matrix seminorms $\|\cdot\|_{q_1,\alpha,q_2}$, where

$$\|T\|^{(n)}_{q_1,\alpha,q_2}=\|T|H^n_\alpha\|_{mb}\quad\text{in }\mathcal{MB}(H_{\alpha,q_1},M_n(K_{\alpha,q_2})),\quad T\in M_n(\mathcal{C}_{q_1,\Lambda,q_2}(\mathcal{D},\Delta)),\alpha\in\Lambda.$$

Obviously, $\mathcal{C}_{q_1,\Lambda,q_2}(\mathcal{D},\Delta)$ is a linear subspace in $\mathcal{C}_{\Lambda}(\mathcal{D},\Delta)$. Moreover,

$$\mathcal{C}_{q_1,\Lambda,q_2}(\mathcal{D},\Delta)=\mathcal{MC}_{\Lambda}(\mathcal{D}_{q_1},\Delta_{q_2})\tag{5.2}$$

(as the linear spaces) by virtue of Theorem 3.2.

Corollary 5.1: The spaces $\mathcal{C}_{\Lambda}(\mathcal{D},\Delta)$ *and* $\mathcal{C}_{c,\Lambda,c}(\mathcal{D},\Delta)$ *are identical as the quantum spaces. Thus,*

$$\mathcal{C}_{\Lambda}(\mathcal{D},\Delta)=\mathcal{C}_{c,\Lambda,c}(\mathcal{D},\Delta).$$

Proof: Take $T\in M_n(\mathcal{C}_{\Lambda}(\mathcal{D},\Delta))$. Then $T|H^n_\alpha\in\mathcal{B}(H^n_\alpha,K^n_\alpha)=M_n(\mathcal{B}(H_\alpha,K_\alpha))$. However, $\mathcal{B}(H_\alpha,K_\alpha)=\mathcal{MB}(H_{\alpha,c},K_{\alpha,c})$ as the quantum normed spaces. Therefore,

$$\begin{aligned}\|T\|^{(n)}_\alpha&=\|T:H^n_\alpha\to K^n_\alpha\|=\|T|H^n_\alpha\|_{M_n(\mathcal{B}(H_\alpha,K_\alpha))}=\|T|H^n_\alpha\|_{M_n(\mathcal{MB}(H_{\alpha,c},K_{\alpha,c}))}=\|T|H^n_\alpha\|_{\mathcal{MB}(H_{\alpha,c},M_n(K_{\alpha,c}))}\\&=\|T:H_{\alpha,c}\to M_n(K_{\alpha,c})\|_{mb}=\|T\|^{(n)}_{c,\alpha,c},\end{aligned}$$

that is, $\|T\|^{(n)}_\alpha=\|T\|^{(n)}_{c,\alpha,c}$. It remains to use Theorem 5.1 (see also Ref. 13, Sec. 3.4.1). □

Now let $\mathfrak{S}=\{\text{ball } M(H_{\alpha,q_1}):\alpha\in\Lambda\}$ be the (matrix total) family in $M(\mathcal{D}_{q_1})$ of all unit sets. Consider the quantum space $\mathcal{MC}_\Lambda(\mathcal{D}_{q_1},\Delta_{q_2})_{\mathfrak{S}}$ equipped with the $\mathfrak{S}$-quantum topology (see Sec. III D).

Theorem 5.2: *Let $\mathcal{E}=\{H_\alpha:\alpha\in\Lambda\}$ and $\mathcal{S}=\{K_\alpha:\alpha\in\Lambda\}$ be quantum domains with their union spaces $\mathcal{D}$ and Δ, respectively. Then $\mathcal{C}_{q_1,\Lambda,q_2}(\mathcal{D},\Delta)=\mathcal{MC}_\Lambda(\mathcal{D}_{q_1},\Delta_{q_2})_{\mathfrak{S}}$ for all quantizations q_1 and q_2. In particular, $\mathcal{C}_\Lambda(\mathcal{D},\Delta)=\mathcal{MC}_\Lambda(\mathcal{D}_c,\Delta_c)_{\mathfrak{S}}$.*

Proof: Let us prove that the family $\{\|\cdot\|_{q_1,\alpha,q_2},\alpha\in\Lambda\}$ of matrix seminorms on $\mathcal{C}_{q_1,\Lambda,q_2}(\mathcal{D},\Delta)$ and the family (4.2) on $\mathcal{MC}_\Lambda(\mathcal{D}_{q_1},\Delta_{q_2})_{\mathfrak{S}}$ are equivalent. Take a matrix $T=[T_{ij}]\in M_n(\mathcal{C}_{q_1,\Lambda,q_2}(\mathcal{D},\Delta))$ being identified with the relevant matrix continuous linear mapping $T:\mathcal{D}_{q_1}\to M_n(\Delta_{q_2})$ [see (5.2) and Lemma 4.1] from $\mathcal{MC}(\mathcal{D}_{q_1},M_n(\Delta_{q_2}))$. Using (3.3) (see to the proof of Theorem 3.1), we derive that

$$\begin{aligned}p_{\rho,\alpha}^{(n)}(T)&=\sup\{p_\rho^{(rn)}(T^{(r)}(v)):v\in\text{ball } M_r(H_{\alpha,q_1}),r\in\mathbb{N}\}\leq\rho_\alpha^{-1}\sup\{\|T^{(r)}(v)\|_{M_{rn}(K_{\alpha,q_2})}:v\\&\in\text{ball } M_r(H_{\alpha,q_1}),r\in\mathbb{N}\}=\rho_\alpha^{-1}\sup\{\|T^{(r)}:M_r(H_{\alpha,q_1})\to M_{rn}(K_{\alpha,q_2})\|:r\in\mathbb{N}\}=\rho_\alpha^{-1}\|T:H_{\alpha,q_1}\\&\to M_n(K_{\alpha,q_2})\|_{mb}=\rho_\alpha^{-1}\|T\|_{q_1,\alpha,q_2}^{(n)}.\end{aligned}$$

Note that $T|H_\alpha^n\in M_n(\mathcal{MB}(H_{\alpha,q_1},K_{\alpha,q_2}))=\mathcal{MB}(H_{\alpha,q_1},M_n(K_{\alpha,q_2}))$. So, $p_{\rho,\alpha}\leq\rho_\alpha^{-1}\|\cdot\|_{q_1,\alpha,q_2}$. It follows that the quantum topology on $\mathcal{C}_{q_1,\Lambda,q_2}(\mathcal{D},\Delta)$ generated by the matrix seminorms $\|\cdot\|_{q_1,\alpha,q_2}$, $\alpha\in\Lambda$, is finer than the $\mathfrak{S}$-quantum topology. Now assume that ρ is a bounded family. Again using the argument used in the proof of Theorem 3.1, we deduce that

$$\begin{aligned}\sup(\rho)^{-1}\|T\|_{q_1,\alpha,q_2}^{(n)}&=\sup\{\sup(\rho)^{-1}\|T^{(r)}(v)\|_{M_{rn}(K_{\alpha,q_2})}:v\in\text{ball } M_r(H_{\alpha,q_1}),r\in\mathbb{N}\}\leq\sup\{p_\rho^{(rn)}(T^{(r)}(v)):v\\&\in\text{ball } M_r(H_{\alpha,q_1}),r\in\mathbb{N}\}=p_{\rho,\alpha}^{(n)}(T).\end{aligned}$$

Consequently, $\{\|\cdot\|_{q_1,\alpha,q_2},\alpha\in\Lambda\}$ and $\{p_{\rho,\alpha}:\rho\in\mathbb{R}_+^\Lambda,\alpha\in\Lambda\}$ are equivalent matrix seminorms, that is, $\mathcal{C}_{q_1,\Lambda,q_2}(\mathcal{D},\Delta)=\mathcal{MC}_\Lambda(\mathcal{D}_{q_1},\Delta_{q_2})_{\mathfrak{S}}$.

If $q_1=q_2=c$ are the same column quantization over all Hilbert spaces then using Corollary 5.1, we conclude that $\mathcal{C}_\Lambda(\mathcal{D},\Delta)=\mathcal{MC}_\Lambda(\mathcal{D}_c,\Delta_c)_{\mathfrak{S}}$. □

Corollary 5.2 If $\mathcal{E}$ and $\mathcal{S}$ admit gradations $\mathcal{N}=\{N_\iota:\iota\in\Xi\}$ and $\mathcal{N}'=\{N'_\iota:\iota\in\Xi\}$, respectively, then $\mathcal{C}_{q_1,\Xi,q_2}(\mathcal{D},\Delta)=\mathcal{MC}_\Xi(\mathcal{D}_{q_1},\Delta_{q_2})_\beta$.

Proof: It suffices to apply (4.3), Theorem 5.2 and Proposition 3.1 (see Sec. IV B). □

Corollary 5.3: Let $\mathcal{E}$ and $\mathcal{S}$ be atomic domains with the finite rank gradations $\mathcal{N}=\{N_\iota:\iota\in\Xi\}$ and $\mathcal{N}'=\{N'_\iota:\iota\in\Xi\}$, respectively. Then

$$\mathcal{C}_\Xi(\mathcal{D},\Delta)=\mathcal{C}_{q_1,\Xi,q_2}(\mathcal{D},\Delta)=\mathcal{MC}_\Xi(\max\mathcal{D},\max\Delta)_\beta$$

for all quantizations q_1 and q_2.

Proof: Indeed, we have $\mathcal{C}_\Xi(\mathcal{D},\Delta)=\mathcal{C}_{c,\Xi,c}(\mathcal{D},\Delta)=\mathcal{MC}_\Xi(\mathcal{D}_c,\Delta_c)_\beta=\mathcal{MC}_\Xi(\max\mathcal{D},\max\Delta)_\beta=\mathcal{MC}_\Lambda(\mathcal{D}_{q_1},\Delta_{q_2})_\beta=\mathcal{C}_{q_1,\Xi,q_2}(\mathcal{D},\Delta)$, thanks to Corollaries 5.2 and 4.1 □

B. Quantum Arens–Mackey scale

Let (V,W) be a dual pair with the duality $\langle\cdot,\cdot\rangle$. We say that a quantum topology $\mathfrak{p}$ in $M(V)$ is *compatible with the duality* (V,W) if $V'=W$ with respect to the (polynormed) topology $\mathfrak{s}=\mathfrak{p}|V$. Thus, all quantizations of the Arens–Mackey scale $\sigma(V,W)\subseteq\mathfrak{s}\subseteq\tau(V,W)$ called *the quantum Arens–Mackey scale of the pair* (V,W) are precisely quantum topologies compatible with the duality (V,W), thanks to Arens–Mackey theorem.

Lemma 5.1: A quantum topology $\mathfrak{p}$ in $M(V)$ *belongs to the quantum Arens-Mackey scale of the pair* (V,W) if and only if $\mathfrak{s}(V,W)\subseteq\mathfrak{p}\subseteq\mathfrak{t}(V,W)$, *where $\mathfrak{s}(V,W)$ is the weak quantum topology and* $\mathfrak{t}(V,W)=\max\tau(V,W)$.

Proof: Put $\mathfrak{s}=\mathfrak{p}|V$. Using Theorem 2.3, Corollary 2.5, and Proposition 2.2, we derive that

$$\mathfrak{s}(V,W) = \max\sigma(V,W) = \min\sigma(V,W) \subseteq \min\mathfrak{s} \subseteq \mathfrak{p} \subseteq \max\mathfrak{s} \subseteq \max\tau(V,W) = \mathfrak{t}(V,W)$$

whenever $\sigma(V,W)\subseteq\mathfrak{s}\subseteq\tau(V,W)$. □

Now let $\mathfrak{p}$ be a quantum topology in $M(V)$ compatible with the duality (V,W), and let $\Xi=\{p\}$ be its defining family of matrix seminorms. In order to avoid some technical details we assume that Ξ is *a saturated family*, that is, it is upward filtered, and $cp\in\Xi$ whenever $c\in\mathbb{R}_+$ and $p\in\Xi$. Consider the disjoint union $\Delta=\vee_p$ ball $p^{\odot}$ of unit sets in $M(W)$ of the dual gauges $p^{\odot}$. For each $w\in\Delta$ we set $N_w=\mathbb{C}^{n(w)}$ such that $n(w)=n$ whenever $w\in M_n(W)$. The Hilbert space sum $N_p=\oplus_{w\in\text{ ball } p^{\odot}}N_w$ is a closed subspace in the Hilbert space $H=\oplus_{w\in\Delta}N_w$. Since $H=\oplus_{p\in\Xi}N_p$, it follows that $\mathcal{N}=\{N_p:p\in\Xi\}$ is a gradation in H, thanks to Lemma 4.3, and $\mathcal{D}=\Sigma_pN_p$ is the union space of the quantum domain associated with $\mathcal{N}$. For each $w\in\Delta$ we have the matrix seminorm $p_w(v)=\|\langle\langle v,w\rangle\rangle\|$, $v\in M(V)$ (see Sec. II E). We have also the atomic gradation $\mathcal{A}=\{N_w:w\in\Delta\}$ in H with the same union space $\mathcal{D}$.

