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Integral Equations and Operator Theory

Regularities in Noncommutative Banach Algebras

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Abstract. In this paper we introduce regularities and subspectra in a unital noncommutative Banach algebra and prove that there is a correspondence between them similar to the commutative case. This correspondence involves a radical on a class of Banach algebras equipped with a subspectrum. Taylor and Slodkowski spectra for noncommutative tuples of bounded linear operators are the main examples of subspectra in the noncommutative case.

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1. Introduction

The regularities play an important role in the (joint) spectral theory of the Banach algebra framework. They generalize the (joint) invertibility in a Banach algebra. It is well known [13] that there is a close relationship between the spectral systems and regularities. In [15] the characterization of those regularities in a commutative Banach algebra related to subspectra has been proposed. A subspectrum in the sense of Zelazko [22] on a commutative Banach algebra A is a set-valued mapping over all tuples in A with the properties to be compact and polynomial spectral mapping. In this paper we introduce a subspectrum on a unital (noncommutative) Banach algebra based upon the properties to be compact and spectral mapping with respect to the noncommutative polynomials (see below Section 3), and establish a correspondence between them and regularities. In the noncommutative case, a subspectrum on A can be determined in terms of Lie algebras generated by the tuples $a = (a_1, \ldots, a_k) \in A^k$ in A using a fixed Banach space representation $\alpha : A \to \mathcal{B}(X)$ [8]. To conduct that approach, one might demand a restrictive condition concerning the Banach algebra A. We meet with the known phenomena [8] when a tuple of noncommutative polynomials $p(a) = (p_1(a), \ldots, p_m(a)) \in A^m$ in elements of a k-tuple $a \in A^k$ generating a finite dimensional nilpotent Lie subalgebra $\mathfrak{L}(a) \subseteq A$, may generate an infinite dimensional Lie subalgebra $\mathfrak{L}(p(a)) \subseteq A$. To avoid these type of problems, we shall assume that A is a nilpotent Lie algebra, that is, its Lie algebra structure determined by the Lie multiplication $[a,b] = ab - ba$, $a,b \in A$, is nilpotent (see Section 6). Such algebra A admits sufficiently many subspectra. So are Slodkowski, Taylor spectra

$$\sigma_{\pi,n}(a) = \sigma_{\pi,n}(\pi(a_1), \ldots, \pi(a_k)), \quad \sigma_{\delta,n}(a) = \sigma_{\delta,n}(\pi(a_1), \ldots, \pi(a_k)), \quad n \geq 0,$$

and Harte type spectrum $\sigma_R(a)$ for tuples a in A. Thus if τ is one of these spectra, then $\tau(a)$ is a nonempty compact subset in $\mathbb{C}^k$ for a k-tuple a in A, $\tau(x)$ is a subset of the usual spectrum $\sigma(x)$ for a singleton $x \in A$, and if $p(a)$ is a m-tuple of noncommutative polynomials in elements of a k-tuple a, then

$$\tau(p(a)) = p(\tau(a))$$

(see [8] and Proposition 6.5). The assumption on A to be a nilpotent Lie algebra is also sustained by the noncommutative functional calculus problem [9]. In Proposition 6.6 we show that the closed associative envelopes of a supernilpotent Lie subalgebra $\mathfrak{g}$ (that is, its commutator $[\mathfrak{g},\mathfrak{g}]$ consists of nilpotent elements) possess that property. But an operator tuple a in $\mathcal{B}(X)$ generating a supernilpotent Lie subalgebra $\mathfrak{g} \subseteq \mathcal{B}(X)$ admits [9], [6], [7] a noncommutative holomorphic functional calculus in a neighborhood of the Taylor spectrum $\sigma_T(a)$, which extends Taylor functional calculus [19]. Thus a noncommutative Banach algebra A which is nilpotent as a Lie algebra has all the favorable spectral properties just as commutative Banach algebras.

A regularity R in a unital Banach algebra A is defined as a nonempty subset $R \subseteq A$ such that $ab \in R$ iff $a,b \in R$ (see Section 4). Each regularity automatically involves a set K_R of characters φ of A such that $R \cap \ker(\varphi) = \emptyset$, and the closed two-sided ideal

$$R(A) = \bigcap \{\ker(\varphi) : \varphi \in K_R\}$$

called the R-radical of A. The set

$$R^{\#} = A \backslash \bigcup \{\ker(\varphi) : \varphi \in K_R\}$$

is called an envelope of R. A regularity R having an open proper envelope $R^{\#}$ is of importance in our consideration. Such regularities appear when we deal with subspectra. Namely, a subspectrum τ on A associates a regularity R_τ in A given by the rule

$$R_\tau = \{a \in A : 0 \notin \tau(a)\}.$$

In this case, R_τ is a nonempty open proper subset in A and $R_\tau^{\#} = R_\tau$ (see [15] for the commutative case). A key role in the noncommutative case plays the τ-radical $\operatorname{Rad}_\tau A$ of A associated to a subspectrum τ on a Banach algebra A. According to the definition

$$\operatorname{Rad}_\tau A = \{a \in A : \tau(a) = \{0\}\}.$$

It is proved (see Corollary 4.9) that $\operatorname{Rad}_\tau A$ is a closed two-sided ideal in A which contains the Jacobson radical $\operatorname{Rad} A$, and A is commutative modulo $\operatorname{Rad}_\tau A$. Moreover, τ determines a subspectrum $\tau^\sim$ on the quotient algebra $A/\operatorname{Rad}_\tau A$. Thus $\tau^\sim$ is a subspectrum in the sense of Zelazko on the commutative (semisimple) Banach algebra $A/\operatorname{Rad}_\tau A$. Furthermore, τ generates a compact subset $\mathcal{K}_\tau$ of the character space $\operatorname{Char}(A)$ of A such that

$$\tau(a_1, \ldots, a_k) = \{(\varphi(a_1), \ldots, \varphi(a_k)) : \varphi \in \mathcal{K}_\tau\}$$

for a k-tuple $(a_1, \ldots, a_k)$ in A.

The process of generating regularities from subspectra can be reversed (see Section 5). Namely, fix a regularity R in A with its open proper envelope $R^\#$; one may define a Harte type spectrum σ_R on A by the rule

$$\sigma_R(a) = \left\{\lambda \in \mathbb{C}^k : A(a-\lambda) \cap R^\# = \emptyset\right\},$$

where a is a k-tuple in A and $A(a-\lambda)$ is the left ideal in A generated by the tuple $a-\lambda$. One can prove that the left ideal in the definition of $\sigma_R(a)$ can be replaced with the right ideal $(a-\lambda)A$ generated by $a-\lambda$, and σ_R is a subspectrum on A. Moreover,

$$R_{\sigma_R} = R^\#$$

and the σ_R-radical is reduced to the R-radical, that is,

$$\operatorname{Rad}_{\sigma_R} A = R(A).$$

Thus the correspondence $\tau \to R_\tau$ between subspectra on A and regularities in A has a right inverse $R \to \sigma_R$. Furthermore $\tau \subseteq \sigma_{R_\tau}$. In the commutative case that relation has been observed in [15]. Note that, in general $\tau \neq \sigma_{R_\tau}$. We investigate that difference in Section 6 by proposing necessary and sufficient condition when the latter inclusion turns out to be an equality.

2. Preliminaries

All considered linear spaces are assumed to be complex and $\mathbb{C}$ denotes the field of all complex numbers. For a unital associative algebra A, $\operatorname{Rad}(A)$ denotes its Jacobson radical and A^* the space of all linear functionals. A unital algebra homomorphism $\lambda : A \to \mathbb{C}$ is said to be a character of A, and the set of all characters of A is denoted by $\operatorname{Char}(A)$. If S is a subset of an associative algebra A, then $A(S)$ (respectively, $(S)A$) denotes the left (respectively, right) ideal in A generated by S. The group of all invertible elements in A is denoted by $G(A)$. If A is a Banach algebra, then as it is well known [4, 1.2], $G(A)$ is an open subset in A and $\operatorname{Char}(A)$ is a compact space with respect to the weak*-topology in the space of all bounded linear functionals on A. We use the denotation $\sigma(a)$ for the spectrum of an element $a \in A$. The Banach algebra of all bounded linear operators on a Banach space X is denoted by $\mathcal{B}(X)$. If $\pi : A \to B$ is an algebra homomorphism, then $\pi^{(n)} : A^n \to B^n$ denotes the mapping $\pi^{(n)}(a_1, \ldots, a_n) = (\pi(a_1), \ldots, \pi(a_n))$ between the n-tuples in A and B.

The following assertion is a well known [14] fact.

Theorem (Gleason, Kahane, Zelazko). *Let A be a unital Banach algebra and let $\varphi : A \to \mathbb{C}$ be a linear functional such that $\varphi(1) = 1$ and $\varphi(a) \neq 0$ for all $a \in G(A)$. Then $\varphi \in \operatorname{Char}(A)$.*

Now let $\mathfrak{F}_n(e)$ be the free associative algebra generated by n elements

$$e = (e_1, \ldots, e_n).$$

Each its element $p(e)$ is a noncommutative polynomial $p(e) = \sum_\nu \alpha_\nu e^\nu$, where $\alpha_\nu \in \mathbb{C}$ and $e^\nu = e_{j_1} \cdots e_{j_k}$ for a finite sequence $\nu = (j_1, \ldots, j_k)$ of elements from the set $\{1, \ldots, n\}$. For a n-tuple $a = (a_1, \ldots, a_n) \in A^n$ in a unital associative algebra A, we have a well defined algebra homomorphism

$$\Gamma_a : \mathfrak{F}_n(e) \to A$$

such that $\Gamma_a(e_i) = a_i$ for all i. If $p(e) \in \mathfrak{F}_n(e)$ is a free polynomial, then $\Gamma_a(p(e)) = p(a)$ is the same polynomial in A taken by a. Indeed,

$$\Gamma_a(p(e)) = \Gamma_a\left(\sum_\nu \alpha_\nu e^\nu\right) = \sum_\nu \alpha_\nu a^\nu = p(a).$$

We say that Γ_a is *a polynomial calculus for the tuple a*. Similarly, each element $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ determines a character $\Gamma_\lambda : \mathfrak{F}_n(e) \to \mathbb{C}$ such that $\Gamma_\lambda(e_i) = \lambda_i$ for all i. We say that Γ_λ is *a point calculus*. Actually, each character of $\mathfrak{F}_n(e)$ is a point calculus. We put

$$p(\lambda) = \Gamma_\lambda(p(e)).$$

If $p(e) = (p_1(e), \ldots, p_m(e))$ is a m-tuple in $\mathfrak{F}_n(e)$ and $\lambda \in \mathbb{C}^n$, then we write $p(\lambda)$ to indicate the m-tuple $(p_1(\lambda), \ldots, p_m(\lambda)) \in \mathbb{C}^m$ in $\mathbb{C}$.

Now let A be a unital Banach algebra and let B be a unital subalgebra of A. Consider the family $\mathcal{I}_A(B)$ of all left ideals I in B such that $I \cap G(A) = \emptyset$. The following assertion was proved in [10] (see also [23]).