Lemma 5.2: The family $\{p_w:w\in\Delta\}$ defines the weak quantum topology $\mathfrak{s}(V,W)$.

Proof: Fix $w\in M_n(W)$. Since $W=\mathcal{C}(V,\mathbb{C})$ with respect to $\mathfrak{p}|V$, it follows (see Sec. II D) that $M_n(W)=\mathcal{C}(V,M_n)=\mathcal{MC}(V,M_n)$. Then $\|w^{(\infty)}(v)\|\leq p(v)$, $v\in M(V)$ for some $p\in\Xi$. Recall that $\mathbb{R}_+\Xi\subseteq\Xi$. It follows that $w\in(\text{ball } p)^{\odot}=\text{ball } p^{\odot}$ thanks to Corollary 2.3, that is, $w\in\Delta$. So, $\{p_w:w\in\Delta\}$ determines the weak quantum topology in $M(V)$ (see Theorem 2.3). □

Now we can prove the following representation theorem.

Theorem 5.3: *Let (V,W) be a dual pair and let $\mathfrak{p}$ be a quantum topology on V compatible with the duality (V,W). There is a topological matrix embedding $(V,\mathfrak{p})\hookrightarrow\mathcal{MC}(\mathcal{D}_c)_\beta$ for a certain quantum domain $\mathcal{E}$ with its union space $\mathcal{D}$. Moreover, there is a topological matrix embedding $(V,\mathfrak{s}(V,W))\hookrightarrow\mathcal{MC}(\max\mathcal{D})_\beta$ for a certain space $\mathcal{D}$ equipped with the finest polynormed topology.*

Proof. Let $\mathfrak{p}$ be a quantum topology in $M(V)$ compatible with the duality (V,W) with its defining family Ξ of matrix seminorms. Consider the linear mapping $\Phi:(V,\mathfrak{p})\to C^*_{\Xi}(\mathcal{D})$, $\Phi(v)=(\langle\langle v,w\rangle\rangle)_{w\in\Delta}$. If $v\in M_n(V)$ and $p\in\Xi$, then

$$\|\Phi^{(n)}(v)|N_p^n\|_{\mathcal{B}(N_p^n)} = \sup\|\langle\langle v,\text{ball } p^{\odot}\rangle\rangle\| = \sup_r\|\langle\langle v,\mathcal{MB}_p(V,M_r)\rangle\rangle\| = p^{(n)}(v),$$

thanks to Proposition 2.2. Thus Φ is a topological matrix embedding of $(V,\mathfrak{p})$ into $C^*_{\Xi}(\mathcal{D})$. However, $C^*_{\Xi}(\mathcal{D})=\mathcal{C}_{\Xi}(\mathcal{D},\mathcal{D})=\mathcal{C}_{c,\Xi,c}(\mathcal{D})=\mathcal{MC}_{\Xi}(\mathcal{D}_c)_\beta\subseteq\mathcal{MC}(\mathcal{D}_c)_\beta$, thanks to Corollaries 5.1 and 5.2, where $\mathcal{D}_c=\text{op}\oplus_{p\in\Xi}N_{p,c}$.

Further, $\|\Phi^{(n)}(v)|N_w^n\|_{\mathcal{B}(N_w^n)}=\|\langle\langle v,w\rangle\rangle\|=p_w^{(n)}(v)$ for all $v\in M_n(V)$, $n\in\mathbb{N}$, $w\in\Delta$. By Lemma 5.2, $\{p_w:w\in\Delta\}$ is a defining family of matrix seminorms for the weak quantum topology $\mathfrak{s}(V,W)$ in $M(V)$. It follows that $\Phi:(V,\mathfrak{s}(V,W))\to C^*_{\Delta}(\mathcal{D})$ is a topological matrix isomorphism onto its range. Using Corollary 5.3, we derive that $C^*_{\Delta}(\mathcal{D})=\mathcal{MC}_{\Delta}(\max\mathcal{D})_\beta\subseteq\mathcal{MC}(\max\mathcal{D})_\beta$, where $\mathcal{D}$ is equipped with the finest polynormed topology. □

Actually, the construction proposed in the proof of Theorem 5.3 allows us to classify all quantum topologies compatible with the given duality. Namely, consider a set J with a partition $J=\vee_{\kappa\in\Xi}J_\kappa$. We can identify J with the family $\{J_\kappa\}$ of sets. A family $I=\{I_\alpha\}$ of sets is said to be *a divisor of* J if for each I_α there corresponds a unique $\kappa\in\Xi$ such that $I_\alpha\subseteq J_\kappa$ and $J=\cup I$. For instance, $A=\{\{w\}:w\in J\}$ is a divisor of J called *the atomic divisor*, and $J=\{J_\kappa\}$ itself called *the top divisor.*

Assume for each $w\in J$ there corresponds *an atomic algebra* M_{n_w} of all scalar n_w-square matrices, where n_w can be thought as a value of a certain function $n:J\to\mathbb{N}$ at the point w. Then for each member J_κ of the family J relates von Neumann algebra $M_{J_\kappa}=\oplus^\infty_{w\in J_\kappa}M_{n_w}$. Consider the direct product $\mathfrak{D}_J=\Pi_{\kappa\in\Xi}M_{J_\kappa}$. Each element $a\in\mathfrak{D}_J$ can be written as a locally bounded family $a=(a_w)_{w\in J}$ with $\sup_{w\in J_\kappa}\|a_w\|<\infty$ for each κ. Each divisor I of J generates a family of matrix seminorms on $\mathfrak{D}_J$. Namely, if $I_\alpha\in I$ then we set

$$\pi_{I_\alpha}(a) = \sup\{\|a_w\| : w \in I_\alpha\}, \quad a \in M(\mathfrak{D}_J).$$

Note that $\pi_{I_\alpha}(a) \leqslant \pi_{J_\kappa}(a) = \|(a_w)_{w\in J_\kappa}\| < \infty$, that is, $\pi_{I_\alpha} \leqslant \pi_{J_\kappa}$ whenever $I_\alpha \subseteq J_\kappa$. If $\pi^{(1)}_{I_\alpha}(a)=0$ for all $I_\alpha \in I$, then $\|a_w\| = \pi^{(1)}_{\{w\}}(a)=0$ for all $w \in \cup I = J$, that is, $a=0$. Hence, the family $\mathfrak{d}_I = \{\pi_{I_\alpha} : I_\alpha \in I\}$ defines a (Hausdorff) quantum topology in $M(\mathfrak{D}_J)$ called *a divided quantum topology*. Let us introduce the notations $\sigma_w = \pi_{\{w\}}$, $w \in J$, and $\tau_\kappa = \pi_{J_\kappa}$, $\kappa \in \Xi$. Put

$$\mathfrak{a} = \{\sigma_w : w \in J\} = \mathfrak{d}_A \quad \text{and} \quad \mathfrak{t} = \{\tau_\kappa : \kappa \in \Xi\} = \mathfrak{d}_J.$$

They define quantum topologies in $M(\mathfrak{D}_J)$ called *the atomic and top quantum topologies* (or *boundaries*), respectively. We use the same notations $\mathfrak{a}$, $\mathfrak{d}_I$, and $\mathfrak{t}$ for the relevant quantum topologies in $M(\mathfrak{D}_J)$ associated with these matrix seminorms. Since $\sigma_w \leqslant \pi_{I_\alpha} \leqslant \tau_\kappa$ whenever $w \in I_\alpha \subseteq J_\kappa$, we obtain the following inclusions:

$$\mathfrak{a} \subseteq \mathfrak{d}_I \subseteq \mathfrak{t}$$

for the divided quantum topologies, where I is a divisor of J. For a linear subspace $X \subseteq \mathfrak{D}_J$, we have a scale of quantum topologies $\mathfrak{d}_I | M(X)$ on X inherited from $\mathfrak{d}_I$.

Theorem 5.4: *Let (V,W) be a dual pair. Then V can be identified with a subspace of $\mathfrak{D}_J$ such that the quantum Arens–Mackey scale of the pair (V,W) is precisely the quantum scale*

$$\mathfrak{a}|M(V) \subseteq \mathfrak{d}_I|M(V) \subseteq \mathfrak{t}|M(V).$$

Proof: Consider a saturated family $\Xi = \{p\}$ of matrix seminorms defining the quantum topology $\mathfrak{t}(V,W)$ (see Lemma 5.1). For each $p \in \Xi$, its unit set ball p is a weakly closed matrix set in $M(V)$. Indeed, each ball$p^{(n)}$ is an absolutely convex (Remark 2.1) closed set in $M_n(V) = (V, \tau(V,W))^{n^2}$. It follows that ball$p^{(n)}$ is closed with respect to the Mackey topology $\tau(M_n(V), M_n(W))$ (see Sec. II C). Then ball$p^{(n)}$ is $\sigma(M_n(V), M_n(W))$-closed, thanks to Mazur's theorem (Ref. 18, Sec. 10.4.9) applied to the dual pair $(M_n(V), M_n(W))$ with respect to the scalar pairing.

Put $J_p = \text{ball}p^{\odot} \subseteq M(W)$, $p \in \Xi$. Note that if $q \leqslant p$ for some p, $q \in \Xi$, then ball $p \subseteq$ ball q, which, in turn, implies that $J_q = \text{ball}q^{\odot} = (\text{ball } q)^{\odot} \subseteq (\text{ball } p)^{\odot} = \text{ball}p^{\odot} = J_p$, thanks to Corollary 2.3. Consider the family $J = \{J_p\}$ which can be identified with the relevant disjoint union, and the linear embedding $\Phi : V \to \mathfrak{D}_J$, $\Phi(v) = (\langle\langle v, w\rangle\rangle)_{w \in J}$ as in the proof of Theorem 5.3. Then $\sigma_w(\Phi^{(\infty)}(v)) = \|\langle\langle v, w\rangle\rangle\| = p_w(v)$, $v \in M(V)$, $w \in J$. By Lemma 5.2, $\{p_w : w \in J\}$ determines the weak quantum topology $\mathfrak{s}(V,W)$ in $M(V)$. Hence, $\mathfrak{a}|M(X)$ is identified with the weak quantum topology, where $X = \Phi(V)$. Further, for each $p \in \Xi$ we have

$$p(v) = \sup\|\langle\langle v, \text{ball } p^{\odot}\rangle\rangle\| = \sup\|\langle\langle v, J_p\rangle\rangle\| = \tau_p(\Phi^{(\infty)}(v)), \quad v \in M(V),$$

by virtue of Proposition 2.2 (see also Theorem 5.3). Note that ball p is weakly closed as we have just proved above. Hence, the upper bound $\mathfrak{t}|M(X) = \{\tau_p | M(X) : p \in \Xi\}$ of the quantum scale of X is identified with $\mathfrak{t}(V,W)$.

Now consider any quantum topology $\mathfrak{q}$ in $M(V)$ compatible with the given duality (V,W). This quantum topology is determined by a saturated family Θ of matrix seminorms on V. By Lemma 5.1, $\mathfrak{s}(V,W) \subseteq \mathfrak{q} \subseteq t(V,W)$. Hence, for each $q \in \Theta$ there corresponds $p \in \Xi$ such that $q \leqslant p$, which, in turn, implies that ball$q^{\odot} \subseteq J_p$ (Corollary 2.3). Put $I = \{\text{ball}q^{\odot} : q \in \Theta\}$. Using Zermelo's axiom of choice, one can assume that I is a family of subsets of the family J. Actually I is a divisor of J. Indeed, it suffices to prove that $\cup I = J$. Take $w \in J$. Since $\mathfrak{s}(V,W) \subseteq \mathfrak{q}$, it follows that $p_w \leqslant q$ for a certain $q \in \Theta$. Then $w \in (\text{ball}p_w)^{\odot} = \text{ball}p_w^{\odot} \subseteq \text{ball}q^{\odot} \subseteq I$ (see Corollary 2.3). So, I is a divisor of J. Moreover, $\mathfrak{q} = \mathfrak{d}_I | M(X)$. Indeed, if $q \in \Theta$ and $v \in M(V)$, then $q(v) = \sup\|\langle\langle v, \text{ball}q^{\odot}\rangle\rangle\| = \pi_{\text{ball}q^{\odot}}(\Phi^{(\infty)}(v))$, thanks to Proposition 2.2, that is, $\mathfrak{q} = \mathfrak{d}_I | M(X)$.

Finally, let us prove that each quantum topology $\mathfrak{d}_L$ in $M(V)$ induced from $\mathfrak{D}_J$ via the mapping $\Phi : V \to \mathfrak{D}_J$ belongs to the quantum Arens–Mackey scale of the pair (V,W), where $L = \{L_\alpha\}$ is a divisor of J. Put $q_{L_\alpha} = \pi_{L_\alpha} \cdot \Phi^{(\infty)}$. Each L_α is a subset of a certain $J_p (p \in \Xi)$ and

$$p_w(v) = \|\langle\langle v,w\rangle\rangle\| \leqslant \sup\|\langle\langle v,L_\alpha\rangle\rangle\| = \pi_{L_\alpha}(\Phi^{(\infty)}(v)) = q_{L_\alpha}(v) \leqslant \sup\|\langle\langle v,J_p\rangle\rangle\| = p(v)$$

for all $v \in M(V)$ and $w \in L_\alpha$. Thus $p_w \leqslant q_{L_\alpha} \leqslant p$ whenever $w \in L_\alpha \subseteq J_p$. It follows that $\mathfrak{s}(V,W) \subseteq \mathfrak{d}_L \subseteq \mathfrak{t}(V,W)$. By Lemma 5.1, $\mathfrak{d}_L$ belongs to the quantum Arens–Mackey scale of the pair (V,W). □

This is a noncommutative Arens–Mackey theorem mentioned in Sec. I.

C. The finite-rank operators in $C^*_\Xi(\mathcal{D})$

Everywhere below we fix a domain $\mathcal{E}=\{H_\alpha : \alpha \in \Lambda\}$ in a Hilbert space H which admits a gradation $\mathcal{N}=\{N_\iota : \iota \in \Xi\}$. So, Λ is the set of all finite subsets of Ξ, $H_\alpha = \oplus_{\iota\in\alpha} N_\iota$ for each $\alpha \in \Lambda$, and $H=\oplus_{\iota\in\Xi} N_\iota$ (see Lemma 4.3). The algebraic orthogonal sum $\mathcal{D}=\Sigma_{\iota\in\Xi} N_\iota \subseteq H$ is the union space of the domain $\mathcal{E}$. Consider the multinormed C^*-algebra $C^*_\Xi(\mathcal{D})$ of all noncommutative continuous functions on $\mathcal{E}$ [see (5.1)]. If $T \in C^*_\Xi(\mathcal{D})$, then $T(N_\kappa) \subseteq N_\kappa$, $\kappa \in \Xi$, therefore T has an infinite diagonal matrix realization $T=\oplus_{\kappa\in\Xi} T_\kappa$, $T_\kappa = T|N_\kappa \in \mathcal{B}(N_\kappa)$, $\kappa \in \Xi$. For $a,b \in H$ we put $a \otimes b \in \mathcal{B}(H)$ to denote the one-rank operator, that is, $(a\otimes b)x = \langle x,b\rangle a$, $x \in H$.

Lemma 5.3: If $a,b \in H\backslash\{0\}$, then $(a\otimes b)|_{\mathcal{D}} \in C^*_\Xi(\mathcal{D})$ if and only if both a and *b belong to the same nest subspace of the gradation $\mathcal{N}$.*

Proof: If $a,b \in N_\kappa$ for a certain κ, then $(a\otimes b)|N_\kappa^\perp = 0$ and $(a\otimes b)N_\kappa = \langle N_\kappa, b\rangle a \subseteq \mathbb{C}a \subseteq N_\kappa$. By Lemma 4.3, $(a\otimes b)|_{\mathcal{D}} \in C^*_\Xi(\mathcal{D})$.

Conversely, assume that $(a\otimes b)|_{\mathcal{D}} \in C^*_\Xi(\mathcal{D})$. Then $(a\otimes b)x \neq 0$ for some $x \in \mathcal{D}$, for in the contrary case $b \perp \mathcal{D}$ which means that $b=0$. Moreover, $\langle x,b\rangle a \in \mathcal{D}$, that is, $a \in \mathcal{D}$. Since $\langle (a \otimes b)x, y\rangle = \langle x, (b\otimes a)y\rangle$ for all $x,y \in H$, it follows that $(b\otimes a)|_{\mathcal{D}} = ((a\otimes b)|_{\mathcal{D}})^* \in C^*_\Xi(\mathcal{D})$, thanks to Proposition 5.1. In particular, $b \in \mathcal{D}$.

Now take $\alpha,\beta \in \Lambda$ such that $a=\Sigma_{\iota\in\alpha} a_\iota \in H_\alpha$ and $b=\Sigma_{\iota\in\beta} b_\iota \in H_\beta$. If $b_\kappa \neq 0$ for a certain $\kappa \in \beta$, then $(a\otimes b)b_\kappa = \Sigma_{\iota\in\alpha}\|b_\kappa\|^2 a_\iota \in N_\kappa$, that is, $a_\iota = 0$ for all ι, $\iota \neq \kappa$. Similarly, $(b\otimes a)a_\kappa = \Sigma_{\iota\in\beta}\|a_\kappa\|^2 b_\iota \in N_\kappa$ implies that $b_\iota = 0$ for all ι, $\iota \neq \kappa$. Whence $a,b \in N_\kappa$. □

On the grounds of the just proved assertion, we introduce the ideal $\mathcal{F}_\Xi(\mathcal{D})$ in $C^*_\Xi(\mathcal{D})$ of all finite-rank operators as a subspace generated by the one-rank operators $(a\otimes b)|_{\mathcal{D}}$ in $C^*_\Xi(\mathcal{D})$. Later we write $a\otimes b$ instead of $(a\otimes b)|_{\mathcal{D}}$. By Lemma 5.3, each $T \in \mathcal{F}_\Xi(\mathcal{D})$ has a diagonal realization $F=\oplus_{\kappa\in\alpha} F_\kappa$ with $F_\kappa = \Sigma_{i=1}^{s_\kappa} a_{i\kappa}\otimes b_{i\kappa} \in \mathcal{F}(N_\kappa)$, $\kappa \in \alpha$, $\alpha \in \Lambda$. We also introduce the ideal $\mathcal{K}_\Xi(\mathcal{D})$ in $C^*_\Xi(\mathcal{D})$ of *locally compact operators*, that is, $K \in \mathcal{K}_\Xi(\mathcal{D})$ whenever $K|N_\kappa \in \mathcal{K}(N_\kappa)$ for all $\kappa \in \Xi$. In particular,

$$\mathcal{K}_\Xi(\mathcal{D}) = \{K = \bigoplus_{\iota\in\Xi} K_\iota \in C^*_\Xi(\mathcal{D}) : K_\iota \in \mathcal{K}(N_\iota), \iota \in \Xi\}$$

and $\mathcal{F}_\Xi(\mathcal{D}) \subseteq \mathcal{K}_\Xi(\mathcal{D})$.

Remark 5.1: The ideal $\mathcal{F}_\Xi(\mathcal{D})$ is dense in $\mathcal{K}_\Xi(\mathcal{D})$. Indeed, take $K=\oplus_{\iota\in\Xi} K_\iota \in \mathcal{K}_\Xi(\mathcal{D})$ and take $F_\iota \in \mathcal{F}(N_\iota)$ with $\|K_\iota - F_\iota\| < \varepsilon$, $\iota \in \Xi$. Put $R_\alpha = \oplus_{\iota\in\alpha} F_\iota \in \mathcal{F}_\Xi(\mathcal{D})$, $\alpha \in \Lambda$. If $\alpha \subseteq \beta$, then

$$\|K - R_\beta\|_\alpha = \|(K-R_\beta)|H_\alpha\| = \|\bigoplus_{\iota\in\alpha}(K_\iota - F_\iota)\| = \max\{\|K_\iota - F_\iota\| : \iota \in \alpha\} < \varepsilon.$$

D. Locally trace class operators

Now we introduce the ideal of all *locally trace class operators on the domain $\mathcal{E}$*. Fix a finite subset $\alpha \in \Lambda$. By Lemma 4.3, $H=H_\alpha \oplus (\oplus_{\kappa\notin\alpha} N_\kappa)$ and we set $\mathcal{T}_{\mathcal{E}}(\alpha) = \{A = A_\alpha \oplus 0 \in \mathcal{B}(H) : A_\alpha = \oplus_{\kappa\in\alpha} A_\kappa \in \mathcal{T}(\oplus_{\kappa\in\alpha} N_k)\} = \oplus^1_{\kappa\in\alpha}\mathcal{T}(N_\kappa)$ (ℓ_1-direct sum), that is, $\mathcal{T}_{\mathcal{E}}(\alpha) \subseteq \mathcal{T}(H)$ is a closed subspace and $\|A\|_t = \|A_\alpha\|_t = \Sigma_{\kappa\in\alpha}\|A_\kappa\|_t$. Evidently, $\mathcal{T}_{\mathcal{E}}(\alpha) \subseteq \mathcal{T}_{\mathcal{E}}(\beta)$ whenever $\alpha \subseteq \beta$, $\alpha,\beta \in \Lambda$. *The space of all locally trace class operators* is defined as

$$\mathcal{T}_{\Xi}(\mathcal{D}) = \bigcup_{\alpha\in\Lambda} \mathcal{T}_{\mathcal{E}}(\alpha) = \bigoplus_{\kappa\in\Xi} \mathcal{T}_{\mathcal{E}}(\kappa)$$

equipped with the inductive topology, or polynormed direct sum topology. So, $\mathcal{T}_{\Xi}(\mathcal{D})$ is an inductive limit of the Banach spaces and $\mathcal{F}_{\Xi}(\mathcal{D})\subseteq\mathcal{T}_{\Xi}(\mathcal{D})\subseteq\mathcal{K}_{\Xi}(\mathcal{D})$. Furthermore, $\mathcal{T}_{\Xi}(\mathcal{D})$ is an ideal in $C^*_{\Xi}(\mathcal{D})$ and we have a well defined *trace functional* $\mathrm{tr}:\mathcal{T}_{\Xi}(\mathcal{D})\to\mathbb{C}$, $\mathrm{tr}(A)=\mathrm{tr}(A_\alpha)=\Sigma_{\kappa\in\alpha}\mathrm{tr}(A_\kappa)$ whenever $A\in\mathcal{T}_{\mathcal{E}}(\alpha)$, $\alpha\in\Lambda$.

Lemma 5.4: If $\mathcal{K}_{\Xi}(\mathcal{D})'_\beta$ is the space of all continuous linear functionals on $\mathcal{K}_{\Xi}(\mathcal{D})$ equipped with the strong dual topology, then the mapping $\Phi:\mathcal{T}_{\Xi}(\mathcal{D})\to\mathcal{K}_{\Xi}(\mathcal{D})'_\beta$, $\Phi_A S=\mathrm{tr}(SA)$, is a topological isomorphism.

Proof: First note that if $A\in\mathcal{T}_{\mathcal{E}}(\alpha)$ and $S=\oplus_{\kappa\in\Xi}S_\kappa\in\mathcal{K}_{\Xi}(\mathcal{D})$, then $\Phi_A S=\Sigma_{\kappa\in\alpha}\mathrm{tr}(S_\kappa A_\kappa)$ and

$$|\Phi_A S| \leq \sum_{\kappa\in\alpha} |\mathrm{tr}(S_\kappa A_\kappa)| \leq \sum_{\kappa\in\alpha} \|S_\kappa\|\|A_\kappa\|_t \leq \|S|H_\alpha\|\|A_\alpha\|_t = \|S\|_\alpha\|A\|_t,$$

that is, $\Phi_A\in\mathcal{K}_{\Xi}(\mathcal{D})'$. Furthermore, if $\Phi_A=0$, then $\mathrm{tr}(A_\kappa(a\otimes b))=0$ for all $a,b\in N_\kappa$, $\kappa\in\alpha$ (see Lemma 5.3), which, in turn, implies that $\langle A_\kappa a,b\rangle=0$ for all $a,b\in N_\kappa$, that is, $A_\kappa=0$, $\kappa\in\alpha$.