Theorem 2.1. *If $aBa^{-1} \subseteq B$ for all $a \in B \cap G(A)$, then $\mathcal{I}_A(B)$ possesses the projection property, that is, for each mutually commuting k-tuple $b = (b_1, \ldots, b_k)$ in $I \in \mathcal{I}_A(B)$ and $c \in B$ commuting with b there correspond $\lambda \in \mathbb{C}$ and $J \in \mathcal{I}_A(B)$ such that $(b_1, \ldots, b_k, c - \lambda) \in J^{k+1}$.*

We shall apply (as in [15]) Theorem 2.1 to the following particular case. Let $A = \mathcal{C}(K)$ be the Banach algebra of all complex continuous functions on a compact topological space K furnished with the uniform norm $\|f\|_\infty = \sup\{|f(x)| : x \in K\}$, and let B be a unital subalgebra of A. For a commutative tuple $a \in B^k$ we put

$$\tau(a) = \left\{\lambda \in \mathbb{C}^k : B(a - \lambda) \cap G(\mathcal{C}(K)) = \emptyset\right\}, \tag{2.1}$$

which is a compact subset in $\mathbb{C}^k$. On account of Theorem 2.1, we infer that τ possesses the projection property, that is, if $a = (a_1, \ldots, a_{k+1}) \in B^{k+1}$ is a $k+1$-tuple and $a' = (a_1, \ldots, a_k)$, then $\tau(a') = \pi(\tau(a))$, where $\pi : \mathbb{C}^{k+1} \to \mathbb{C}^k$ is the canonical projection onto the first k coordinates. Actually, the projection property involves (see [10]) the polynomial spectral mapping property

$$\tau\left(p\left(a\right)\right)=p\left(\tau\left(a\right)\right),$$

where p is a family of polynomials in several complex variables. In this case it is said that τ is a subspectrum on B (see below Section 3). Thus (2.1) determines a subspectrum on B. This type of subspectra were characterized by A. Wawrzynczyk in [23].

Finally, if A is an associative algebra, then it is also a Lie algebra with respect to the canonical Lie multiplication $[a,b]=ab-ba$, $a,b\in A$. To indicate this Lie algebra structure we use the denotation $A_{\mathfrak{lie}}$, thus $A_{\mathfrak{lie}}$ is the same algebra A considered with respect to the Lie multiplication called *the attendant Lie algebra*. Let us recall that a Lie algebra L is said to be nilpotent if its lower central series $\left\{L^{(n)}\right\}_{n\in\mathbb{N}}$ (where $L^{(n)}=\left[L,L^{(n-1)}\right]$) is vanishing, that is, $L^{(k)}=\{0\}$ for a certain k. Thus each operator $\operatorname{ad}x:L\to L$, $(\operatorname{ad}x)(y)=[x,y]$ $(x\in L)$ of the adjoint representation is nilpotent. If $k=1$, the Lie algebra L is commutative. A finite-dimensional nilpotent Lie algebra L with $L^{(2)}=\{0\}$ is called a Heisenberg algebra. A typical example is a Lie algebra $\mathfrak{g}$ with a basis e_1,e_2,e_3 such that $[e_1,e_2]=e_3$ and $[e_i,e_3]=0$ for all i. Further, note that $A_{\mathfrak{lie}}^{(1)}=[A,A]=A^{(1)}$ and $A_{\mathfrak{lie}}^{(n)}=\left[A,A_{\mathfrak{lie}}^{(n-1)}\right]=\left[A,A^{(n-1)}\right]=A^{(n)}$, $n>1$.

Let A be a unital associative algebra. A subalgebra $B\subseteq A$ is said to be *an inverse closed subalgebra* if any invertible in A element of B is invertible in B. Since the inverse closed subalgebras are stable with respect to arbitrary intersections, it can be defined *an inverse closed envelope of a subset in A.*

The following assertion is well known [20], [1].

Lemma (Turovskii). *Let A be a unital Banach algebra which is the closure of the inverse closed envelope of a (not necessarily finite-dimensional) nilpotent Lie algebra. Then A is commutative modulo its Jacobson radical* $\operatorname{Rad}A$*. In particular, so is a Banach algebra A with its nilpotent attendant Lie algebra $A_{\mathfrak{lie}}$.*

Note that the closed associative envelope in A generated by $A_{\mathfrak{lie}}$ is obviously reduced to the whole algebra A. Therefore if $A_{\mathfrak{lie}}$ is a nilpotent Lie algebra, then A as a closed associative envelope of $A_{\mathfrak{lie}}$ is commutative modulo the Jacobson radical thanks to Turovskii's lemma.

3. Subspectra

In this section we consider purely algebraic case. We introduce a subspectrum in a noncommutative algebra and show that it generates a subspectrum on the quotient algebra modulo suitable radical.

Let A be a unital associative algebra. As in the commutative case [21], a subspectrum τ on A is a mapping which associates to every k-tuple $a = (a_1, \dots, a_k) \in A^k$ a nonempty compact set $\tau(a) \subseteq \mathbb{C}^k$ such that $\tau(a) \subseteq \prod_{i=1}^{k} \sigma(a_i)$ and it possesses the spectral mapping property

$$\tau(p(a)) = p(\tau(a)) \tag{3.1}$$

for an m-tuple $p(e) = (p_1(e), \dots, p_m(e)) \in \mathfrak{F}_k(e)^m$. Of course, we have assumed that the usual spectrum $\sigma(a)$ of each element $a \in A$ is nonvoid. That is true whenever A is a Banach algebra. Note that the equality (3.1) establishes a relation between the polynomial calculus Γ_a and point calculi Γ_λ, $\lambda \in \tau(a)$. Namely,

$$p(\tau(a)) = \{p(\lambda) : \lambda \in \tau(a)\} = \left\{\Gamma_\lambda^{(m)}(p(e)) : \lambda \in \tau(a)\right\} = \tau\left(\Gamma_a^{(m)}(p(e))\right).$$

For subspectra τ and σ on A we put $\tau \subseteq \sigma$ if $\tau(a) \subseteq \sigma(a)$ for all tuples a in A.

Now let τ be a subspectrum on A. We put

$$\operatorname{Rad}_\tau(A) = \{a \in A : \tau(a) = \{0\}\}.$$

We say that $\operatorname{Rad}_\tau(A)$ is the *τ-radical in A*.

Lemma 3.1. *Let τ be a subspectrum on A. Then $\operatorname{Rad}_\tau(A)$ is a two-sided ideal in A and the quotient algebra $A/\operatorname{Rad}_\tau(A)$ is commutative. Moreover,*

$$\operatorname{Rad}(A) \subseteq \operatorname{Rad}_\tau(A),$$

and the inclusion turns out to be an equality whenever $\tau(a) = \sigma(a)$ for all $a \in A$.

Proof. By assumption, $\tau(a) \subseteq \sigma(a)$ is a nonempty subset for each $a \in A$. Therefore $\tau(a) = \{0\}$ if $\sigma(a) = \{0\}$. Take $a \in \operatorname{Rad}(A)$. Then $\lambda - a$ is invertible in A for all λ, $\lambda \neq 0$ (see [5, 8.6.3]). It follows that $\sigma(a) = \{0\}$, that is, $a \in \operatorname{Rad}_\tau(A)$. Thus $\operatorname{Rad}(A) \subseteq \operatorname{Rad}_\tau(A)$.

Take $a, b \in \operatorname{Rad}_\tau(A)$. Then

$$\tau(a + b) = \{\lambda + \mu : (\lambda, \mu) \in \tau(a, b)\}$$

and $\tau(a, b) \subseteq \tau(a) \times \tau(b) = \{0\}$. Whence $\tau(a + b) = \{0\}$ and $a + b \in \operatorname{Rad}_\tau(A)$. Using a similar argument, we conclude that $ca, ac \in \operatorname{Rad}_\tau(A)$ for any $c \in A$. Thus $\operatorname{Rad}_\tau(A)$ is a two-sided ideal in A. Now take $a, b \in A$ and let $p(e_1, e_2) = e_1 e_2 - e_2 e_1 \in \mathfrak{F}_2(e)$. Then $p(a, b) = [a, b] \in A$ and

$$\tau(p(a, b)) = p(\tau(a, b)) = \{\lambda\mu - \mu\lambda : (\lambda, \mu) \in \tau(a, b)\} = \{0\}.$$

Hence $[a, b] \in \operatorname{Rad}_\tau(A)$ and therefore A is commutative modulo $\operatorname{Rad}_\tau(A)$. □

Consider the quotient linear mapping

$$\pi_\tau : A \to A/\operatorname{Rad}_\tau(A) \ \ \pi_\tau(a) = a^\sim.$$

If $a = (a_1, \dots, a_k)$ is a k-tuple in A, then $a^\sim = (a_1^\sim, \dots, a_k^\sim) = \pi_\tau^{(k)}(a)$ is a k-tuple in $A/\operatorname{Rad}_\tau(A)$.

Lemma 3.2. *Let τ be a subspectrum on A. Then to each k-tuple $a^{\sim} = (a_1^{\sim}, \dots, a_k^{\sim})$ in $A/\operatorname{Rad}_{\tau}(A)$ there corresponds a subset $\tau^{\sim}(a^{\sim}) \subseteq \mathbb{C}^k$ such that*

$$\tau^{\sim}(a^{\sim}) = \tau(b)$$

for a k-tuple $b = (b_1, \dots, b_k) \in A^k$ with $a_i^{\sim} = b_i^{\sim}$ for all i.

Proof. One has to prove that $\tau(a+x) = \tau(a)$ for any k-tuple $x = (x_1, \dots, x_k)$ in $\operatorname{Rad}_{\tau}(A)$. Take $\lambda \in \tau(a)$. Then $(\lambda, \mu) \in \tau(a, x)$ for some $\mu \in \mathbb{C}^k$. But $\mu \in \tau(x) \subseteq \prod_{i=1}^{k} \tau(x_i) = \{0\}$ and $\lambda + \mu \in \tau(a+x)$, that is, $\lambda \in \tau(a+x)$. Thus $\tau(a) \subseteq \tau(a+x)$. Since $-x \in \operatorname{Rad}_{\tau}(A)^k$, it follows that $\tau(a+x) \subseteq \tau(a+x-x) = \tau(a)$. It remains to put $\tau^{\sim}(a^{\sim}) = \tau(a)$. □

Lemma 3.3. *Let τ be a subspectrum on A. Then $\tau^{\sim}(a^{\sim}) \subseteq \sigma(a^{\sim})$ for a singleton $a \in A$, where $\sigma(a^{\sim})$ is the usual spectrum of $a^{\sim}$ in the algebra $A/\operatorname{Rad}_{\tau}(A)$.*

Proof. If $\lambda \notin \sigma(a^{\sim})$, then $(a-\lambda)b = 1 + x$ for some $x \in \operatorname{Rad}_{\tau}(A)$ and $b \in A$. But $\tau((a-\lambda)b) = \tau(1+x) = \{1\}$ and

$$\tau((a-\lambda)b) = \{zw : (z, w) \in \tau(a-\lambda, b)\}.$$

Then $0 \notin \tau(a-\lambda)$, for in the contrary case $(0, w) \in \tau(a-\lambda, b)$ for some $w \in \mathbb{C}$, which in turn implies that $0 = 0w \in \tau((a-\lambda)b)$. So, $\lambda \notin \tau(a)$. On account of Lemma 3.2, $\tau^{\sim}(a^{\sim}) = \tau(a) \subseteq \sigma(a^{\sim})$. Whence $\lambda \notin \tau^{\sim}(a^{\sim})$. □

Using Lemmas 3.2 and 3.3, we obtain that

$$\tau^{\sim}(a_1^{\sim}, \dots, a_k^{\sim}) = \tau(a_1, \dots, a_k) \subseteq \prod_{i=1}^{k} \tau(a_i) = \prod_{i=1}^{k} \tau(a_i^{\sim}) \subseteq \prod_{i=1}^{k} \sigma(a_i^{\sim})$$

is a nonempty compact subset.