Conversely, take $\varphi\in\mathcal{K}_{\Xi}(\mathcal{D})$. Then $|\varphi(S)|\leq C\max\{\|S\|_\iota:\iota\in\alpha\}$, that is, $|\varphi(S)|\leq C\|S\|_\alpha$ for some positive constant C and a finite subset $\alpha\subseteq\Xi$. It follows that

$$\varphi(\bigoplus_{\kappa\in\Xi} S_\kappa) = \varphi(\bigoplus_{\kappa\in\alpha} S_\kappa) = \sum_{\kappa\in\alpha}\varphi(S_\kappa)$$

for all $\oplus_{\kappa\in\Xi}S_\kappa\in\mathcal{K}_{\Xi}(\mathcal{D})$. Since $S_\kappa\in\mathcal{K}(N_\kappa)$, we conclude that $\varphi(S_\kappa)=\mathrm{tr}(A_\kappa S_\kappa)$ for a certain $A_\kappa\in\mathcal{T}(N_\kappa)$ (see Ref. 4, Sec. 3.19.1), $\kappa\in\alpha$. Then $A=A_\alpha\oplus 0\in\mathcal{T}_{\mathcal{E}}(\alpha)$ with $A_\alpha=\oplus_{\kappa\in\alpha}A_\kappa$, and $\varphi(S)=\mathrm{tr}(AS)=\Phi_A(S)$ for all $S\in\mathcal{K}_{\Xi}(\mathcal{D})$.

Finally, if $U_\varepsilon=\Pi_{\kappa\in\Xi}\varepsilon_\kappa^{-1}\,\mathrm{ball}\,\mathcal{K}(N_\kappa)$ is a bounded set in $\mathcal{K}_{\Xi}(\mathcal{D})$, then $U_\varepsilon^\circ=\mathrm{abc}(\cup_\kappa\varepsilon_\kappa\,\mathrm{ball}\,\mathcal{T}_{\mathcal{E}}(\kappa))$ (see Ref. 23, Sec. 4.4.2) for each $\varepsilon\in\mathbb{R}_+^{\Xi}$. It follows that $\Phi:\mathcal{T}_{\Xi}(\mathcal{D})\to\mathcal{K}_{\Xi}(\mathcal{D})'_\beta$ is a topological isomorphism. □

Lemma 5.5: The mapping $\Psi:C^*_{\Xi}(\mathcal{D})\to\mathcal{T}_{\Xi}(\mathcal{D})'_\beta$, $\Psi_T A=\mathrm{tr}(TA)$, is a topological isomorphism.

Proof: If $A\in\mathcal{T}_{\mathcal{E}}(\alpha)$ and $T=\oplus_{\kappa\in\Xi}T_\kappa\in C^*_{\Xi}(\mathcal{D})$, then $TA=T(A_\alpha\oplus 0)=(\oplus_{\kappa\in\alpha}T_\kappa A_\kappa)\oplus 0=(TA)_\alpha\oplus 0\in\mathcal{T}_{\mathcal{E}}(\alpha)$ and

$$|\Psi_T A| \leq \sum_{\kappa\in\alpha}|\mathrm{tr}(T_\kappa A_\kappa)| \leq \sum_{\kappa\in\alpha}\|T_\kappa\|\|A_\kappa\|_t \leq \|T|H_\alpha\|\|A\|_t.$$

The latter means that $\Psi_{T|H_\alpha}\in\mathcal{T}_{\mathcal{E}}(\alpha)^*$. Hence, $\Psi_T\in\mathcal{T}_{\Xi}(\mathcal{D})'$. If $\Psi_T=0$, then $\mathrm{tr}(T_\kappa(a\otimes b))=0$ for all $a,b\in N_\kappa$, $\kappa\in\Xi$, which, in turn, implies that $\langle T_\kappa a,b\rangle=0$ for all $a,b\in N_\kappa$, that is, $T_\kappa=0$ for all κ.

Now take $\psi\in\mathcal{T}_{\Xi}(\mathcal{D})'$. Then $|\psi|_{\mathcal{T}_{\mathcal{E}}(\kappa)}(A)|\leq C_\kappa|A|_t$, $A\in\mathcal{T}_{\mathcal{E}}(\kappa)$, $\kappa\in\Xi$. It follows that $\psi(A_\kappa)=\mathrm{tr}(A_\kappa T_\kappa)$ for a certain $T_\kappa\in\mathcal{B}(N_\kappa)$ (see Ref. 4, Sec. 3.19.2). Put $T=\oplus_{\kappa\in\Xi}T_\kappa\in C^*_{\Xi}(\mathcal{D})$. If $A\in\mathcal{T}_{\Xi}(\mathcal{D})$, then $A\in\mathcal{T}_{\mathcal{E}}(\alpha)$ for some $\alpha\in\Lambda$, and $\psi(A)=\Sigma_{\kappa\in\alpha}\psi(A_\kappa)=\mathrm{tr}(AT)$. Consequently, $\psi=\Psi_T$.

Finally, one can easily verify that $|\alpha|^{-1}\,\mathrm{ball}\|\cdot\|_\alpha\subseteq(\mathrm{ball}\,\mathcal{T}_{\mathcal{E}}(\alpha))^\circ\subseteq\mathrm{ball}\|\cdot\|_\alpha$, $\alpha\in\Lambda$. Hence, $\Psi:C^*_{\Xi}(\mathcal{D})\to\mathcal{T}_{\Xi}(\mathcal{D})'_\beta$ is a topological isomorphism. □

In particular, the spaces $\mathcal{T}_{\Xi}(\mathcal{D})$ and $C^*_{\Xi}(\mathcal{D})$ are in the natural duality with respect to the scalar pairing

$$\langle\cdot|\cdot\rangle:\mathcal{T}_{\Xi}(\mathcal{D})\times C^*_{\Xi}(\mathcal{D})\to\mathbb{C},\quad \langle A|T\rangle=\mathrm{tr}(AT).$$

Therefore, $C^*_{\Xi}(\mathcal{D})$ is equipped with *the weak* topology given by the family* $\{w_A\}$ of seminorms $w_A(T)=|\mathrm{tr}(AT)|$, $A\in\mathcal{T}_{\Xi}(\mathcal{D})$, $T\in C^*_{\Xi}(\mathcal{D})$. In particular, $\mathcal{T}_{\Xi}(\mathcal{D})=\mathcal{C}^\sigma(C^*_{\Xi}(\mathcal{D}),\mathbb{C})$ is the subspace in $C^*_{\Xi}(\mathcal{D})'$ of all weak* continuous functionals $A:C^*_{\Xi}(\mathcal{D})\to\mathbb{C}$, $A(T)=\mathrm{tr}(AT)$.

E. The quantum topologies on $\mathcal{T}_{\Xi}(\mathcal{D})$

Note that each $\mathcal{T}_{\mathcal{E}}(\alpha)$ is a closed subspace of the operator space $\mathcal{T}(H)$. Let us remind that (Ref. 13, Sec. 3.2.3) the matrix norm in $\mathcal{T}(H)$ is inherited from the operator dual $\mathcal{B}(H)^*$ by means of the identification

$$\mathcal{T}(H)=\mathcal{C}^{\sigma}(\mathcal{B}(H),\mathbb{C})\subseteq\mathcal{B}(H)^{*},\quad A(T)=\operatorname{tr}(AT),\quad A\in\mathcal{T}(H),T\in\mathcal{B}(H).$$

Therefore, each $\mathcal{T}_{\mathcal{E}}(\alpha)$ turns out to be a quantum normed space $\mathcal{T}_{\mathcal{E}}(\alpha)_t$ whose matrix norm is denoted by t_α. If $\alpha=\{\kappa\}$ is a singleton, then we write t_κ instead of $t_{\{\kappa\}}$. Below we shall prove that the index t can be thought as a quantization over the class $\{\mathcal{T}_{\mathcal{E}}(\alpha):\alpha\in\Lambda\}$ of Banach spaces. The space $\mathcal{T}_{\Xi}(\mathcal{D})$ being an inductive limit of the quantum normed spaces $\mathcal{T}_{\mathcal{E}}(\alpha)_t$ turns out to be a quantum space. Namely,

$$\mathcal{T}_{\Xi}(\mathcal{D})_t=\underset{\rightarrow}{\operatorname{oplim}}\{\mathcal{T}_{\mathcal{E}}(\alpha)_t:\alpha\in\Lambda\}.$$

Note that $M_n(\mathcal{T}_{\Xi}(\mathcal{D}))\subseteq\mathcal{C}(C^{*}_{\Xi}(\mathcal{D}),M_n)=\mathcal{MC}(C^{*}_{\Xi}(\mathcal{D}),M_n)$, $n\in\mathbb{N}$ [see (2.5)]. More precisely,

$$M_n(\mathcal{T}_{\Xi}(\mathcal{D}))=\mathcal{MC}^{\sigma}(C^{*}_{\Xi}(\mathcal{D}),M_n)$$

is the subspace of all matrix continuous linear mappings $C^{*}_{\Xi}(\mathcal{D})\to M_n$, which are weak* continuous. Indeed, if a linear mapping $F=[F_{ij}]:C^{*}_{\Xi}(\mathcal{D})\to M_n$, $[F_{ij}]T=[F_{ij}(T)]$, is weak* continuous, then $\|F(T)\|\leqslant c\max\{w_{A_k}(T)\}$ for some positive c and a finite subset $\{A_k\}\subseteq\mathcal{T}_{\Xi}(\mathcal{D})$. In particular, $|F_{ij}(T)|\leqslant\|F(T)\|\leqslant c\max\{w_{A_k}(T)\}$, that is, $F_{ij}\in\mathcal{C}^{\sigma}(C^{*}_{\Xi}(\mathcal{D}),\mathbb{C})=\mathcal{T}_{\Xi}(\mathcal{D})$ and $[F_{ij}]\in M_n(\mathcal{T}_{\Xi}(\mathcal{D}))$. Conversely, if all $F_{ij}\in\mathcal{C}^{\sigma}(C^{*}_{\Xi}(\mathcal{D}),\mathbb{C})$, then $F=[F_{ij}]\in\mathcal{C}^{\sigma}(C^{*}_{\Xi}(\mathcal{D}),M_n)=\mathcal{MC}^{\sigma}(C^{*}_{\Xi}(\mathcal{D}),M_n)$ [see (2.5)].

Lemma 5.6: If $T\in M_n(C^{*}_{\Xi}(\mathcal{D}))$, then $T|H^n_\alpha=\oplus_{\kappa\in\alpha}T_\kappa$ with $T_\kappa\in M_n(\mathcal{B}(N_\kappa))$, $\kappa\in\alpha$, up to the canonical (isometry) identification $H^n_\alpha=\oplus_{\kappa\in\alpha}N^n_\kappa$. Moreover, if $A\in M_n(\mathcal{T}_{\mathcal{E}}(\alpha))$, then $A=\oplus_{\kappa\in\alpha}A_\kappa$ and $A_\kappa\in M_n(\mathcal{T}(N_\kappa))$ for all $\kappa\in\alpha$.

Proof: If $T=[T_{ij}]\in M_n(C^{*}_{\Xi}(\mathcal{D}))$, then each $T_{ij}|H_\alpha=\oplus_{\kappa\in\alpha}T_{ij,\kappa}\in\mathcal{B}(\oplus_{\kappa\in\alpha}N_\kappa)$, whereas the matrix $T|H^n_\alpha$ is identified with the operator $T:H^n_\alpha\to H^n_\alpha$, $T(x_i)_i=(\Sigma_jT_{ij}x_j)_i$. However, the correspondence $H^n_\alpha\to\oplus_{\kappa\in\alpha}N^n_\kappa$, $((x_{i\kappa})_\kappa)_i\mapsto((x_{i\kappa})_i)_\kappa$ (here each $(x_{i\kappa})_\kappa\in\oplus_{\kappa\in\alpha}N_\kappa$) implements an isometrical identification (replacement of the brackets). Within the latter identification, one may write

$$T(x_i)_i=\left(\sum_j T_{ij}x_j\right)_i=\left(\sum_j T_{ij}(x_{j\kappa})_\kappa\right)_i=\left(\left(\sum_j T_{ij,\kappa}x_{j\kappa}\right)_\kappa\right)_i=\left(\left(\sum_j T_{ij,\kappa}x_{j\kappa}\right)_i\right)_\kappa=([T_{ij,\kappa}]_{i,j}(x_{i\kappa})_i)_\kappa=(\bigoplus_{\kappa\in\alpha}[T_{ij,\kappa}]_{i,j})((x_{i\kappa})_i)_\kappa=(\bigoplus_{\kappa\in\alpha}[T_{ij,\kappa}]_{i,j})(x_i)_i,$$

that is, $T=\oplus_{\kappa\in\alpha}T_\kappa$ with $T_\kappa=[T_{ij,\kappa}]_{i,j}\in M_n(\mathcal{B}(N_\kappa))$.

Finally, if $A=[A_{ij}]\in M_n(\mathcal{T}_{\mathcal{E}}(\alpha))$, then each A_{ij} is identified with $(A_{ij})_\alpha=\oplus_{\kappa\in\alpha}A_{ij,\kappa}\in\mathcal{T}(\oplus_{\kappa\in\alpha}N_\kappa)$. The rest is clear. □

If $\mathcal{P}_\kappa:\mathcal{T}_{\mathcal{E}}(\alpha)\to\mathcal{T}_{\mathcal{E}}(\kappa)$ is the canonical projection onto $\mathcal{T}(N_\kappa)$ with respect to the decomposition $\mathcal{T}_{\mathcal{E}}(\alpha)=\oplus^1_{\kappa\in\alpha}\mathcal{T}(N_\kappa)$, then using Lemma 5.6, we obtain that $\mathcal{P}^{(n)}_\kappa(A)=[\mathcal{P}_\kappa(A_{ij})]_{i,j}=[A_{ij,\kappa}]_{i,j}=A_\kappa$ for all $A\in M_n(\mathcal{T}_{\mathcal{E}}(\alpha))$.

Theorem 5.5: *Each canonical projection $\mathcal{P}_\beta:\mathcal{T}_{\mathcal{E}}(\alpha)\to\mathcal{T}_{\mathcal{E}}(\beta)$, $\beta\subseteq\alpha$, is a matrix contraction, therefore t is a quantization over the Banach space class $\{\mathcal{T}_{\mathcal{E}}(\alpha):\alpha\in\Lambda\}$. In particular,*

$$\mathcal{T}_{\Xi}(\mathcal{D})_t=\operatorname{op}\bigoplus_{\iota\in\Xi}\mathcal{T}_{\mathcal{E}}(\iota)_t\quad\textit{and}\quad(\mathcal{T}_{\Xi}(\mathcal{D})_t)'_\beta=C^{*}_{\Xi}(\mathcal{D}).$$

Proof: Take $A=\oplus_{\kappa\in\alpha}A_\kappa\in M_n(\mathcal{T}_{\mathcal{E}}(\alpha)_t)$. Recall that $M_n(\mathcal{T}_{\mathcal{E}}(\alpha)_t)\subseteq M_n(\mathcal{T}(H))\subseteq M_n(\mathcal{B}(H)^{*})$ and $M_n(\mathcal{T}(H))=\mathcal{MB}^{\sigma}(\mathcal{B}(H),M_n)$ is the space of all matrix bounded linear mappings $\mathcal{B}(H)\to M_n$, which are weak* continuous. Using the dual matrix norm [see Sec. IV C, and also Ref. 13, (3.2.3)], we deduce that

$$t_\alpha(A)=\sup\|\langle\langle A|\text{ball }M(\mathcal{B}(H))\rangle\rangle\|,\quad A\in M(\mathcal{T}_{\mathcal{E}}(\alpha)_t).\tag{5.3}$$

Take $S=[S_{vl}]\in M_m(\mathcal{B}(H))$ with $\|S\|\leqslant1$. Put $T=P^{\oplus n}_\alpha SP^{\oplus n}_\alpha$, where $P_\alpha\in\mathcal{B}(H)$ is the projection onto H_α. Then $T=[T_{vl}]_{v,l}=[[T^{\kappa\tau}_{vl}]_{\kappa,\tau}]_{v,l}=[[T^{\kappa\tau}_{vl}]_{v,l}]_{\kappa,\tau}=[T^{\kappa\tau}]_{\kappa,\tau}$ up to the isometric identification $H^n_\alpha=\oplus_{\kappa\in\alpha}N^n_\kappa$ used in Lemma 5.6, where $T^{\kappa\tau}=[T^{\kappa\tau}_{vl}]_{v,l}:N^n_\tau\to N^n_\kappa$ is a bounded linear operator. In particular, $T^{\kappa\kappa}\in M_n(\mathcal{B}(N_\kappa))$, $\|T^{\kappa\kappa}\|\leqslant1$. It follows that

$$\langle\langle A|S\rangle\rangle=[\langle A_{ij}|S_{vl}\rangle]=[\mathrm{tr}(A_{ij}S_{vl})]=[\mathrm{tr}(A_{ij}P_\alpha S_{vl})]=[\mathrm{tr}(P_\alpha S_{vl}A_{ij})]=[\mathrm{tr}(A_{ij}T_{vl})]=[\mathrm{tr}([A_{ij,\kappa}T_{vl}^{\kappa\tau}]_{\kappa,\tau})]$$
$$=\left[\sum_{\kappa\in\alpha}\mathrm{tr}(A_{ij,\kappa}T_{vl}^{\kappa\kappa})\right]=\sum_{\kappa\in\alpha}\langle\langle A_\kappa|T^{\kappa\kappa}\rangle\rangle,$$

where each $A_{ij,\kappa}T_{vl}^{\kappa\tau}:N_\tau\to N_\kappa$, and $\langle\langle A_\kappa|T^{\kappa\kappa}\rangle\rangle$ indicates the matrix duality associated with the pairing $\langle\cdot|\cdot\rangle:\mathcal{T}(N_\kappa)\times\mathcal{B}(N_\kappa)\to\mathbb{C}$, $\langle A|T\rangle=\mathrm{tr}(AT)$. Using (5.3), we conclude that

$$t_\alpha^{(n)}(A)=\sup\|\langle\langle A|\mathrm{ball}\ M(X_\alpha)\rangle\rangle\|, \tag{5.4}$$

where $X_\alpha=\oplus_{\kappa\in\alpha}^\infty\mathcal{B}(N_\kappa)$ is the operator sum of the operator spaces $\mathcal{B}(N_\kappa)$, $\kappa\in\alpha$, and $\langle\langle A|T\rangle\rangle=\Sigma_{\kappa\in\alpha}\langle\langle A_\kappa|T_\kappa\rangle\rangle$ for each $T=\oplus_{\kappa\in\alpha}T_\kappa\in\mathrm{ball}\ M(X_\alpha)$. Thus, $M_n(\mathcal{T}_\mathcal{E}(\alpha)_t)=\mathcal{MB}^\sigma(X_\alpha,M_n)$, $n\in\mathbb{N}$.

Now take $\beta\in\Lambda$ with $\beta\subseteq\alpha$. Then $\mathcal{P}_\beta^{(n)}:M_n(\mathcal{T}_\mathcal{E}(\alpha)_t)\to M_n(\mathcal{T}_\mathcal{E}(\beta)_t)$ is reduced to the restriction mapping $\mathcal{MB}^\sigma(X_\alpha,M_n)\to\mathcal{MB}^\sigma(X_\beta,M_n)$, $A\mapsto A|_{X_\beta}$. Since $\|A|_{X_\beta}\|_{mb}\le\|A\|_{mb}$, it follows that $\mathcal{P}_\beta$ is a matrix contraction. Therefore t is a quantization over the class $\{\mathcal{T}_\mathcal{E}(\alpha):\alpha\in\Lambda\}$ (see Sec. III A). Using Corollary 3.1, we conclude that $\mathcal{T}_\Xi(\mathcal{D})_t=\mathrm{op}\oplus_{\iota\in\Xi}\mathcal{T}_\mathcal{E}(\iota)_t$. In particular, each matrix bounded set $\mathfrak{B}$ in $M(\mathcal{T}_\Xi(\mathcal{D})_t)$ is contained in a certain $M(\mathcal{T}_\mathcal{E}(\alpha)_t)$ and it is matrix norm bounded there thanks to Corollary 3.2. So, the strong quantum dual $(\mathcal{T}_\Xi(\mathcal{D})_t)'_\beta$ is generated by the dual matrix seminorms $t_\alpha^\odot$, $\alpha\in\Lambda$ (see Sec. IV C). If $T\in M_n(C_\Xi^*(\mathcal{D}))$ and $A\in M_r(\mathcal{T}_\mathcal{E}(\alpha)_t)$ then as above $\langle\langle A|T\rangle\rangle=\Sigma_{\kappa\in\alpha}\langle\langle A_\kappa|T_\kappa\rangle\rangle$ (see Lemmas 5.5 and 5.6). Hence,

$$t_\alpha^\odot(T)=\sup\|\langle\langle\mathrm{ball}\ M(\mathcal{T}_\mathcal{E}(\alpha)_t)|T\rangle\rangle\|=t_\alpha^\odot(\underset{\kappa\in\alpha}{\oplus}T_\kappa)=t_\alpha^\odot(TP_\alpha^{\oplus n}). \tag{5.5}$$

Let us prove that $t_\alpha^\odot(T)=\|T\|_\alpha^{(n)}$, $T\in M_n(C_\Xi^*(\mathcal{D}))$, $n\in\mathbb{N}$. For each $T\in M_n(C_\Xi^*(\mathcal{D}))$ we have $T|H_\alpha^n=\oplus_{\kappa\in\alpha}T_\kappa\in M_n(X_\alpha)$. In particular, $\|T\|_\alpha^{(n)}=\|T|H_\alpha^n\|=\max_{\kappa\in\alpha}\|T_\kappa\|$. Based upon (5.4), we infer that the canonical embedding $\mathcal{T}_\mathcal{E}(\alpha)_t\to X_\alpha^*$ is a matrix isometry. Then its dual mapping $X_\alpha^{**}\to\mathcal{T}_\mathcal{E}(\alpha)_t^*$ is a matrix contraction. Taking into account that the canonical embedding $X_\alpha\hookrightarrow X_\alpha^{**}$ is a matrix isometry (Ref. 13, Proposition 3.2.1), we derive that the canonical mapping $X_\alpha\to\mathcal{T}_\mathcal{E}(\alpha)_t^*$, $T\mapsto\Psi_T$, $\Psi_TA=\mathrm{tr}(TA)$ (see Lemma 5.5), is a matrix contraction (see also Ref. 1, Sec. 2.2.14). Hence $t_\alpha^\odot(T)=t_\alpha^\odot(\oplus_{\kappa\in\alpha}T_\kappa)\le\max_{\kappa\in\alpha}\|T_\kappa\|=\|T\|_\alpha^{(n)}$. Finally, using Theorem 3.2.3 of Ref. 13 and (5.4) and (5.5), we derive that

$$\|T_\kappa\|=\sup\{\|\langle\langle A_\kappa|T_\kappa\rangle\rangle\|:A_\kappa\in\mathrm{ball}\ M_n(\mathcal{T}(N_\kappa))\}=\sup\{\|\langle\langle\cdots0\oplus A_\kappa\oplus0\cdots,T|H_\alpha^n\rangle\rangle\|:t_\alpha^{(n)}(\cdots0\oplus A_\kappa$$
$$\oplus0\cdots)\le1\}\le t_\alpha^\odot(T),$$

that is, $\|T\|_\alpha^{(n)}\le t_\alpha^\odot(T)$. Therefore, the linear isomorphism $\Psi:C_\Xi^*(\mathcal{D})\to\mathcal{T}_\Xi(\mathcal{D})'$, $\Psi_TA=\mathrm{tr}(TA)$, proposed in Lemma 5.5, implements a topological matrix isomorphism of the relevant quantum spaces. □

Consider the quantum space $\mathfrak{D}_J=\Pi_{\kappa\in\Xi}M_{J_\kappa}$ with its family $\mathfrak{t}=\{\tau_\kappa:\kappa\in\Xi\}$ of matrix seminorms (see Sec. V B). Then, $\mathcal{T}_J=\mathrm{op}\oplus_{\kappa\in\Xi}\mathcal{T}_{J_\kappa}$ is a quantum space, where each $\mathcal{T}_{J_\kappa}=\oplus_{w\in J_\kappa}^1\mathcal{T}_{n_w}$ is the operator space of all trace class matrices in $M_{J_\kappa}=\oplus_{w\in J_\kappa}^\infty M_{n_w}$. Note that $\mathfrak{D}_J$ is a quantum subspace in $C_\Xi^*(\mathcal{D})$, where $\mathcal{D}=\Sigma_{\kappa\in\Xi}N_\kappa$, $N_\kappa=\oplus_{w\in J_\kappa}\mathbb{C}^{n_w}$ and $H=\oplus_{w\in J}\mathbb{C}^{n_w}$. In particular, $\mathcal{T}_J$ is a subspace in $\mathcal{T}_\Xi(\mathcal{D})$.