Theorem 3.4. *Let τ be a subspectrum on a unital associative algebra A. The correspondence $\tau^{\sim}$ over all tuples in $A/\operatorname{Rad}_{\tau}(A)$ is a subspectrum on the commutative algebra $A/\operatorname{Rad}_{\tau}(A)$.*

Proof. Take a k-tuple $a^{\sim} = (a_1^{\sim}, \dots, a_k^{\sim})$ in $A/\operatorname{Rad}_{\tau}(A)$ and a noncommutative polynomial $p(e) = \sum_{\nu} \alpha_{\nu} e^{\nu} \in \mathfrak{F}_k(e)$. Then

$$\begin{aligned} p(a^{\sim}) &= \Gamma_{a^{\sim}}(p(e)) = \sum_{\nu} \alpha_{\nu} (a^{\sim})^{\nu} = \sum_{\nu} \alpha_{\nu} \pi_{\tau}^{(k)}(a)^{\nu} = \sum_{\nu} \alpha_{\nu} \pi_{\tau}(a^{\nu}) \\ &= \pi_{\tau}\left(\sum_{\nu} \alpha_{\nu} a^{\nu}\right) = \pi_{\tau}(\Gamma_a(p(e))) = p(a)^{\sim}. \end{aligned}$$

If $p(e) = (p_1(e), \dots, p_m(e))$ is a m-tuple in $\mathfrak{F}_k(e)$, then

$$\tau^{\sim}(p(a^{\sim})) = \tau^{\sim}(p(a)^{\sim}) = \tau(p(a)) = p(\tau(a)) = p(\tau^{\sim}(a^{\sim})).$$

It remains to appeal to Lemmas 3.1, 3.2 and 3.3. □

In particular, if τ is a subspectrum on a Banach algebra A and $\operatorname{Rad}_\tau(A)$ is closed, then $\tau^\sim$ is a subspectrum on the commutative Banach algebra $A/\operatorname{Rad}_\tau(A)$. In the next section we prove that $\operatorname{Rad}_\tau(A)$ is closed for each subspectrum on a Banach algebra A.

Definition 3.5. Let τ be a subspectrum on a unital associative algebra A. We say that a subspace $I \subseteq A$ is *τ-singular* if $0 \in \tau(c)$ for any tuple $c \in I^k$, $k \in \mathbb{N}$. A linear functional $\varphi \in A^*$ is said to be *a τ-singular* if its kernel $\ker(\varphi)$ is a τ-singular subspace in A. The set of all τ-singular functionals on A is denoted by K_τ.

The concept of a τ-singular subspace is motivated by the key reasoning in [15, Lemma 2.3] for the commutative Banach algebra case.

The following simple lemma will be useful later.

Lemma 3.6. *If $I \subseteq A$ is a τ-singular subspace, then $1 \notin I$.*

Proof. Being $\tau(1) \subseteq \sigma(1) = \{1\}$ a nonempty subset, we conclude that $\tau(1) = \{1\}$. Then $0 \notin \tau(1)$ and therefore $1 \notin I$. □

Proposition 3.7. *Each τ-singular subspace in A is annihilated by a τ-singular functional on A.*

Proof. Let I is a τ-singular subspace in A. One has to prove that $\varphi(I) = \{0\}$ for a certain τ-singular functional $\varphi \in A^*$. Consider a family $\mathcal{E}$ of all τ-singular subspaces $F \subseteq A$ such that $I \subseteq F$, that is, $0 \in \tau(c)$ for any tuple $c = (c_1, \ldots, c_k) \in F^k$, $k \in \mathbb{N}$ (Definition 3.5). If $\{F_\alpha\}$ is a linearly ordered family in $\mathcal{E}$, then $\cup_\alpha F_\alpha \in \mathcal{E}$. By Zorn's lemma, there is a maximal element in $\mathcal{E}$, say J. By Lemma 3.6, $1 \notin J$.

Let us prove that $A = J \oplus \mathbb{C}1$. If the latter does not hold, then $\mathbb{C}u \cap (J \oplus \mathbb{C}1) = \{0\}$ for a certain $u \in A$, that is, $J \cap (\mathbb{C}u \oplus \mathbb{C}1) = \{0\}$. Let $c = (c_1, \ldots, c_k) \in J^k$ be a k-tuple in J, and let $(c, u) = (c_1, \ldots, c_k, u) \in A^{k+1}$. Since $0 \in \tau(c)$, it follows that $(0, \lambda) \in \tau(c, u)$ or $0 \in \tau(c, u - \lambda)$ for some $\lambda \in \mathbb{C}$. Thus

$$K(c) = \{\mu \in \mathbb{C} : 0 \in \tau(c, u - \mu)\}$$

is a nonempty compact subset in $\mathbb{C}$. Using (3.1) (namely, the Projection Property), we obtain that $K(c, b) \subseteq K(c) \cap K(b)$ for all tuples c, b in J. Hence there is a common point $\lambda_0 \in K(c)$ for all tuples c in J. Put $x = u - \lambda_0 1 \in \mathbb{C}u \oplus \mathbb{C}1$. Thus $x \notin J$ and $0 \in \tau(c, x)$ for all tuples c in J. Consider the subspace $\overline{J} = J \oplus \mathbb{C}x \subseteq A$. If $c + \xi x = (c_1 + \xi_1 x, \ldots, c_k + \xi_k x) \in \overline{J}^k$ (herein $\xi = (\xi_1, \ldots, \xi_k) \in \mathbb{C}^k$) is a k-tuple in $\overline{J}$, then

$$\tau(c + \xi x) = \{\lambda + \mu\xi : (\lambda, \mu) \in \tau(c, x)\},$$

which in turn implies that $0 \in \tau(c + \xi x)$. Thus $\overline{J} \in \mathcal{E}$ and $J \neq \overline{J}$, a contradiction.

Consequently, $A = J \oplus \mathbb{C}1$, that is, $J = \ker(\varphi)$ for some $\varphi \in A^*$. But φ is a τ-singular functional, for $J \in \mathcal{E}$. It remains to note that $I \subseteq J$. □

Now let τ be a subspectrum on A and let

$$R_\tau = \{a \in A : 0 \notin \tau(a)\}. \tag{3.2}$$

Note that $R_\tau \cap \ker(\varphi) = \emptyset$ for each $\varphi \in K_\tau$.

Corollary 3.8. $A \backslash R_\tau = \bigcup \{\ker(\varphi) : \varphi \in K_\tau\}$.

Proof. Take $a \in A \backslash R_\tau$. Then $0 \in \tau(a)$. Consider the subspace $\mathbb{C}a$ in A generated by a and let $\xi a = (\xi_1 a, \ldots, \xi_k a)$ be a k-tuple in $\mathbb{C}a$, where $\xi = (\xi_1, \ldots, \xi_k) \in \mathbb{C}^k$. Let $p_i(e) = \xi_i e$, $1 \leq i \leq k$, be polynomials in $\mathfrak{F}_1(e)$, and let $p(e) = (p_1(e), \ldots, p_k(e)) \in \mathfrak{F}_1(e)^k$. Then

$$\tau(\xi a) = \tau(p(a)) = p(\tau(a)) = \{p(\lambda) : \lambda \in \tau(a)\} = \{\lambda \xi : \lambda \in \tau(a)\} \subseteq \mathbb{C}^k.$$

In particular, $0 \in \tau(\xi a)$, that is, $\mathbb{C}a$ is a τ-singular subspace in A. By Proposition 3.7, $a \in \ker(\varphi)$ for some τ-singular functional $\varphi \in A^*$. Thus $a \in \ker(\varphi)$ for some $\varphi \in K_\tau$. □

Corollary 3.9. *If τ is a subspectrum on A, then*

$$\operatorname{Rad}_\tau(A) = \bigcap \{\ker(\varphi) : \varphi \in K_\tau\}.$$

Proof. Take $a \notin \operatorname{Rad}_\tau(A)$. Then $\lambda \in \tau(a)$ for some nonzero $\lambda \in \mathbb{C}$, that is, $0 \in \tau(a - \lambda)$. The latter means that $a - \lambda \in A \backslash R_\tau$. Using Corollary 3.8, infer that $a - \lambda \in \ker(\varphi)$ for some $\varphi \in K_\tau$. It follows that $\varphi(a) = \lambda \varphi(1) \neq 0$ by virtue of Lemma 3.6. Thereby $a \notin \ker(\varphi)$. So,

$$\bigcap \{\ker(\varphi) : \varphi \in K_\tau\} \subseteq \operatorname{Rad}_\tau(A).$$

Conversely, take $a \notin \ker(\varphi)$ for some $\varphi \in K_\tau$. Then $\varphi(a) \neq 0$ and

$$\varphi\left(a - \varphi(a)\varphi(1)^{-1}\right) = \varphi(a) - \varphi(a)\varphi(1)^{-1}\varphi(1) = 0,$$

that is, $a - \varphi(a)\varphi(1)^{-1} \in \ker(\varphi)$ (see Lemma 3.6). Since $\varphi \in K_\tau$, it follows that $0 \in \tau\left(a - \varphi(a)\varphi(1)^{-1}\right)$. This in turn implies that $\varphi(a)\varphi(1)^{-1} \in \tau(a)$, that is, $\tau(a) \neq \{0\}$ or $a \notin \operatorname{Rad}_\tau(A)$. Thus $\operatorname{Rad}_\tau(A) \subseteq \bigcap \{\ker(\varphi) : \varphi \in K_\tau\}$. □

Corollary 3.10. *Let τ be a subspectrum on A. Then $\tau^\sim$ is a subspectrum on the algebra $A/\operatorname{Rad}_\tau(A)$ with the properties $\pi_\tau(R_\tau) = R_{\tau^\sim}$ and $\pi_\tau^*(K_{\tau^\sim}) = K_\tau$, where*

$$\pi_\tau^* : (A/\operatorname{Rad}_\tau(A))^* \to A^*$$

is the dual of the quotient mapping $\pi_\tau : A \to A/\operatorname{Rad}_\tau(A)$.

Proof. Using Lemma 3.1, infer that $A/\operatorname{Rad}_\tau(A)$ is commutative. Moreover, $\tau^\sim$ is a subspectrum on $A/\operatorname{Rad}_\tau(A)$ as we have shown in Theorem 3.4. The equality $\pi_\tau(R_\tau) = R_{\tau^\sim}$ follows directly from (3.2) and Lemma 3.2.

Now take $\psi \in K_{\tau^\sim}$ and put $\varphi = \psi \cdot \pi_\tau$. If a is a k-tuple in $\ker(\varphi)$, then $a^\sim$ is a k-tuple in $\ker(\psi)$ and $0 \in \tau^\sim(a^\sim) = \tau(a)$ by virtue of Lemma 3.2. Thus φ is a τ-singular functional. Therefore $\pi_\tau^*(K_{\tau^\sim}) \subseteq K_\tau$. Conversely, take $\varphi \in K_\tau$. By Corollary 3.9, $\varphi = \psi \cdot \pi$ for some $\psi \in (A/\operatorname{Rad}_\tau(A))^*$. Again by Lemma 3.2, ψ is $\tau^\sim$-singular. □

A result of Zelazko [22] asserts that for each subspectrum τ on a commutative Banach algebra B there corresponds a unique compact subset $K \subseteq \operatorname{Char}(B)$ such that $\tau(a) = \left\{\varphi^{(k)}(a) : \varphi \in K\right\}$ for any k-tuple a in B. In the pure algebraic context this result has the following generalization.