Corollary 5.4: The identification $(\mathcal{T}_\Xi(\mathcal{D})_t)'_\beta=C_\Xi^*(\mathcal{D})$ restricted to the subspace $\mathfrak{D}_J$ implements the quantum space isomorphism $(\mathcal{T}_J)'_\beta=\mathfrak{D}_J$.

Proof: If $T=(T_w)_{w\in J}\in\mathfrak{D}_J$ and $A_\kappa=(A_w)_{w\in J_\kappa}\in\mathcal{T}_{J_\kappa}$, then $TA=(T_wA_w)_{w\in J_\kappa}\in\mathcal{T}_{J_\kappa}$ and

$$|\Psi_TA|\le\sum_{w\in J_\kappa}|\mathrm{tr}(T_wA_w)|\le\sum_{w\in J_\kappa}\|T_w\|\|A_w\|_t\le\tau_\kappa(T)\|A\|_t$$

as in Lemma 5.5. Hence, $\Psi_T\in(\mathcal{T}_J)'$. Conversely, if $\psi\in(\mathcal{T}_J)'$, then for each $\kappa\in\Xi$ there correspondence $T_\kappa\in M_{J_\kappa}$ such that $\psi(A_\kappa)=\mathrm{tr}(A_\kappa T_\kappa)$. Therefore, $\psi=\Psi_T$ with $T=(T_\kappa)_{\kappa\in\Xi}\in\mathfrak{D}_J$. Hence the mapping $\Psi:C_\Xi^*(\mathcal{D})\to\mathcal{T}_\Xi(\mathcal{D})'$, $\Psi_TA=\mathrm{tr}(TA)$, implements an isomorphism $(\mathcal{T}_J)'_\beta=\mathfrak{D}_J$. Finally, if t_κ is the matrix norm on $\mathcal{T}_{J_\kappa}$ then $t_\kappa^\odot=\tau_\kappa$ (Ref. 1, Sec. 2.2.14). Using Corollary 3.2, we conclude that the quantum space identification $(\mathcal{T}_\Xi(\mathcal{D})_t)'_\beta = C^*_\Xi(\mathcal{D})$ from Theorem 5.5 restricted to the subspace $\mathfrak{D}_J$ associates the quantum space isomorphism $(\mathcal{T}_J)'_\beta = \mathfrak{D}_J$. □

Thus, $\mathfrak{D}_J$ is a locally W^*-algebra.[15,11] Note that $\mathcal{T}_J$ being a subspace of all trace class operators turns out to be a normed space. It can be proved[11] that $\mathcal{T}_J$ is unique up to an isometry. So, $\mathcal{T}_J$ is the predual of the local von Neumann algebra $\mathfrak{D}_J$.

By Theorem 5.5, $M(\mathcal{T}_\Xi(\mathcal{D})) = \oplus_{\kappa\in\Xi} M(\mathcal{T}_\mathcal{E}(\kappa)_t)$ is the quantum direct sum. In particular, $M(\mathcal{T}_\Xi(\mathcal{D})_t)$ has a neighborhood filter base $\{\mathfrak{B}_\varepsilon : \varepsilon \in \mathbb{R}_+^\Xi\}$ (see Sec. III), where $\mathfrak{B}_\varepsilon = \mathrm{amc} \cup_{\kappa\in\Xi} \varepsilon_\kappa \text{ ball } t_\kappa$. On the grounds of Lemma 5.4, we have the weak* topology $\sigma(\mathcal{T}_\Xi(\mathcal{D}), \mathcal{K}_\Xi(\mathcal{D}))$ in $\mathcal{T}_\Xi(\mathcal{D})$. We denote the weak* closure of the neighborhood $\mathfrak{B}_\varepsilon \subseteq M(\mathcal{T}_\Xi(\mathcal{D}))$ by $\mathfrak{B}_\varepsilon^-$.

Lemma 5.7: Let $\mathfrak{A}_\varepsilon = \Pi_{\kappa\in\Xi} \varepsilon_\kappa^{-1} \text{ ball } M(\mathcal{K}(N_\kappa))$ *be the matrix bounded set in* $M(\mathcal{K}_\Xi(\mathcal{D}))$, *where* $\varepsilon \in \mathbb{R}_+^\Xi$. *Then* $\mathfrak{B}_\varepsilon^- = \mathfrak{A}_\varepsilon^\odot$, *where* $\mathfrak{A}_\varepsilon^\odot \subseteq M(\mathcal{T}_\Xi(\mathcal{D}))$ *is the absolute matrix polar of* $\mathfrak{A}_\varepsilon$ *with respect to the dual pair* $(\mathcal{K}_\Xi(\mathcal{D}), \mathcal{T}_\Xi(\mathcal{D}))$. *In particular, the family* $\{\mathfrak{B}_\varepsilon^- : \varepsilon \in \mathbb{R}_+^\Xi\}$ *is a neighborhood filter base of a certain quantization of* $\mathcal{T}_\Xi(\mathcal{D})$ *denoted by* $\mathcal{T}_\Xi(\mathcal{D})_{t^*}$.

Proof: First, take $A_\kappa \in M_n(\mathcal{T}_\mathcal{E}(\kappa)_t)$, $\kappa \in \Xi$. Since $\mathcal{B}(N_\kappa^n) = \mathcal{K}(N_\kappa^n)^{**}$, it follows that the unit ball of $M_n(\mathcal{K}(N_\kappa)) = \mathcal{K}(N_\kappa^n)$ is weak* dense in ball $M_n(\mathcal{B}(N_\kappa))$ (see, for instance, Ref. 4, Sec. 5.4.1). Using (5.4), we derive

$$t_\kappa(A) = \sup\|\langle\langle A|\text{ball } M(\mathcal{B}(N_\kappa))\rangle\rangle\| = \sup\|\langle\langle A|\text{ball } M(\mathcal{K}(N_\kappa))\rangle\rangle\| = \sup\|\langle\langle A|\text{ball } M(\mathcal{K}_\Xi(\mathcal{D}))\rangle\rangle\| = \|A\|_\kappa^\odot,$$

that is, $t_\kappa = \|\cdot\|_\kappa^\odot$, where $\|\cdot\|_\kappa$ is the matrix seminorm on $\mathcal{K}_\Xi(\mathcal{D})$. Using the pairing $\langle\cdot|\cdot\rangle : \mathcal{K}_\Xi(\mathcal{D}) \times \mathcal{T}_\Xi(\mathcal{D}) \to \mathbb{C}$, $\langle K|A\rangle = \mathrm{tr}(AK)$ (see Lemma 5.4), just proved equality $t_\kappa = \|\cdot\|_\kappa^\odot$, and Corollary 2.3, we conclude that

$$\left(\text{ball}\|\cdot\|_\kappa\right)^\odot = \left\{A \in M(\mathcal{T}_\Xi(\mathcal{D})) : \sup\|\langle\langle \text{ball}\|\cdot\|_\kappa|A\rangle\rangle\| \leqslant 1\right\} = \left\{A \in M(\mathcal{T}_\Xi(\mathcal{D})) : \sup\|\langle\langle A|\text{ball}\|\cdot\|_\kappa\rangle\rangle\| \leqslant 1\right\} = \text{ball } \|\cdot\|_\kappa^\odot = \text{ball } t_\kappa$$

(see Sec. IV A). Using Lemma 2.4 and bipolar Theorem 2.1, we obtain that

$$\mathfrak{B}_\varepsilon^\odot = (\mathrm{amc} \bigcup_{\kappa\in\Xi} \varepsilon_\kappa \text{ ball } t_\kappa)^\odot = (\bigcup_{\kappa\in\Xi} (\varepsilon_\kappa^{-1} \text{ ball}\|\cdot\|_\kappa)^\odot)^\odot = \bigcap_{\kappa\in\Xi} (\varepsilon_\kappa^{-1} \text{ ball}\|\cdot\|_\kappa)^{\odot\odot} = \bigcap_{\kappa\in\Xi} \varepsilon_\kappa^{-1} \text{ ball}\|\cdot\|_\kappa = \prod_{\kappa\in\Xi} \varepsilon_\kappa^{-1} \text{ ball } M(\mathcal{K}(N_\kappa)) = \mathfrak{A}_\varepsilon.$$

Note that ball$\|\cdot\|_\kappa$, $\kappa \in \Xi$, are weakly closed absolutely matrix convex sets in $M(\mathcal{K}_\Xi(\mathcal{D}))$. Using again the bipolar Theorem 2.1, we derive that $\mathfrak{B}_\varepsilon^- = \mathfrak{B}_\varepsilon^{\odot\odot} = \mathfrak{A}_\varepsilon^\odot$.

Put $\mathfrak{B}_\varepsilon^- = (\mathfrak{b}_{\varepsilon,n}^-)$ and $\mathfrak{A}_\varepsilon = (\mathfrak{u}_{\varepsilon,n})$, which are matrix sets in $M(\mathcal{T}_\Xi(\mathcal{D}))$ and $M(\mathcal{K}_\Xi(\mathcal{D}))$, respectively. Note that $\mathfrak{u}_{\varepsilon,1} = \Pi_{\kappa\in\Xi} \varepsilon_\kappa^{-1} \text{ ball } \mathcal{K}(N_\kappa) = U_\varepsilon$ (see to the proof of Lemma 5.4). Using Corollary 2.2, we derive that $\mathfrak{u}_{\varepsilon,1}^\odot = \mathfrak{u}_{\varepsilon,1}^\circ = U_\varepsilon^\circ = \mathfrak{b}_{\varepsilon,1}^-$. However, $U_\varepsilon^\circ = \mathrm{abc}(\cup_{\kappa\in\Xi} \varepsilon_\kappa \text{ ball } \mathcal{T}_\mathcal{E}(\kappa))$ (see to the proof of Lemma 5.4). Thus $\{\mathfrak{b}_{\varepsilon,1}^-\}$ is a neighborhood filter base of the original inductive topology in $\mathcal{T}_\Xi(\mathcal{D})$, that is, $\mathcal{T}_\Xi(\mathcal{D})_{t^*}$ is a quantization of $\mathcal{T}_\Xi(\mathcal{D})$. □

Theorem 5.6: *The linear isomorphism* $\Phi : \mathcal{T}_\Xi(\mathcal{D}) \to \mathcal{K}_\Xi(\mathcal{D})'$, $\Phi_A S = \mathrm{tr}(SA)$, *implements the topological matrix isomorphism*

$$\mathcal{T}_\Xi(\mathcal{D})_{t^*} = \mathcal{K}_\Xi(\mathcal{D})'_\beta.$$

If Ξ *is a countable set then* $\mathcal{T}_\Xi(\mathcal{D})_{t^*} = \mathcal{T}_\Xi(\mathcal{D})_t$.

Proof: Since the quantum topology in $M(\mathcal{K}_\Xi(\mathcal{D}))$ is precisely the direct product topology from $\Pi_{\kappa\in\Xi} M(\mathcal{K}(N_\kappa))$, it follows that all matrix bounded sets in $M(\mathcal{K}_\Xi(\mathcal{D}))$ are exhausted by the matrix sets $\mathfrak{A}_\varepsilon$, $\varepsilon \in \mathbb{R}_+^\Xi$. Hence $\{\mathfrak{A}_\varepsilon^\odot : \varepsilon \in \mathbb{R}_+^\Xi\}$ is a neighborhood filter base in $M(\mathcal{K}_\Xi(\mathcal{D})'_\beta)$ (see Sec. IV C). By Lemma 5.7, $\{\mathfrak{A}_\varepsilon^\odot : \varepsilon \in \mathbb{R}_+^\Xi\}$ is a neighborhood filter base of the quantization $\mathcal{T}_\Xi(\mathcal{D})_{t^*}$. Whence $\mathcal{T}_\Xi(\mathcal{D})_{t^*} = \mathcal{K}_\Xi(\mathcal{D})'_\beta$ up to the topological matrix isomorphism.

Finally, assume that $\Xi = \mathbb{N}$. Obviously, the inductive quantum topology in $M(\mathcal{T}_\Xi(\mathcal{D})_t)$ is finer than one in $M(\mathcal{T}_\Xi(\mathcal{D})_{t^*})$. Conversely, fix $\varepsilon = (\varepsilon_n) \in \mathbb{R}_+^\mathbb{N}$ and choose a sequence $\delta \in \mathbb{R}_+^\mathbb{N}$ such that $\Sigma_n \varepsilon_n^{-1} \delta_n \leqslant 1$. Let us prove that $\mathfrak{B}_\delta^- \subseteq \mathfrak{B}_\varepsilon$. Take $A \in \mathfrak{B}_\delta^-$. By Lemma 5.6, $A = \Sigma_{n\in\alpha} A_n \in M(\mathcal{T}_\mathcal{E}(\alpha))$ for a certain $\alpha \in \Lambda$. Using Lemma 5.7, we conclude that $A \in \mathfrak{A}_\delta^{\odot}$, that is, $\|\Sigma_{n\in\alpha}\langle\langle A_n | K_n\rangle\rangle\| \leqslant 1$ for all $K=(K_n)\in\mathfrak{A}_\delta$. In particular, $\sup\|\langle\langle A_n | \delta_n^{-1} \text{ ball } M(\mathcal{K}(N_n))\rangle\rangle\| \leqslant 1$, that is, $A_n \in (\delta_n^{-1} \text{ ball}\|\cdot\|_n)^{\odot} = \delta_n \text{ ball } t_n$. It follows that $C_n = \varepsilon_n \delta_n^{-1} A_n \in \varepsilon_n \text{ ball } t_n$ for all $n\in\alpha$. Moreover, $A=\Sigma_{n\in\alpha}\varepsilon_n^{-1}\delta_n C_n$ and $\Sigma_{n\in\alpha}\varepsilon_n^{-1}\delta_n \leqslant 1$. Thus $A \in \text{abc}(\cup_{n\in\mathbb{N}}\varepsilon_n \text{ ball } M(\mathcal{T}_{\mathcal{E}}(n))) \subseteq \mathfrak{B}_\varepsilon$. In particular, the quantum topology in $M(\mathcal{T}_\Xi(\mathcal{D})_t)$ is coarser than one in $M(\mathcal{T}_\Xi(\mathcal{D})_{t^*})$. Therefore, $\mathcal{T}_\Xi(\mathcal{D})_{t^*}=\mathcal{T}_\Xi(\mathcal{D})_t$. □

Corollary 5.5: If Ξ is countable then the embedding $\mathcal{K}_\Xi(\mathcal{D}) \subseteq C_\Xi^*(\mathcal{D})$ is just the canonical embedding $\mathcal{K}_\Xi(\mathcal{D}) \hookrightarrow (\mathcal{K}_\Xi(\mathcal{D})'_\beta)'_\beta$ of $\mathcal{K}_\Xi(\mathcal{D})$ into its second strong quantum dual.

Proof: It suffices to apply Theorems 5.6 and 5.5. □

F. The dual realization

Let V be a quantum space with a family $\mathfrak{S}=\{\mathfrak{I}_\kappa : \kappa\in\Xi\}$ of matrix bounded sets in $M(V)$ such that $M(V)$ is the matrix hull of $\cup\mathfrak{S}$. Without loss of the generality, it can be assumed that $\mathfrak{S}$ is an upward filtered family of weakly closed and absolutely matrix convex sets. Thus, the $\mathfrak{S}$-quantum dual $V'_{\mathfrak{S}}$ has the defining family $\{p_\kappa^{\odot} : \kappa\in\Xi\}$ of matrix seminorms, where $p_\kappa = p_{\mathfrak{B}_\kappa}$ is the Minkowski functional of $\mathfrak{B}_\kappa$ (see Sec. IV C). Note that $\mathfrak{B}_\kappa = \text{ball } p_\kappa$ thanks to Proposition 2.1. We use the notation $\mathfrak{S}(V',V)$ for this quantum topology in $M(V')$. If $\mathfrak{S}$ is the family of all matrix bounded sets in $M(V)$ then we have the strong quantum dual topology in $M(V')$ denoted by $\beta(V',V)$. Based on the natural duality between the spaces V and V', one may treat V as the space $\mathcal{C}^\sigma(V'_{\mathfrak{S}},\mathbb{C})$ of all weak* continuous linear functionals on $V'_{\mathfrak{S}}$, and $v(f)=\langle\langle v,f\rangle\rangle = f^{(n)}(v)$ whenever $v\in M_n(V)$ and $f\in M(V')$. The family $p_v(f)=\|\langle\langle v,f\rangle\rangle\|$, $v\in M(V)$, $f\in M(V')$, of matrix seminorms defines the weak* quantum topology $\mathfrak{s}(V',V)$ in $M(V')$. Thus we have the inclusions $\mathfrak{s}(V',V) \subseteq \mathfrak{S}(V',V) \subseteq \beta(V',V)$ of the quantum topologies in $M(V')$.

Let us consider the mapping $n:\mathfrak{S}\to\mathbb{N}$, $v\mapsto n_v$, such that $n_v=n$ whenever $v\in M_n(V)$. As in Sec. V B, we have the quantum space $\mathfrak{D}_{\mathfrak{S}} = \Pi_{\kappa\in\Xi} M_{\mathfrak{I}_\kappa}$, $M_{\mathfrak{I}_\kappa} = \oplus_{v\in\mathfrak{I}_\kappa}^\infty M_{n_v}$, with its lower and upper quantum boundaries $\mathfrak{a}=\{\sigma_v : v\in\mathfrak{S}\}$ and $\mathfrak{t}=\{\tau_\kappa : \kappa\in\Xi\}$ (see Sec. V B). By Corollary 5.4, $\mathfrak{D}_{\mathfrak{S}} = (\mathcal{T}_{\mathfrak{S}})'_\beta$, therefore $\mathfrak{D}_{\mathfrak{S}}$ possesses the weak* topology $\sigma(\mathfrak{D}_{\mathfrak{S}},\mathcal{T}_{\mathfrak{S}})$ too. The latter topology is determined by the family of seminorms $\{\omega_a : a\in\mathcal{T}_{\mathfrak{S}}\}$ with $\omega_a(b)=|\text{tr}(ab)|$, $b\in\mathfrak{D}_{\mathfrak{S}}$. In particular, it admits precisely one quantization, which is the weak* quantum topology $\mathfrak{s}(\mathfrak{D}_{\mathfrak{S}},\mathcal{T}_{\mathfrak{S}})$, thanks to Theorem 2.3.

Now consider the linear mapping

$$\Phi: V'_{\mathfrak{S}} \to \mathfrak{D}_{\mathfrak{S}}, \quad \Phi(f) = (\langle\langle v,f\rangle\rangle)_{v\in\mathfrak{S}}.$$

If $\Phi(f)=0$, then $f^{(\infty)}(v)=\langle\langle v,f\rangle\rangle=0$ for all $v\in\mathfrak{S}$, that is, $f^{(\infty)}(\cup\mathfrak{S})=\{0\}$. However, the matrix hull of $\cup\mathfrak{S}$ is the whole matrix space $M(V)$, therefore $f^{(\infty)}=0$ or $f=0$ (see Remark 4.1). Hence, $\Phi: V'_{\mathfrak{S}} \to \mathfrak{D}_{\mathfrak{S}}$ is a linear isomorphism onto its range.

Lemma 5.8: If V is complete, then $\Phi: V'_{\mathfrak{S}} \to \mathfrak{D}_{\mathfrak{S}}$ is a weak homeomorphism.*

Proof: Fix $a=(a_v)_{v\in\mathfrak{I}_\kappa} \in \mathcal{T}_{\mathfrak{I}_\kappa} = \oplus_{v\in\mathfrak{I}_\kappa}^1 \mathcal{T}_{n_v}$ for a certain $\kappa\in\Xi$. Then $a_v = \Sigma_{m=1}^{n_v} g_{m,v}\otimes h_{m,v}$ for some orthogonal sequences $\{g_{m,v}\}$, $\{h_{m,v}\}$ in $\mathbb{C}^{n_v}$, and $\|a\|_t = \Sigma_{v\in\mathfrak{I}_\kappa}\Sigma_{m=1}^{n_v}\|g_{m,v}\|\|h_{m,v}\|$ (see, for instance, Ref. 4, Sec. 3.18.13). If $b\in\mathfrak{D}_{\mathfrak{S}}$, then

$$\text{tr}(ba) = \text{tr}((b_v a_v)_{v\in\mathfrak{I}_\kappa}) = \sum_{v\in\mathfrak{I}_\kappa} \text{tr}(b_v a_v) = \sum_{v\in\mathfrak{I}_\kappa}\sum_{m=1}^{n_v} \langle b_v g_{m,v}, h_{m,v}\rangle.$$

Note that $g_{m,v}=(g_{m,v_i})_i$, $h_{m,v}=(h_{m,v_i})_i \in \mathbb{C}^{n_v} (=M_{n_v,1})$ are columns if $v=[v_{ij}]\in\mathfrak{I}_\kappa$. In particular, if $b=\Phi(f)$ for a certain $f\in V'$, then

$$\begin{aligned}\mathrm{tr}(ba) &= \sum_{v\in\mathfrak{J}_\kappa}\sum_{m=1}^{n_v}\langle[f(v_{ij})](g_{m,v_j})_i,(h_{m,v_i})_i\rangle = \sum_{v\in\mathfrak{J}_\kappa}\sum_{m=1}^{n_v}\sum_{i,j=1}^{n_v} f(v_{ij})g_{m,v_j}\overline{h_{m,v_i}}\\ &= \sum_{v\in\mathfrak{J}_\kappa}\sum_{m=1}^{n_v}\sum_{i,j=1}^{n_v} f(\overline{h_{m,v_i}}v_{ij}g_{m,v_j}) = \sum_{v\in\mathfrak{J}_\kappa} f\left(\sum_{m=1}^{n_v} h^*_{m,v}vg_{m,v}\right).\end{aligned}$$

Consider the following series $x_a=\Sigma_{v\in\mathfrak{J}_\kappa}\Sigma_{m=1}^{n_v}h^*_{m,v}vg_{m,v}$ in V, which is a limit of some matrix combinations of $\mathfrak{J}_\kappa$. If p is a continuous matrix seminorm on V then $\sup p(\mathfrak{J}_\kappa)=c_\kappa<\infty$, for $\mathfrak{J}_\kappa$ is a matrix bounded set. It follows that

$$\sum_{v\in\mathfrak{J}_\kappa}\sum_{m=1}^{n_v} p^{(1)}(h^*_{m,v}vg_{m,v}) \leqslant \sum_{v\in\mathfrak{J}_\kappa}\sum_{m=1}^{n_v}\|h^*_{m,v}\|p^{(n_v)}(v)\|g_{m,v}\| \leqslant c_\kappa\|a\|_t.$$

Being V a complete space, we conclude that x_a as a sum of the absolutely convergent series belongs to V and $p^{(1)}(x_a)\leqslant c_\kappa\|a\|_t$. Using the continuity of f, we derive that $\mathrm{tr}(\Phi(f)a)=\Sigma_{v\in\mathfrak{J}_\kappa}f(\Sigma_{m=1}^{n_v}h^*_{m,v}vg_{m,v})=f(x_a)$. Recall that the weak* topology $\sigma(\mathfrak{D}_\mathfrak{S},\mathcal{T}_\mathfrak{S})$ is given by the family of seminorms $\omega_a(b)=|\mathrm{tr}(ab)|$, $a\in\mathcal{T}_\mathfrak{S}$, $b\in\mathfrak{D}_\mathfrak{S}$, whereas the weak* topology $\sigma(V',V)$ is given by the family $w_x(g)=|g(x)|$, $x\in V$, $g\in V'$. Take $a\in\mathcal{T}_\mathfrak{S}$. Then $a=\oplus_{\kappa\in\alpha}a_\kappa$ for a certain finite subset $\alpha\subseteq\Xi$ with $a_\kappa\in\mathcal{T}_{\mathfrak{J}_\kappa}$, $\kappa\in\alpha$. Put $x_a=\Sigma_{\kappa\in\alpha}x_{a_\kappa}\in V$. It follows that x_a belongs to the matrix hull of the union $\cup\mathfrak{S}$, and

$$\omega_a(\Phi(f)) = |\mathrm{tr}(\Phi(f)a)| = \left|\sum_{\kappa\in\alpha}\mathrm{tr}(\Phi(f)a_\kappa)\right| = \left|\sum_{\kappa\in\alpha}f(x_{a_\kappa})\right| = |f(x_a)| = w_{x_a}(f) \tag{5.6}$$

for all $f\in V'$. Whence Φ is a weak* continuous linear mapping.