Theorem 3.11. *Let τ be a subspectrum on a unital associative algebra A. Then*

$$\tau(a_1, \ldots, a_k) = \{(\overline{\varphi}(a_1), \ldots, \overline{\varphi}(a_k)) : \varphi \in K_\tau\}$$

for all tuples $(a_1, \ldots, a_k)$ in A, where $\overline{\varphi} = \varphi(1)^{-1}\varphi$ (see Lemma 3.6).

Proof. Let $a = (a_1, \ldots, a_k)$ and take $\mu \in \tau(a)$. Then $0 \in \tau(a - \mu)$. Consider a subspace F in A generated by $a - \mu$. Each element of F has the form $p(a - \mu)$ for some $p(e) \in \mathfrak{F}_k(e)$ such that $p(0) = 0$. Using the Spectral Mapping Property (3.1), we conclude that F is a τ-singular subspace in A. On account of Proposition 3.7, $F \subseteq \ker(\varphi)$ for some τ-singular functional $\varphi \in A^*$. Thus $\varphi(a_i - \mu_i) = 0$ or $\varphi(a_i) = \mu_i \varphi(1)$, $1 \leq i \leq k$. It follows that $\overline{\varphi}(a_i) = \mu_i$ or $\overline{\varphi}^{(k)}(a) = \mu$. Conversely, take $\varphi \in K_\tau$. Then $a_i - \overline{\varphi}(a_i) \in \ker(\varphi)$ for all i. Being φ a τ-singular functional, we deduce that $0 \in \tau\left(a - \overline{\varphi}^{(k)}(a)\right)$ or $\overline{\varphi}^{(k)}(a) \in \tau(a)$. Thus $\tau(a) = \left\{\overline{\varphi}^{(k)}(a) : \varphi \in K_\tau\right\}$. □

4. Regularities

In this section we introduce regularities in a unital associative algebra and investigate their properties.

Let A be a unital associative algebra and let R be a nontrivial subset in A. As in [15], we introduce the *envelope* $R^\#$ of R in A as

$$A \backslash R^\# = \bigcup \{\ker(\varphi) : \varphi \in A^*, R \cap \ker(\varphi) = \emptyset\}.$$

By its very definition, $R \subseteq R^\#$.

Lemma 4.1. $R^{\#\#} = R^\#$.

Proof. Since $R^\# \subseteq R^{\#\#}$, it suffices to prove that $R^{\#\#} \subseteq R^\#$. Take $a \in A \backslash R^\#$. Then $a \in \ker(\varphi)$, $R \cap \ker(\varphi) = \emptyset$, for some $\varphi \in A^*$. But $\ker(\varphi) \subseteq A \backslash R^\#$, that is, $\ker(\varphi) \cap R^\# = \emptyset$. The latter in turn implies that $a \in A \backslash R^{\#\#}$. □

The following definition plays a key role in our consideration.

Definition 4.2. Let A be a unital algebra. A nonempty subset $R \subseteq A$ is said to be *a regularity in* A if it possesses the following property

$$ab \in R \quad \text{iff} \quad a, b \in R.$$

The following two assertions provide us with examples of regularities.

Lemma 4.3. *The set $G(A)$ of all invertible elements in a unital algebra A is a regularity in A whenever A is commutative modulo its Jacobson radical. Moreover, $G(A) = G(A)^\#$ if A is a Banach algebra.*

Proof. If $a, b \in A$, then $\sigma(ab) = \sigma((ab)^{\sim})$, where $(ab)^{\sim}$ is the image of ab in the commutative algebra $A/\operatorname{Rad}(A)$. But $(ab)^{\sim} = a^{\sim}b^{\sim}$. Therefore, $0 \notin \sigma(ab)$ iff both $a^{\sim}$ and $b^{\sim}$ are invertible in $A/\operatorname{Rad}(A)$. Thus $0 \notin \sigma(a^{\sim}) = \sigma(a)$ and $0 \notin \sigma(b^{\sim}) = \sigma(b)$, that is, $a, b \in G(A)$. Thus $G(A)$ is a regularity in A.

Now assume that A is a Banach algebra. Take $a \notin G(A)$. Then $a^{\sim}$ is not invertible in $A/\operatorname{Rad}(A)$ and therefore belongs to a maximal ideal I of $A/\operatorname{Rad}(A)$. But $I = \ker(\phi)$ for some character ϕ of the commutative algebra $A/\operatorname{Rad}(A)$ [4, 1.3.2]. It follows that $a \in \ker(\varphi)$, where $\varphi = \phi\pi \in \operatorname{Char}(A)$. But $G(A) \cap \ker(\varphi) = \emptyset$. Therefore $a \notin G(A)^{\#}$. Thus $G(A) = G(A)^{\#}$. □

Proposition 4.4. *Let τ be a subspectrum on A and let $R_\tau = \{a \in A : 0 \notin \tau(a)\}$ (see (3.2)). Then R_τ is a regularity in A and $R_\tau = R_\tau^{\#}$.*

Proof. With $\tau(0) = \{0\}$ in mind, infer $0 \notin R_\tau$. Moreover, $1 \in R_\tau$, for $\tau(1) \subseteq \sigma(1) = \{1\}$. Thus $\emptyset \neq R_\tau \neq A$. Using Corollary 3.8, we obtain that

$$A \backslash R_\tau = \bigcup \{\ker(\varphi) : \varphi \in K_\tau\}.$$

But $\bigcup \{\ker(\varphi) : \varphi \in K_\tau\} \subseteq A \backslash R_\tau^{\#}$, that is, $A \backslash R_\tau \subseteq A \backslash R_\tau^{\#}$. Thus $R_\tau = R_\tau^{\#}$.

Since $\tau(ab) = \{\lambda\mu : (\lambda, \mu) \in \tau(a, b)\}$, it follows that $0 \notin \tau(ab)$ iff $0 \notin \tau(a)$ and $0 \notin \tau(b)$. Whence R_τ is a regularity in A. □

Now let us prove the main result of this section.

Theorem 4.5. *Let R be a regularity in A. Then $G(A) \subseteq R$. Moreover, if A is a Banach algebra and $\varphi \in A^*$ is such that $R \cap \ker(\varphi) = \emptyset$, then $\ker(\varphi) = \ker(\phi)$ for some $\phi \in \operatorname{Char}(A)$. In particular,*

$$A \backslash R^{\#} = \bigcup \{\ker(\varphi) : \varphi \in \operatorname{Char}(A), R \cap \ker(\varphi) = \emptyset\}.$$

Proof. Take $a \in R$. Then $a = a \cdot 1 \in R$ and therefore $1 \in R$. Further, if $a \in G(A)$, then $1 = aa^{-1} \in R$, which in turn implies that $a \in R$. Thus $G(A) \subseteq R$.

Now suppose A is a Banach algebra and let $\varphi \in A^*$ such that $R \cap \ker(\varphi) = \emptyset$. Taking into account that $G(A) \subseteq R$, we obtain $G(A) \cap \ker(\varphi) = \emptyset$. Using the Gleason-Kahane-Zelazko theorem, we deduce that $\phi = \varphi(1)^{-1}\varphi \in \operatorname{Char}(A)$. But $\ker(\varphi) = \ker(\phi)$.

Finally, the union over all the characters indicated above belongs to $A \backslash R^{\#}$. Conversely, take $a \in A \backslash R^{\#}$. Then $a \in \ker(\varphi)$ and $R \cap \ker(\varphi) = \emptyset$ for some functional $\varphi \in A^*$. But $\ker(\varphi) = \ker(\phi)$ for some $\phi \in \operatorname{Char}(A)$. □

Corollary 4.6. *If R is a regularity in a Banach algebra A, then so is $R^{\#}$.*

Proof. Take $a, b \in R^{\#}$. If $ab \notin R^{\#}$, then $ab \in \ker(\varphi)$ for some $\varphi \in \operatorname{Char}(A)$, $R \cap \ker(\varphi) = \emptyset$, by virtue of Theorem 4.5. Then

$$0 = \varphi(ab) = \varphi(a)\varphi(b).$$

Hence a or b belongs to $\ker(\varphi)$, that is, a or b does not belong $R^{\#}$, a contradiction.

Conversely, if $a \notin R^{\#}$ or $b \notin R^{\#}$, then $\varphi(ab) = \varphi(a)\varphi(b) = 0$ for some $\varphi \in \mathrm{Char}(A)$, $R \cap \ker(\varphi) = \emptyset$ (Theorem 4.5), that is, $ab \notin R^{\#}$. Whence $R^{\#}$ is a regularity in A. □

Now let τ be a subspectrum on a unital Banach algebra A and let $\mathcal{K}_\tau \subseteq \mathrm{Char}(A)$ be the set of all τ-singular characters of A. Note that $\mathcal{K}_\tau \subseteq K_\tau$ and $\ker(\varphi) \cap R_\tau = \emptyset$ for all $\varphi \in K_\tau$ (see Section 3).

Corollary 4.7. *Let τ be a subspectrum on a unital Banach algebra A. If $\varphi \in K_\tau$, then* $\ker(\varphi) = \ker(\phi)$ *for some $\phi \in \mathcal{K}_\tau$. In particular,*

$$A \backslash R_\tau = \bigcup \{\ker(\varphi) : \varphi \in \mathcal{K}_\tau\}.$$

Moreover, R_τ is open in A. Thus R_τ has an open proper envelope $R_\tau^{\#}$ which coincides with itself.

Proof. Take $\varphi \in K_\tau$. Using Proposition 4.4 and Theorem 4.5, we infer that $\ker(\varphi) = \ker(\phi)$ for some $\phi \in \mathrm{Char}(A)$. But $\ker(\varphi)$ is a τ-singular subspace, therefore $\phi \in \mathcal{K}_\tau$. Thus $A \backslash R_\tau = \bigcup \{\ker(\varphi) : \varphi \in \mathcal{K}_\tau\}$.

Finally, take $a \in R_\tau$. Then $s_a = \min\{|\lambda| : \lambda \in \tau(a)\} > 0$ and if $\|b\| < s_a$, then $a + b \in R_\tau$. □

Corollary 4.8. *Let τ be a subspectrum on a unital Banach algebra A. Then*

$$\tau(a) = \left\{\varphi^{(k)}(a) : \varphi \in \mathcal{K}_\tau\right\}$$

for all tuples $a \in A^k$, $k \in \mathbb{N}$. Moreover, $\mathcal{K}_\tau$ is a compact subspace in $\mathrm{Char}(A)$.

Proof. First note that $\overline{\phi}(x) = \phi(1)^{-1}\phi(x) = \phi(x)$ for all $\phi \in \mathrm{Char}(A)$ and $x \in A$. Thus

$$\left\{\varphi^{(k)}(a) : \varphi \in \mathcal{K}_\tau\right\} \subseteq \left\{\overline{\varphi}^{(k)}(a) : \varphi \in K_\tau\right\}.$$

On account of Theorem 3.11, it suffices to prove that for any $\varphi \in K_\tau$ there corresponds $\phi \in \mathcal{K}_\tau$ such that $\overline{\varphi} = \phi$. By Corollary 4.7, $\ker(\varphi) = \ker(\phi)$ for some $\phi \in \mathcal{K}_\tau$. Then $\phi = \alpha\varphi$ for some $\alpha \in \mathbb{C}$. But $1 = \phi(1) = \alpha\varphi(1)$, that is, $\alpha = \varphi(1)^{-1}$ (Lemma 3.6). Thus $\phi = \overline{\varphi}$.

Now let us prove that $\mathcal{K}_\tau$ is compact with respect to the topology inherited from $\mathrm{Char}(A)$. It suffices to prove $\mathcal{K}_\tau$ is closed in $\mathrm{Char}(A)$. Take $\varphi \in \mathrm{Char}(A) \backslash \mathcal{K}_\tau$. Then $\ker(\varphi)$ is not τ-singular, that is, $0 \notin \tau(b)$ for some (say k-)tuple b in $\ker(\varphi)$. Let $\varepsilon > 0$ be the distance between the origin and $\tau(b)$ in $\mathbb{C}^k$, and let

$$U_{b,\eta}(\varphi) = \left\{\phi \in \mathrm{Char}(A) : \max_{1 \le i \le k} |\phi(b_i) - \varphi(b_i)| < \eta\right\}$$

be a neighborhood of φ in $\mathrm{Char}(A)$, where $\eta = (2k)^{-1/2}\varepsilon$. Take $\phi \in U_{b,\eta}(\varphi)$. Then $\max|\phi(b_i)| < (2k)^{-1/2}\varepsilon$. Moreover,

$$(\phi(b_1), \ldots, \phi(b_k)) \notin \tau(b),$$

for in the contrary case we would have $\varepsilon^2 \le \sum_{i=1}^k |\phi(b_i)|^2 \le k \max|\phi(b_i)|^2 < \varepsilon^2/2$. It follows that $\phi \notin \mathcal{K}_\tau$. Thus $U_{b,\eta}(\varphi) \cap \mathcal{K}_\tau = \emptyset$. □

Corollary 4.9. *Let τ be a subspectrum on a unital Banach algebra A. Then*

$$\operatorname{Rad}_{\tau}(A) = \bigcap \{\ker(\varphi) : \varphi \in \mathcal{K}_{\tau}\} .$$

In particular, $\operatorname{Rad}_{\tau}(A)$ *is a closed two-sided ideal in A.*

Proof. It suffices to apply Corollary 3.9 and Corollary 4.7. □

Corollary 4.10. *Let τ be a subspectrum on a unital Banach algebra A. Then $\tau^{\sim}$ is a subspectrum on the commutative Banach algebra $A/\operatorname{Rad}_{\tau}(A)$ and $\pi_{\tau}^{*}(\mathcal{K}_{\tau^{\sim}}) = \mathcal{K}_{\tau}$.*

Proof. Using Lemma 3.1 and Corollary 4.9, we infer that $A/\operatorname{Rad}_{\tau}(A)$ is a commutative Banach algebra. Moreover, $\tau^{\sim}$ is a subspectrum on $A/\operatorname{Rad}_{\tau}(A)$ as we have shown in Lemma 3.2.

Now take $\psi \in \mathcal{K}_{\tau^{\sim}}$. Then $\psi \in \operatorname{Char}(A/\operatorname{Rad}_{\tau}(A))$ such that ψ is $\tau^{\sim}$-singular. Put $\varphi = \pi_{\tau}^{*}(\psi) = \psi \cdot \pi_{\tau}$. Evidently, $\varphi \in \operatorname{Char}(A)$ and $\varphi \in K_{\tau}$ by virtue of Corollary 3.10. So $\varphi \in \mathcal{K}_{\tau}$. Conversely, take $\varphi \in \mathcal{K}_{\tau}$. By Corollary 4.9,

$$\varphi = \psi \cdot \pi_{\tau}, \ \ \psi \in \operatorname{Char}(A/\operatorname{Rad}_{\tau}(A)) .$$

Again by Corollary 3.10, $\psi \in K_{\tau^{\sim}}$. Thus $\pi_{\tau}^{*}(\mathcal{K}_{\tau^{\sim}}) = \mathcal{K}_{\tau}$. □

5. Subspectrum associated to a regularity in a Banach algebra

In this section we construct a subspectrum called Harte type spectrum by means of a regularity in a Banach algebra. That will reverse (in a certain sense) the process of creating regularities from subspectra.

Let R be a regularity in a unital Banach algebra A and let

$$K_R = \{\varphi \in \operatorname{Char}(A) : R \cap \ker(\varphi) = \emptyset\} .$$

According to Theorem 4.5,

$$A \backslash R^{\#} = \bigcup \{\ker(\varphi) : \varphi \in K_R\} . \tag{5.1}$$

Let us introduce a closed two-sided ideal

$$R(A) = \bigcap \{\ker(\varphi) : \varphi \in K_R\}$$

in A. Evidently, $\operatorname{Rad} A \subseteq R(A)$. Moreover, since $\varphi([x,y]) = 0$, $x, y \in A$, $\varphi \in \operatorname{Char}(A)$, it follows that $A/R(A)$ is a semisimple commutative Banach algebra.

Lemma 5.1. *If R has an open proper envelope $R^{\#}$ in A, then K_R is a nonempty compact space.*

Proof. Since $R^{\#}$ is proper, it follows that $A/R^{\#} \neq \emptyset$ and therefore $K_R \neq \emptyset$. It remains to prove that K_R is a closed subset in $\operatorname{Char}(A)$. Take a net $\{\varphi_{\iota}\}$ in K_R which tends to $\varphi \in \operatorname{Char}(A)$. If $\varphi \notin K_R$, then $\varphi(x) = 0$ for a certain $x \in R$. Note that $x - \varphi_{\iota}(x) \in \ker(\varphi_{\iota})$ and $\ker(\varphi_{\iota}) \cap R^{\#} = \emptyset$ (see Lemma 4.1). Thus $\{x - \varphi_{\iota}(x)\}$ is a net in the closed set $A \backslash R^{\#}$ and it tends to x. Consequently, $x \in A \backslash R^{\#}$ and thereupon $x \notin R$, a contradiction. Thus $\varphi \in K_R$ and K_R is closed in $\operatorname{Char}(A)$. □

Now let $a = (a_1, \dots, a_k) \in A^k$ be a k-tuple in A. It determines a continuous mapping

$$\widehat{a} = (\widehat{a}_1, \dots, \widehat{a}_k) : K_R \to \mathbb{C}^k, \quad \widehat{a}(\varphi) = \varphi^{(k)}(a) = (\varphi(a_1), \dots, \varphi(a_k)).$$

Actually, it is the restriction to K_R of the continuous mapping $\widehat{a} : \mathrm{Char}(A) \to \mathbb{C}^k$, $\widehat{a}(\varphi) = \varphi^{(k)}(a)$ determined by the Gelfand transform. Put

$$\tau_R(a) = \mathrm{im}(\widehat{a}),$$

which is a nonempty compact subset in $\mathbb{C}^k$ thanks to Lemma 5.1.

Lemma 5.2. *Let $a \in A^k$ and let $x \in R(A)^k$. Then $\tau_R(a + x) = \tau_R(a)$.*

Proof. Note that $\widehat{x}(\varphi) = 0$ for all $\varphi \in K_R$. Then $\tau_R(a + x) = \mathrm{im}\left(\widehat{a + x}\right) = \mathrm{im}(\widehat{a} + \widehat{x}) = \mathrm{im}(\widehat{a}) = \tau_R(a)$. □

Proposition 5.3. *Let R be a regularity in a Banach algebra A whose envelope is open and proper. Then the correspondence τ_R over all tuples in A is a subspectrum on A. Moreover, $R_{\tau_R} = R^{\#}$ and $\mathrm{Rad}_{\tau_R} A = R(A)$.*

Proof. Take a k-tuple a in A, and m-tuple $p(e)$ in $\mathfrak{F}_k(e)$. Then $p(a)$ is a m-tuple in A and

$$\begin{aligned}\widehat{p(a)}(\varphi) &= \varphi^{(m)}(p(a)) = (\varphi(p_1(a)), \dots, \varphi(p_m(a))) = (p_1(\widehat{a}(\varphi)), \dots, p_m(\widehat{a}(\varphi))) \\ &= p(\widehat{a}(\varphi)).\end{aligned}$$

Therefore

$$\tau_R(p(a)) = \mathrm{im}\left(\widehat{p(a)}\right) = p(\mathrm{im}(\widehat{a})) = p(\tau_R(a)),$$

that is, (3.1) holds. Further,

$$\begin{aligned}R_{\tau_R} &= \{a \in A : 0 \notin \tau_R(a)\} = \{a \in A : \widehat{a}(\varphi) \neq 0, \varphi \in K_R\} \\ &= \{a \in A : \varphi(a) \neq 0, \varphi \in K_R\} = \{a \in A : a \notin \ker(\varphi), \varphi \in K_R\} \\ &= A \setminus \bigcup \{\ker(\varphi) : \varphi \in K_R\} = R^{\#},\end{aligned}$$

that is, $R_{\tau_R} = R^{\#}$.

Finally,

$$\begin{aligned}\mathrm{Rad}_{\tau_R} A &= \{a \in A : \tau_R(a) = \{0\}\} = \{a \in A : \varphi(a) = 0, \varphi \in K_R\} \\ &= \bigcap \{\ker(\varphi) : \varphi \in K_R\} = R(A),\end{aligned}$$

that is, $R(A) = \mathrm{Rad}_{\tau_R} A$. □

By Lemma 3.1, Corollary 4.9 and Proposition 5.3, infer that $A/R(A)$ is a commutative semisimple Banach algebra. The mapping

$$T : A \to \mathcal{C}(K_R), \quad T(a) = \widehat{a}, \quad \widehat{a}(\varphi) = \varphi(a),$$

is a contractive homomorphism. Evidently, $\ker(T) = R(A)$ and it can be factored as a composition of the quotient mapping $A \to A/R(A)$ and a contractive homomorphism $A/R(A) \to \mathcal{C}(K_R)$ (see Lemma 5.2). Denote the range of T by B, which is a unital subalgebra in $\mathcal{C}(K_R)$.

Lemma 5.4. $T\left(R^{\#}\right) = B \cap G(\mathcal{C}(K_R))$.

Proof. If $a \in R^{\#}$, then $\widehat{a}(\varphi) = \varphi(a) \neq 0$ for all $\varphi \in K_R$, that is, the function $\widehat{a}$ is invertible in $\mathcal{C}(K_R)$. Conversely, if $\widehat{a}$ is invertible in $\mathcal{C}(K_R)$, then $\varphi(a) \neq 0$ for all $\varphi \in K_R$. It follows that $0 \notin \tau_R(a)$, that is, $a \in R_{\tau_R}$. Using Proposition 5.3, infer that $a \in R^{\#}$. Thus $T\left(R^{\#}\right) = B \cap G(\mathcal{C}(K_R))$. □

For a k-tuple $\widehat{a}$ in B we set

$$\gamma_R(\widehat{a}) = \left\{\lambda \in \mathbb{C}^k : B(\widehat{a} - \lambda) \cap G(\mathcal{C}(K_R)) = \emptyset\right\}.$$

Note that γ_R is a subspectrum on B (see (2.1)).

Lemma 5.5. *If R is a regularity in a unital Banach algebra A, then $R^{\#} = R^{\#} + R(A)$. In particular, if $A(M)$ (respectively, $(M)A$) is the left ideal (respectively, right) in A generated by a subset $M \subseteq A$, then $A(M) \cap R^{\#} = \emptyset$ iff $(M)A \cap R^{\#} = \emptyset$.*

Proof. If $a + x \notin R^{\#}$ for some $a \in R^{\#}$ and $x \in R(A)$, then $\varphi(a + x) = 0$ for some $\varphi \in K_R$ thanks to (5.1). Thereby $\varphi(a) = 0$, that is, $a \in A/R^{\#}$. So, $R^{\#} = R^{\#} + R(A)$.