Finally, the set $\{x_a:a\in\mathcal{T}_\mathfrak{S}\}$ spans V. Indeed, take $x\in V$. Since $M(V)$ is the matrix hull of $\cup\mathfrak{S}$, it follows that $x=\Sigma_{i=1}^s g_iv_ih_i$ is a matrix combination with $v_i\in\cup\mathfrak{S}$. Hence we can assume that $x=gvh$ with $g\in M_{1,r}$, $v\in\mathfrak{J}_\kappa\cap M_r(V)$ and $h\in M_{r,1}$ for some $\kappa\in\Xi$ and $r\in\mathbb{N}$. Then $h\in\mathbb{C}^r$ and $g^*\in\mathbb{C}^r$ are columns and $a=h\otimes g^*\in\mathcal{T}_{\mathfrak{J}_\kappa}$. Moreover, $x_a=g^{**}vh=gvh=x$.

In particular, $\{w_{x_a}:a\in\mathcal{T}_\mathfrak{S}\}$ is a defining family of seminorms of the weak* topology $\sigma(V',V)$. Hence, $\Phi:V'_\mathfrak{S}\to\mathfrak{D}_\mathfrak{S}$ is a weak* homeomorphism, thanks to (5.6). □

Now we prove the dual realization theorem for a quantum space.

Theorem 5.7: *If* $\mathfrak{a}$ and $\mathfrak{t}$ *are the quantum boundaries in* $\mathfrak{D}_\mathfrak{S}$, *then*

$$\mathfrak{a}\subseteq\mathfrak{s}(\mathfrak{D}_\mathfrak{S},\mathcal{T}_\mathfrak{S}) \quad \textit{and} \quad \beta(\mathfrak{D}_\mathfrak{S},\mathcal{T}_\mathfrak{S})=\mathfrak{t}.$$

Moreover, if V is a complete quantum space, then its $\mathfrak{S}$*-quantum dual* $V'_\mathfrak{S}$ *can be identified with a subspace in* $\mathfrak{D}_\mathfrak{S}$ *such that*

$$\mathfrak{s}(V',V)=\mathfrak{s}(\mathfrak{D}_\mathfrak{S},\mathcal{T}_\mathfrak{S})|M(V')=\mathfrak{a}|M(V') \quad \textit{and} \quad \mathfrak{S}(V',V)=\mathfrak{t}|M(V').$$

In this case, V is a barreled space and V′ is a weak closed subspace in* $\mathfrak{D}_\mathfrak{S}$ *if and only if V is a complete bornological space.*

Proof: By Corollary 5.4, $\beta(\mathfrak{D}_\mathfrak{S},\mathcal{T}_\mathfrak{S})=\mathfrak{t}$. Let us prove that $\mathfrak{a}\subseteq\mathfrak{s}(\mathfrak{D}_\mathfrak{S},\mathcal{T}_\mathfrak{S})$. Note that $\{\sigma_v^{(1)}:v\in\mathfrak{S}\}$ is a defining family of seminorms for the polynormed topology $\sigma=\mathfrak{a}|\mathfrak{D}_\mathfrak{S}$ in $\mathfrak{D}_\mathfrak{S}$ determined by the atomic quantum topology $\mathfrak{a}$. If $v\in\mathfrak{S}$ and $b\in\mathfrak{D}_\mathfrak{S}$ then $b_v=[b_{v,i,j}]\in M_{n_v}$ and $\sigma_v^{(1)}(b)=\|b_v\|\leqslant\Sigma_{i,j=1}^{n_v}|b_{v,i,j}|=\Sigma_{i,j=1}^{n_v}|\mathrm{tr}(\epsilon_{ji}b_v)|=\Sigma_{i,j=1}^{n_v}\omega_{\epsilon_{ji}}(b)$, where $\epsilon_{ji}=\epsilon_j^*\epsilon_i\in\mathcal{T}_{n_v}$ for all i,j. Hence, $\sigma\subseteq\sigma(\mathfrak{D}_\mathfrak{S},\mathcal{T}_\mathfrak{S})$. It follows that $\max\sigma\subseteq\max\sigma(\mathfrak{D}_\mathfrak{S},\mathcal{T}_\mathfrak{S})$ thanks to Corollary 2.5. However, $\mathfrak{s}(\mathfrak{D}_\mathfrak{S},\mathcal{T}_\mathfrak{S})$ is the unique quantization of $\sigma(\mathfrak{D}_\mathfrak{S},\mathcal{T}_\mathfrak{S})$ by Theorem 2.3. Using Proposition 2.2, we obtain that $\mathfrak{a}\subseteq\max\sigma\subseteq\mathfrak{s}(\mathfrak{D}_\mathfrak{S},\mathcal{T}_\mathfrak{S})$.

Now assume that V is a complete quantum space and consider the linear mapping $\Phi:V'_\mathfrak{S}\to\mathfrak{D}_\mathfrak{S}$, $\Phi(f)=(\langle\langle v,f\rangle\rangle)_{v\in\mathfrak{S}}$, considered in Lemma 5.8. Note that

$$\tau_\kappa(\Phi(f)) = \|(\langle\langle v,f\rangle\rangle)_{v\in\mathfrak{J}_\kappa}\| = \sup\|\langle\langle \mathfrak{J}_\kappa,f\rangle\rangle\| = \sup\|\langle\langle \text{ball } p_\kappa,f\rangle\rangle\| = p_\kappa^{\odot}(f)$$

(see Sec. IV C) for all $\kappa\in\Xi$ and $f\in M(V')$. Hence, $\Phi: V'_{\mathfrak{S}}\to(\mathfrak{D}_{\mathfrak{S}},\mathfrak{t})$ is a topological matrix isomorphism onto its range. We identify $V'_{\mathfrak{S}}$ with its range in $\mathfrak{D}_{\mathfrak{S}}$. Thereby $\mathfrak{S}(V',V)=\mathfrak{t}|M(V')$. Furthermore, $\sigma(V',V)=\sigma(\mathfrak{D}_{\mathfrak{S}},\mathcal{T}_{\mathfrak{S}})|V'$, thanks to Lemma 5.8. However, the weak* topology $\sigma(V',V)$ admits precisely one quantization, thanks to Theorem 2.3. It follows that $\mathfrak{s}(V',V)=\mathfrak{s}(\mathfrak{D}_{\mathfrak{S}},\mathcal{T}_{\mathfrak{S}})|M(V')$. In particular, $\mathfrak{a}|M(V')\subseteq\mathfrak{s}(\mathfrak{D}_{\mathfrak{S}},\mathcal{T}_{\mathfrak{S}})|M(V')=\mathfrak{s}(V',V)$. Conversely, take $v\in M_n(V)$. Since $M_n(V)\subseteq\mathcal{MC}(V'_{\mathfrak{S}},M_n)$ (see the argument used in Sec. V E), it follows that $\|v^{(\infty)}\times(f)\|\leq cp_\kappa^{\odot}(f)$, $f\in M(V')$ for some positive constant c and $\kappa\in\Xi$. Hence $\sup\|\langle\langle c^{-1}v,\text{ball } p_\kappa^{\odot}\rangle\rangle\|\leq 1$. However, ball $p_\kappa^{\odot}=(\text{ball } p_\kappa)^{\odot}=\mathfrak{J}_\kappa^{\odot}$, thanks to Corollary 2.3. Hence, $c^{-1}v\in\mathfrak{J}_\kappa^{\odot\odot}$. Using the bipolar Theorem 2.1, we conclude that $c^{-1}v\in\mathfrak{J}_\kappa$ and $p_{c^{-1}v}=c^{-1}p_v$. Thus $\{p_v: v\in\mathfrak{S}\}$ determines the weak* quantum topology $\mathfrak{s}(V',V)$. Note that $\sigma_v(\Phi^{(\infty)}(f))=\|\langle\langle v,f\rangle\rangle\|=p_v(f)$ for all $f\in M_n(V')$, $v\in\mathfrak{S}$. Consequently, $\mathfrak{a}|M(V')=\mathfrak{s}(V',V)$.

Finally, let us prove that V is barreled and V' is a weak* closed subspace in $\mathfrak{D}_{\mathfrak{S}}$ if and only if V is a bornological space. First, assume V is a bornological space. Then V is barreled being a complete bornological space (Ref. 23, Sec. 2.8). Let us prove that V' is a weak* closed subspace in $\mathfrak{D}_{\mathfrak{S}}$. Take a net $(f_t)\subseteq V'$ such that $\Phi(f_t)\to b$, $b\in\mathfrak{D}_{\mathfrak{S}}$, with respect to the weak* topology $\sigma(\mathfrak{D}_{\mathfrak{S}},\mathcal{T}_{\mathfrak{S}})$. Since $\mathfrak{a}\subseteq\mathfrak{s}(\mathfrak{D}_{\mathfrak{S}},\mathcal{T}_{\mathfrak{S}})$, it follows that $\sigma_v^{(1)}(b-\Phi(f_t))\to 0$ for all $v\in\mathfrak{S}$. Thus $\lim f_t(v)=b_v$ for all $v\in V$. Using the uniform boundedness principle (see, for instance, Ref. 12, Sec. 7.1.4), we derive that $b_v=f(v)$ for a certain $f\in V'$. Then $\langle\langle v,f_t\rangle\rangle\to\langle\langle v,f\rangle\rangle$ for all $v\in M(V)$. It means that $\Phi(f_t)\to\Phi(f)$ with respect to $\mathfrak{a}|V'$. However, $\mathfrak{a}|V'=\mathfrak{s}(\mathfrak{D}_{\mathfrak{S}},\mathcal{T}_{\mathfrak{S}})|V'=\sigma(\mathfrak{D}_{\mathfrak{S}},\mathcal{T}_{\mathfrak{S}})|V'$. Hence, $\Phi(f)=b$, that is, V' is a weak* closed subspace in $\mathfrak{D}_{\mathfrak{S}}$. Conversely, assume that V is barreled and V' is a weak* closed subspace in $\mathfrak{D}_{\mathfrak{S}}$. Then $\Phi=\varphi'$ for the uniquely defined weakly continuous linear mapping $\varphi:\mathcal{T}_{\mathfrak{S}}\to V$. Since Φ is the weak* isomorphism onto its range (Lemma 5.8), it follows that φ is onto (Ref. 12, Sec. 8.6.4). The space $\mathcal{T}_{\mathfrak{S}}$ being an inductive limit of Banach spaces is a complete bornological space (Ref. 23, Sec. 2.8.2). In particular, it is barreled. Using Ref. 12, Sec. 8.6.2, we conclude that $\varphi:\mathcal{T}_{\mathfrak{S}}\to V$ is a continuous linear mapping. Furthermore, it is weakly open, thanks to Ref. 12, Sec. 8.6.3. Since V is barreled, it follows that $\varphi:\mathcal{T}_{\mathfrak{S}}\to V$ is open too (Ref. 12, Sec. 8.6.10). Thus, V being a quotient of a bornological space $\mathcal{T}_{\mathfrak{S}}$ turns out to be a bornological space. □

Remark 5.2: The weakly open mapping $\varphi:\mathcal{T}_{\mathfrak{S}}\to V$ considered in the proof of Theorem 5.7 is matrix weakly open, that is, the mapping $\varphi^{(\infty)}: M(\mathcal{T}_{\mathfrak{S}})\to M(V)$ is open with respect to the weak quantum topologies. Indeed, since φ is weakly open and onto the space V can be identified with the quotient space $\mathcal{T}_{\mathfrak{S}}/\ker(\varphi)$. In particular, it possesses a new quotient quantum topology whose restriction to V is reduced to the weak topology $\sigma(V,V')$. However, the latter polynormed topology admits precisely one quantization $\mathfrak{s}(V,V')$.[9] Whence $(M(V),\mathfrak{s}(V,V'))$ is just the quotient of $(M(\mathcal{T}_{\mathfrak{S}}),\mathfrak{s}(\mathcal{T}_{\mathfrak{S}},\mathfrak{D}_{\mathfrak{S}}))$, that is, $\varphi^{(\infty)}$ is weakly open (see Ref. 10).

Corollary 5.6: If V is a complete quantum space, then $V'\subseteq\mathcal{D}_{\mathfrak{S}}$ and

$$\beta(V',V)=\beta(\mathfrak{D}_{\mathfrak{S}},\mathcal{T}_{\mathfrak{S}})|M(V').$$

In the normed case Corollary 5.6 is reduced to Blecher's result[1] (see also Ref. 4) on the dual realization of an operator space.

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