Now let M be a subset in A such that $A(M) \cap R^{\#} = \emptyset$. Then

$$(M)A \subseteq A(M) + [A, M] \subseteq A(M) + R(A)$$

(see Corollary 4.9). If $x \in (M)A \cap R^{\#}$, then $x = y + z$ for some $y \in A(M)$ and $z \in R(A)$. It follows that $y = x - z \in R^{\#} + R(A) = R^{\#}$, a contradiction. □

Lemma 5.6. *If φ is a γ_R-singular functional on B, then $A(a) \cap R^{\#} = \emptyset$ for any tuple a in* $\ker(\varphi T)$.

Proof. If a is a k-tuple in $\ker(\varphi T)$, then so is $\widehat{a}$ in $\ker(\varphi)$. Being φ a γ_R-singular, we obtain that $0 \in \gamma_R(\widehat{a})$, that is, $B(\widehat{a}) \cap G(\mathcal{C}(K_R)) = \emptyset$. The latter in turn implies that $A(a) \cap R^{\#} = \emptyset$, for $T\left(A(a) \cap R^{\#}\right) = B(\widehat{a}) \cap G(\mathcal{C}(K_R))$ thanks to Lemmas 5.4 and 5.5. □

Now let R be a regularity in a unital Banach algebra A whose envelope $R^{\#}$ is open and proper. By a *Harte type spectrum σ_R associated with R* we mean a set-valued function over all tuples in A determined by the rule

$$\sigma_R(a) = \left\{\lambda \in \mathbb{C}^k : A(a - \lambda) \cap R^{\#} = \emptyset\right\}$$

for a k-tuple a in A. Using Lemma 4.1, we deduce that

$$\begin{aligned}\sigma_{R^{\#}}(a) &= \left\{\lambda \in \mathbb{C}^k : A(a - \lambda) \cap R^{\#\#} = \emptyset\right\}\\ &= \left\{\lambda \in \mathbb{C}^k : A(a - \lambda) \cap R^{\#} = \emptyset\right\}\\ &= \sigma_R(a).\end{aligned}$$

Furthermore, the left ideal $A(a-\lambda)$ generated by $a-\lambda$ in the definition of $\sigma_R(a)$ can be replaced with the right one as follows from Lemma 5.5. If $R = G(A)$ the set $\sigma_R(a)$ is the known [13, 1.8.1] Harte spectrum of the tuple a.

Theorem 5.7. *Let R be a regularity in a Banach algebra A whose envelope $R^{\#}$ is open and proper. Then*

$$\sigma_R(a) = \tau_R(a)$$

for any k-tuple a in A. In particular, σ_R is a subspectrum on A. Moreover,

$$R_{\sigma_R} = R^{\#} \text{ and } \operatorname{Rad}_{\sigma_R} A = R(A).$$

Proof. Take $\widehat{a}(\varphi) \in \tau_R(a)$, where $\varphi \in K_R$. Since φ is a character of A, it follows that $A(a-\widehat{a}(\varphi)) \subseteq \ker(\varphi)$. Therefore $A(a-\widehat{a}(\varphi)) \cap R^{\#} = \emptyset$, that is, $\widehat{a}(\varphi) \in \sigma_R(a)$. Hence $\tau_R(a) \subseteq \sigma_R(a)$. Conversely, assume that $0 \in \sigma_R(a)$. Then $A(a) \cap R^{\#} = \emptyset$. Using Lemmas 5.4 and 5.5, we infer that $B(\widehat{a}) \cap G(\mathcal{C}(K_R)) = \emptyset$. Whence $0 \in \gamma_R(\widehat{a})$. Since γ_R is a subspectrum on B, we deduce that $0 \in \gamma_R\left(\widehat{b}\right)$ for any tuple in the subspace in B generated by $\widehat{a}$. Thus the subspace generated by $\widehat{a}$ is γ_R-singular. Using Proposition 3.7, we obtain that $\widehat{a}$ is a tuple in $\ker(\phi)$ for some γ_R-singular functional $\phi \in B^*$. According to Lemma 5.6, $A(b) \cap R^{\#} = \emptyset$ for any tuple b in $\ker(\varphi)$, where $\varphi = \phi T$. In particular, $\ker(\varphi) \cap R = \emptyset$. By Theorem 4.5, one can assume that $\varphi \in \operatorname{Char}(A)$, that is, $\varphi \in K_R$. Moreover, a is a tuple in $\ker(\varphi)$. Therefore $0 = \widehat{a}(\varphi) \in \tau_R(a)$, that is, $\sigma_R(a) \subseteq \tau_R(a)$. Thus $\sigma_R(a) = \tau_R(a)$.

It remains to apply Proposition 5.3. □

6. Slodkowski and Harte type spectra

In this section we compare various subspectra and investigate when a subspectrum on a Banach algebra is reduced to the Harte type spectrum.

Let A be a unital Banach algebra. In Section 4, a correspondence

$$\tau \to R_\tau$$

between subspectra on A and regularities in A has been proposed. Further, in Section 5, we have considered a correspondence

$$R \to \sigma_R$$

between regularities and Harte type spectra, which can regarded as a right inverse of the first one, since $R_{\sigma_R} = R^{\#}$ by virtue of Theorem 5.7. Thus $R_{\sigma_R} = R$ whenever $R^{\#} = R$.

Theorem 6.1. *Let τ be a subspectrum on a unital Banach algebra A. Then $\tau \subseteq \sigma_{R_\tau}$. Moreover, $\tau = \sigma_R$ for a regularity R that has an open and proper envelope $R^{\#}$ iff each $\varphi \in \operatorname{Char}(A)$ with $\ker(\varphi) \cap R = \emptyset$ is a τ-singular functional on A.*

Proof. Put $K_{R_\tau} = \{\varphi \in \mathrm{Char}\,(A) : \ker(\varphi) \cap R_\tau = \emptyset\}$ and let a be a k-tuple in A. By Theorem 5.7, $\sigma_{R_\tau}(a) = \tau_{R_\tau}(a) = \widehat{a}\,(K_{R_\tau})$. Moreover,

$$A/R_\tau^{\#} = \bigcup \{\ker(\varphi) : \varphi \in K_{R_\tau}\}$$

thanks to Theorem 4.5. But $R_\tau^{\#} = R_\tau$ due to Proposition 4.4. Furthermore, on account of Corollary 4.7, $A \backslash R_\tau = \bigcup \{\ker(\varphi) : \varphi \in \mathcal{K}_\tau\}$, where $\mathcal{K}_\tau \subseteq K_{R_\tau}$ is the subset of all τ-singular functionals. Using Corollary 4.8, infer that

$$\tau(a) = \left\{\varphi^{(k)}(a) : \varphi \in \mathcal{K}_\tau\right\} = \widehat{a}\,(\mathcal{K}_\tau) \subseteq \widehat{a}\,(K_{R_\tau}) = \sigma_{R_\tau}(a),$$

that is, $\tau \subseteq \sigma_{R_\tau}$.

Now assume that $\tau = \sigma_R$ for some regularity R in A that has an open and proper envelope $R^{\#}$. Take $\varphi \in K_R$. We want to prove that $\varphi \in \mathcal{K}_\tau$. If a is a k-tuple in $\ker(\varphi)$, then

$$0 = \varphi^{(k)}(a) = \widehat{a}\,(\varphi) \in \tau_R(a) = \sigma_R(a) = \tau(a)$$

by virtue of Theorem 5.7. Thus $0 \in \tau(a)$ for each tuple a in $\ker(\varphi)$, that is, φ is τ-singular. Further, if $\varphi \in \mathcal{K}_\tau$, then $0 \in \tau(a)$ for each $a \in \ker(\varphi)$. It follows that $0 \in \sigma_R(a)$, that is, $A(a) \cap R^{\#} = \emptyset$. In particular, $a \notin R$. Thus $\ker(\varphi) \cap R = \emptyset$, which means that $\varphi \in K_R$. Thus $K_R = \mathcal{K}_\tau$.

Conversely, assume that $\mathcal{K}_\tau = K_R$. Then

$$\sigma_R(a) = \widehat{a}\,(K_R) = \widehat{a}\,(\mathcal{K}_\tau) = \tau(a)$$

by virtue of Theorem 5.7 and Corollary 4.8. Thus $\sigma_R = \tau$. □

Note that the inclusion $\tau \subseteq \sigma_{R_\tau}$ stated in Theorem 6.1 can be proper, that is, there are Harte type spectrum σ_R and subspectrum τ with the same regularity R (that is, $R_\tau = R$) such that $\tau \neq \sigma_R$. That can be characterized in terms of characters. Consider a regularity R in A that has an open and proper envelope $R^{\#}$. Then we have a nonempty compact subset $K_R \subseteq \mathrm{Char}\,(A)$ (see Lemma 5.1) of all the σ_R-singular functionals (see Theorem 5.7). For a closed subset $K \subseteq K_R$ we define its A-rationally convex hull in $\mathrm{Char}\,(A)$ as

$$\widetilde{K} = \left\{\varphi \in \mathrm{Char}\,(A) : \ker(\varphi) \subseteq \bigcup \{\ker(\phi) : \phi \in K\}\right\}.$$

Note that $\bigcup \{\ker(\phi) : \phi \in K\} \subseteq \bigcup \{\ker(\phi) : \phi \in K_R\} = A \backslash R^{\#}$. It follows that $\ker(\varphi) \cap R = \emptyset$ for all $\varphi \in \widetilde{K}$, that is, $\widetilde{K} \subseteq K_R$. In terms of the Gelfand transform we have

$$\begin{aligned}\widetilde{K} &= \left\{\varphi \in \mathrm{Char}\,(A) : \forall x \in A, \quad x \in \ker(\varphi) \Longrightarrow \quad x \in \bigcup \{\ker(\phi) : \phi \in K\}\right\} \\ &= \{\varphi \in \mathrm{Char}\,(A) : \forall x \in A, \quad \widehat{x}\,(\varphi) = 0 \Longrightarrow \quad 0 \in \widehat{x}\,(K)\}\end{aligned}$$

(see [15], [23]). Since $R_\tau = R$, it follows that $A \backslash R = \bigcup \{\ker(\varphi) : \varphi \in \mathcal{K}_\tau\}$ (see Corollary 4.7). But

$$A \backslash R \supseteq A \backslash R^{\#} = \bigcup \{\ker(\varphi) : \varphi \in K_R\}.$$

Therefore $\ker(\varphi) \subseteq \bigcup \{\ker(\phi) : \phi \in \mathcal{K}_\tau\}$ for all $\varphi \in K_R$. Thus

$$\widetilde{\mathcal{K}_\tau} = K_R \tag{6.1}$$

for a subspectrum with the regularity R.

Theorem 6.2. *Let R be a regularity in A that has an open and proper envelope $R^{\#}$. Assume that $\mathcal{K}$ is a nonempty closed subset in K_R such that $\widetilde{\mathcal{K}} = K_R$. Then there is a subspectrum τ on A such that $R_\tau = R^{\#}$, $\mathcal{K}_\tau = \mathcal{K}$ and $\tau \subseteq \sigma_R$. Namely, $\tau(a) = \widehat{a}(\mathcal{K})$ for a tuple a in A. Moreover, $\tau \neq \sigma_R$ iff $\mathcal{K} \neq K_R$.*

Proof. Evidently, the relation $\tau(a) = \widehat{a}(\mathcal{K})$ determines a subspectrum on A (see the proof of Proposition 5.3). Moreover,

$$\begin{aligned} R_\tau &= \{a \in A : 0 \notin \tau(a)\} = \{a \in A : 0 \notin \widehat{a}(\mathcal{K})\} \\ &= \{a \in A : \varphi(a) \neq 0, \varphi \in \mathcal{K}\} = A \backslash \bigcup \{\ker(\varphi) : \varphi \in \mathcal{K}\} \\ &= A \backslash \bigcup \left\{\ker(\varphi) : \varphi \in \widetilde{\mathcal{K}}\right\} = A \backslash \bigcup \{\ker(\varphi) : \varphi \in K_R\} \\ &= R^{\#}. \end{aligned}$$

By Theorem 6.1, $\tau \subseteq \sigma_{R^{\#}} = \sigma_R$.

Now let us prove that $\mathcal{K}_\tau = \mathcal{K}$. Clearly $\mathcal{K} \subseteq \mathcal{K}_\tau$. Take $\varphi \in \mathcal{K}_\tau$. So, φ is a τ-singular character. We shall show that φ belongs to the closure of $\mathcal{K}$. Take a neighborhood

$$U_{a,\varepsilon}(\varphi) = \left\{\phi \in \operatorname{Char}(A) : \max_{1 \leq i \leq k} |\phi(a_i) - \varphi(a_i)| < \varepsilon\right\}$$

of φ in $\operatorname{Char}(A)$, where $a = (a_1, \ldots, a_k)$ is a k-tuple in A. Obviously, $a - \varphi^{(k)}(a)$ is a k-tuple in $\ker(\varphi)$. Therefore $0 \in \tau\left(a - \varphi^{(k)}(a)\right)$, which in turn implies that

$$\phi^{(k)}\left(a - \varphi^{(k)}(a)\right) = 0$$

for some $\phi \in \mathcal{K}$. Thus $\phi^{(k)}(a) = \varphi^{(k)}(a)$. It follows that $\phi \in U_{a,\varepsilon}(\varphi) \cap \mathcal{K}$ for any ε. Taking into account that $\mathcal{K}$ is closed, infer that $\varphi \in \mathcal{K}$.

Finally, if $\mathcal{K}$ is a proper subset in K_R, then $\mathcal{K}_\tau \neq K_R$, for $\mathcal{K} = \mathcal{K}_\tau$ as we have just proven. By Theorem 6.1, $\tau \neq \sigma_R$. Conversely, if $\tau \neq \sigma_R$, then $\tau(a) \neq \sigma_R(a)$ for a k-tuple a in A. It follows that $0 \in \sigma_R(b) \backslash \tau(b)$ for a k-tuple b in A. According to Theorem 5.7, $b \in \ker(\varphi)^k$ for some $\varphi \in K_R$. But $0 \notin \tau(b)$. It follows that φ is not τ-singular, that is, $\varphi \notin \mathcal{K}_\tau$. Thereby $\varphi \notin \mathcal{K}$. □

Thus each regularity may generate a family of subspectra different from Harte type spectrum. Let us illustrate this by an example.

Example. Let A be the algebra of all continuous functions on the closed ball

$$\overline{B}(0,1) = \left\{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 \leq 1\right\}$$

centered at the origin that are holomorphic on its interior, and let $B(0,1)$ be the unit open ball in $\mathbb{C}^2$ centered at the origin. The algebra A furnished with the uniform norm on $\overline{B}(0,1)$ is a commutative semisimple Banach algebra. Moreover,

$$\operatorname{Char}(A) = \overline{B}(0,1).$$

Take a closed subset $\mathcal{K} \subseteq \overline{B}(0,1)$ containing the topological (or Shilov) boundary of $\overline{B}(0,1)$. Then $\widetilde{\mathcal{K}} = \overline{B}(0,1)$. Indeed, first note that

$$\overline{B}(0,1) \setminus \mathcal{K} \subseteq B(0,1)$$

by assumption. Take $(z,w) \in \overline{B}(0,1)$. Prove that $(z,w) \in \widetilde{\mathcal{K}}$. Assume that $(z,w) \notin \mathcal{K}$. Then $(z,w) \in B(0,1)$. If $f(z,w) = 0$ for a function $f \in A$, then $f(z_0,w_0) = 0$ for some (z_0,w_0), $|z_0|^2 + |w_0|^2 = 1$, by the known property of holomorphic functions. But $(z_0,w_0) \in \mathcal{K}$, therefore $0 \in f(\mathcal{K})$. It follows that $(z,w) \in \widetilde{\mathcal{K}}$. Thus $\widetilde{\mathcal{K}} = \overline{B}(0,1)$. Further, demonstrate that if τ is a subspectrum on A associated with $\mathcal{K}$ (see Theorem 6.2), then $\mathcal{K}$ is exactly the set of all τ-singular characters. Indeed, take $(a,b) \in \overline{B}(0,1)$ which is a τ-singular character. We have to prove that $(a,b) \in \mathcal{K}$. It suffices to assume that $(a,b) \in B(0,1)$. Consider the polynomials

$$p(z,w) = z - a \quad \text{and} \quad q(z,w) = w - b.$$

Obviously, $p(a,b) = q(a,b) = 0$. Since (a,b) is a τ-singular character, it follows that

$$0 \in \tau(p,q) = \{(p(z,w), q(z,w)) : (z,w) \in \mathcal{K}\}.$$

Then $p(z_0,w_0) = q(z_0,w_0) = 0$ for some $(z_0,w_0) \in \mathcal{K}$. Whence $z_0 = a$ and $w_0 = b$, or $(a,b) = (z_0,w_0) \in \mathcal{K}$.

The example can be modified by extending the boundary as in [18].

Now we apply Theorem 6.2 to demonstrate a difference between Slodkowski and Harte type spectra. Let us start with simple assertions.

Lemma 6.3. *Let $\alpha : A \to B$ be a unital algebra homomorphism between unital Banach algebras A and B, and let R be a regularity in B. Then so is $\alpha^{-1}(R)$ and*

$$\alpha^{-1}(R)^{\#} \subseteq \alpha^{-1}\left(R^{\#}\right).$$

In particular, $\alpha^{-1}\left(R^{\#}\right)$ is a regularity in A such that $\alpha^{-1}\left(R^{\#}\right)^{\#} = \alpha^{-1}\left(R^{\#}\right)$. Moreover, if R has a proper envelope $R^{\#}$ in B, then $\alpha^{-1}(R)$ has a proper envelope too.

Proof. Take $a, b \in A$. Then $ab \in \alpha^{-1}(R)$ iff $\alpha(ab) = \alpha(a)\alpha(b) \in R$ which in turn is possible (see Definition 4.2) iff both $\alpha(a), \alpha(b) \in R$, that is, $a, b \in \alpha^{-1}(R)$. Further, take $a \in A \setminus \alpha^{-1}\left(R^{\#}\right)$. Then $\alpha(a) \notin R^{\#}$. So, $\psi(\alpha(a)) = 0$ for some $\psi \in B^*$, $\ker(\psi) \cap R = \emptyset$. It follows that $a \in \ker(\varphi)$ and $\ker(\varphi) \cap \alpha^{-1}(R) = \emptyset$, where $\varphi = \psi\alpha$. Thus $a \notin \alpha^{-1}(R)^{\#}$ and therefore $\alpha^{-1}(R)^{\#} \subseteq \alpha^{-1}\left(R^{\#}\right)$.

Using Corollary 4.6, infer that $R^{\#}$ is a regularity in B. Therefore $\alpha^{-1}\left(R^{\#}\right)$ is a regularity in A. Moreover, on account of the inclusion that we have just proven and Lemma 4.1, we deduce that

$$\alpha^{-1}\left(R^{\#}\right) \subseteq \alpha^{-1}\left(R^{\#}\right)^{\#} \subseteq \alpha^{-1}\left(R^{\#\#}\right) = \alpha^{-1}\left(R^{\#}\right),$$

that is, $\alpha^{-1}\left(R^{\#}\right)^{\#} = \alpha^{-1}\left(R^{\#}\right)$.

Finally, if $R^{\#} \neq B$, then $0 \notin R^{\#}$ and therefore $0 \notin \alpha^{-1}\left(R^{\#}\right)$. Since $\alpha^{-1}(R)^{\#} \subseteq \alpha^{-1}\left(R^{\#}\right)$, it follows that $\alpha^{-1}(R)^{\#}$ is proper. □

Corollary 6.4. (see [15, Proposition 3.3]) *If $\alpha : A \to B$ is a bounded algebra homomorphism and R is a regularity in B such that $R = R^{\#}$ is open, then $\alpha^{-1}(R) = \alpha^{-1}(R)^{\#}$ is an open regularity too.*

Now assume that A is a unital Banach algebra such that its attendant Lie algebra $A_{\mathfrak{lie}}$ is nilpotent, and let $\alpha : A \to \mathcal{B}(X)$ be a unital bounded algebra homomorphism, that is, a bounded representation of A on the Banach space X. Put

$$R_{\alpha} = \alpha^{-1}\left(G\left(\mathcal{B}\left(X\right)\right)\right). \tag{6.2}$$

If B is the closure of the inverse closed envelope of the nilpotent Lie algebra $\alpha(A)$ in $\mathcal{B}(X)$, then B is an inverse closed Banach algebra, which is commutative modulo its Jacobson radical thanks to Turovskii's lemma. By Lemma 4.3, the set $G(B)$ is a regularity in B and $G(B) = G(B)^{\#}$. Furthermore, $R_{\alpha} = \alpha^{-1}(B \cap G(\mathcal{B}(X)) = \alpha^{-1}(G(B))$. Using Lemma 6.3 and Corollary 6.4, infer that R_{α} is a regularity in A such that $R_{\alpha} = R_{\alpha}^{\#}$ is an open proper subset in A.

If a is a s-tuple in A, then the Lie subalgebra $\mathfrak{L}(a)$ in $A_{\mathfrak{lie}}$ generated by a is nilpotent. Being a finitely generated nilpotent Lie algebra, $\mathfrak{L}(a)$ has a finite dimension. In particular, if $p(e) \in \mathfrak{F}_k(e)^m$ is a m-tuple of polynomials, then $\mathfrak{L}(p(a))$ is a finite dimensional nilpotent Lie subalgebra in $A_{\mathfrak{lie}}$. Consider a unital bounded representation $\alpha : A \to \mathcal{B}(X)$ and a s-tuple a in A. Then $\pi|_{\mathfrak{L}(a)} : \mathfrak{L}(a) \to \mathcal{B}(X)$ is a Lie representation. Using the Koszul complex generated by the $\mathfrak{L}(a)$-module $\left(X, \pi|_{\mathfrak{L}(a)}\right)$, it is defined a family of Slodkowski spectra

$$\{\sigma_{\pi,k}(a), \sigma_{\delta,k}(a) : k \geq 0\}$$

(see [8]), which are compact subsets in $\mathbb{C}^k$ (see [3]). So, we have a family

$$\mathfrak{S} = \{\sigma_{\pi,k}, \sigma_{\delta,k} : k \geq 0\}$$

of set-valued functions over all tuples in A.

Proposition 6.5. *Each $\tau \in \mathfrak{S}$ is a subspectrum on A.*

Proof. Fix a s-tuple a in A and let $A(\mathfrak{L}(a))$ be the associative subalgebra in A generated by the nilpotent Lie algebra $\mathfrak{L}(a)$. The algebra $A(\mathfrak{L}(a))$ furnished with the finest locally convex topology is dominating over the module $\left(X, \pi|_{\mathfrak{L}(a)}\right)$ in the sense of [8, Definition 4], we write

$$A(\mathfrak{L}(a)) \succ \left(X, \pi|_{\mathfrak{L}(a)}\right).$$

Taking into account that $A(\mathfrak{L}(a))$ is comprising polynomials in elements of $\mathfrak{L}(a)$, we may apply the noncommutative spectral mapping theorems from [8]. As we have confirmed above each tuple $p(a)$ in $A(\mathfrak{L}(a))$ generates a finite-dimensional nilpotent Lie subalgebra $\mathfrak{L}(p(a))$, therefore

$$\tau(p(a)) = p(\tau(a))$$

for all $\tau \in \mathfrak{S}$, due to [8, Propostion 6 and Corollary 8]. Thus τ is a subspectrum on A. □

Take $\tau \in \mathfrak{S}$ and $x \in A$. Then $\tau(x) = \sigma(\alpha(x))$ for all $k > 1$. Therefore

$$\begin{aligned} R_\tau &= \{x \in A : 0 \notin \tau(x)\} = \{x \in A : 0 \notin \sigma(\alpha(x))\} \\ &= \{x \in A : \alpha(x) \in G(\mathcal{B}(X))\} = \alpha^{-1}(G(\mathcal{B}(X))) \\ &= R_\alpha \end{aligned}$$

(see (6.2)). Thus $R_\alpha = R_\tau$ for all $\tau \in \mathfrak{S}$. By Theorem 6.1, $\tau \subseteq \sigma_{R_\alpha}$. Moreover, using Corollary 4.8, infer that $\tau(a) = \{\varphi^{(s)}(a) : \varphi \in \mathcal{K}_\tau\}$. Therefore $\widetilde{\mathcal{K}_\tau} = K_{R_\alpha}$ by (6.1). In particular, we have a chain

$$\mathcal{K}_0 \subseteq \mathcal{K}_1 \subseteq \cdots \subseteq \mathcal{K}_n \subseteq \mathcal{K}_{n+1} \subseteq \cdots$$

of increasing compact subsets in K_{R_α} such that $\widetilde{\mathcal{K}_n} = K_{R_\alpha}$ for all $n \geq 0$, where $\mathcal{K}_n = \mathcal{K}_{\sigma_{\pi,n}}$. The known [16] example by Z. Slodkowski shows that spectra $\sigma_{\pi,n}$ are different for a Hilbert space representation α of a commutative Banach algebra. So, if $\sigma_{\pi,n} \neq \sigma_{\pi,n+1}$, then $\mathcal{K}_n \neq \mathcal{K}_{n+1}$ and $\mathcal{K}_n$ turns out to be a nonempty proper closed subset in K_{R_α} such that $\widetilde{\mathcal{K}_n} = K_{R_\alpha}$ by virtue of Theorem 6.2.

Finally, let us consider the Taylor spectrum σ_T which is defined as

$$\sigma_T(a) = \sigma_{\pi,n}(a)$$

if a is a n-tuple in A. By Proposition 6.5, σ_T is a subspectrum on A, therefore

$$\sigma_T(a) = \left\{\varphi^{(n)}(a) : \varphi \in \mathcal{K}_{\sigma_T}\right\},$$

where $\mathcal{K}_{\sigma_T}$ is a closed subset in K_{R_α} such that $\widetilde{\mathcal{K}_{\sigma_T}} = K_{R_\alpha}$. Moreover, $\mathcal{K}_n \subseteq \mathcal{K}_{\sigma_T}$ for all n. But again $\mathcal{K}_{\sigma_T}$ may be a proper subset of K_{R_α} as shows the example in [2] by R. Berntzen and A. Soltysiak. Namely, there are commuting Banach space operators $a, b \in \mathcal{B}(X)$ such that $\sigma_{G(\mathcal{B}(X))}(a, b)$ is not contained in $\sigma_T(a, b)$. If A is the closed unital associative subalgebra in $\mathcal{B}(X)$ generated by a and b, and α is the identical representation $A \to \mathcal{B}(X)$, then

$$\sigma_{G(\mathcal{B}(X))}(a, b) \subseteq \sigma_{A \cap G(\mathcal{B}(X))}(a, b).$$

Therefore $\sigma_{A \cap G(\mathcal{B}(X))}(a, b)$ is not contained in $\sigma_T(a, b)$. Thus $\mathcal{K}_{\sigma_T}$ is a proper closed subset in K_{R_α} such that $\widetilde{\mathcal{K}_{\sigma_T}} = K_{R_\alpha}$.

We end the paper by proposing an example of a noncommutative Banach algebra A with its nilpotent attendant Lie algebra $A_{\mathfrak{lie}}$. That will demonstrate a gap between commutative and noncommutative cases. For the sake of generality, we consider the case of an Arens-Michael (locally multiplicatively associative) algebra reducing it to a Banach algebra. Fix a Heisenberg algebra $\mathfrak{g}$ with its generators e_1, e_2, e_3 (see Section 2). Thus $[e_1, e_2] = e_3$ and $[e_i, e_3] = 0$ for all i. Let A be an Arens-Michael algebra contained the Heisenberg algebra $\mathfrak{g}$ as a Lie subalgebra (in $A_{\mathfrak{lie}}$) and the associative subalgebra in A generated by $\mathfrak{g}$ is dense in it. If $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, then the canonical embedding $\iota : \mathfrak{g} \to A$ is extended up to a canonical algebra homomorphism $\widetilde{\iota} : \mathcal{U}(\mathfrak{g}) \to A$ with the dense range.

Proposition 6.6. *If* $\ker(\widetilde{\iota}) \neq \{0\}$*, then* e_3 *is a nilpotent element in* A*. In particular,* $A_{\mathfrak{lie}}$ *is a nilpotent Lie algebra.*

Proof. Let $\mathcal{C} = \operatorname{im}(\widetilde{\iota})$. By assumption, $\mathcal{C}$ is a dense subalgebra in A. One can easily verify that the k-th term $\mathcal{C}_{\mathfrak{lie}}^{(k)}$ of the lower central series of the Lie subalgebra $\mathcal{C}_{\mathfrak{lie}}$ is contained in $\mathcal{C}e_3^k$, $k \in \mathbb{N}$. Moreover, $A_{\mathfrak{lie}}^{(k)}$ is included into the closure $\overline{\mathcal{C}_{\mathfrak{lie}}^{(k)}}$. To establish that $A_{\mathfrak{lie}}$ is nilpotent, it suffices to prove that e_3 is nilpotent in A.

Let $\{\|\cdot\|_\nu : \nu \in \Lambda\}$ be a family of multiplicative seminorms on A defining its locally convex topology and let $I_\nu = \{a \in A : \|a\|_\nu = 0\}$. Then I_ν is a two-sided ideal in A and A/I_ν is a normed algebra with respect to the quotient norm induced by $\|\cdot\|_\nu$. Let A_ν be its norm-completion. The family of Banach algebras $\{A_\nu\}$ generates an inverse system and its inverse limit is topologically isomorphic to A [12, 5.2.17]. It is obvious that the associative subalgebra in A_ν generated by the nilpotent Lie subalgebra $\pi_\nu(\mathfrak{g})$ is dense in A_ν, where $\pi_\nu : A \to A_\nu$ is the canonical map, $\nu \in \Lambda$. By Turovskii's lemma, A_ν is commutative modulo its Jacobson radical, thereupon $\pi_\nu(e_3)$ is a quasinilpotent element in A_ν. On that account we conclude that $\sigma(e_3) = \{0\}$, for

$$\sigma(e_3) = \bigcup_{\nu \in \Lambda} \sigma(\pi_\nu(e_3))$$

(see [12, 5.2.12]). To prove that e_3 is nilpotent, one suffices to demonstrate that $q(e_3) = 0$ for a certain nonzero polynomial $q(z)$ of one complex variable z [11, Problem 97].

By assumption $\widetilde{\iota}(p) = 0$ for some nonzero $p \in \mathcal{U}(\mathfrak{g})$. By Poincare-Birkhoff-Witt theorem,

$$p = \sum_{m=0}^{n} p_m(e_1, e_2)\, e_3^m$$

for some polynomials $p_m = p_m(z, w)$ in two complex variables. Assume that $i = \max\{\deg(p_m)\}$, where $\deg(p_m)$ is the degree (maximum of the homogeneous degrees) of p_m. Put $i = \deg(p_{m_1}) = \cdots = \deg(p_{m_s})$ for some m_j, $0 \leq m_1 < \cdots < m_s \leq n$. Consider the polynomial p_{m_1}. Then

$$p_{m_1}(e_1, e_2) = \lambda_{km_1} e_1^k e_2^{i-k} + q_{m_1}(e_1, e_2)$$

such that $\lambda_{km_1} \neq 0$ and $q_{m_1}(z, w)$ is a polynomial without the monomial $z^k w^{i-k}$, for some k. Let λ_{km_j} be the coefficient of $z^k w^{i-k}$ in $p_{m_j}(z, w)$, $2 \leq j \leq s$. So

$p_{m_1}(e_1,e_2) = \lambda_{km_j} e_1^k e_2^{i-k} + q_{m_j}(e_1,e_2)$, where $q_{m_j}(z,w)$ is a polynomial without $z^k w^{i-k}$. Consider a linear operator

$$T_{i,k} = (-1)^k (\operatorname{ad} e_2)^k (\operatorname{ad} e_1)^{i-k} : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}).$$

Note that

$$T_{i,k}\left(e_1^s e_2^t e_3^r\right) = T_{i,k}\left(e_1^s e_2^t\right) e_3^r = \frac{s!t!}{(s-k)!\,(t-i+k)!} e_1^{s-k} e_2^{t-i+k} e_3^{i+r}$$

(here we have assumed that $e_p^{-q} = 0$, $p = 1,2$, $q \in \mathbb{N}$), thereby $T_{i,k}(e_1^s e_2^t e_3^r) \neq 0$ only when $s \geq k$ and $t \geq i-k$. It follows that

$$\begin{aligned} T_{i,k}\left(p_{m_j}(e_1,e_2)\right) &= T_{i,k}\left(\lambda_{km_j} e_1^k e_2^{i-k}\right) + T_{i,k}\left(q_{m_j}(e_1,e_2)\right) = \lambda_{km_j} T_{i,k}\left(e_1^k e_2^{i-k}\right) \\ &= k!\,(i-k)!\lambda_{km_j} e_3^i, \end{aligned}$$

for all j, $1 \leq j \leq s$. Thus

$$T_{i,k}(p) = \sum_{j=1}^{s} T_{i,k}\left(p_{m_j}(e_1,e_2)\right) e_3^{m_j} = k!\,(i-k)! \sum_{j=1}^{s} \lambda_{km_j} e_3^{i+m_j}.$$

We set $q = k!\,(i-k)! \sum_{j=1}^{s} \lambda_{km_j} e_3^{i+m_j}$, which is a nonzero polynomial in $\mathcal{U}(\mathfrak{g})$. Prove that $q(e_3) = 0$ in A. Being a two-sided ideal in $\mathcal{U}(\mathfrak{g})$, the subspace $\ker(\widetilde{\imath}) \subseteq \mathcal{U}(\mathfrak{g})$ is invariant with respect to the operator $T_{i,k}$. With $p \in \ker(\widetilde{\imath})$ in mind, infer that $q = T_{i,k}(p) \in \ker(\widetilde{\imath})$. Therefore $q(e_3) = \widetilde{\imath}(q) = 0$. Thus e_3 is a nilpotent element in A. It follows that $A_{\mathfrak{lie}}$ is a nilpotent Lie algebra. □

The assertion stated in Proposition 6.6 can be proved for arbitrary nilpotent Lie algebra. If A is a closed associative envelope of a finite-dimensional nilpotent Lie algebra $\mathfrak{g}$ and all elements from $[\mathfrak{g},\mathfrak{g}]$ are nilpotent in A, then $A_{\mathfrak{lie}}$ is nilpotent. We omit the details.

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