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A Representation Theorem for Local Operator Spaces

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Abstract. In this note, we generalize Ruan's representation theorem and propose an Arveson–Hanh–Banach–Webster theorem for local operator spaces. Further, we investigate the decomposability of a complete contraction acting from a unital multinormed C^*-algebra to a local operator system into a product of contractions and a unital contractive $*$-representation, and we study injectivity in both local operator space and local operator system contexts.

Key words: local operator space, local operator system, multinormed C^*-algebra.

The known representation theorem [3, Theorem 2.3.5] for operator spaces states that each abstract operator space V can be realized as a subspace of the space $\mathscr{B}(H)$ of all bounded linear operators on a Hilbert space H. By realization we mean a matrix isometry $\Phi\colon V \to \mathscr{B}(H)$ of V onto the subspace $\Phi(V) \subseteq \mathscr{B}(H)$. This result plays a central role in the operator space theory and provides an abstract characterization of a linear space of bounded linear operators on a Hilbert space. Physically well motivated, operator spaces can be thought of as quantized normed spaces, where we have replaced functions with operators, thus regarding classical normed spaces as abstract function spaces. Another motivation is predicted by the domination property observed in a noncommutative functional calculus problem [1, Sec. 4], which suggests that (joint) spectral properties of elements in an operator algebra might be expressed in terms of matrices over the original algebra. The implementation of this proposal would lead to a reasonable joint spectral theory in an operator algebra. To have a more solid justification of quantum physics and noncommutative function theory, it is necessary to consider operator analogs of locally convex spaces, that is, quantizations of locally convex spaces. Namely, one might consider linear spaces of unbounded Hilbert space operators or, more generally, projective limits of operator spaces. In recent years, Effros and Webster [4] started to develop this theory under the name "local operator spaces." A central, subtle result of their investigations is an operator version of the classical bipolar theorem [4, Proposition 4.1].

The goal of the present note is to give an intrinsic description of local operator spaces similar to the above-mentioned characterization of operator spaces. We show that each local operator space can be realized as a subspace of unbounded operators on a Hilbert space. Moreover, if the given local operator space has a bounded locally convex topology, then it can be realized by bounded operators on a Hilbert space. This result generalizes Ruan's representation theorem for operator spaces. To restore the natural connection between local operator spaces and unital multinormed C^*-algebras, as it is in the normed case, we introduce local operator systems motivated by the representation theorem for local operator spaces. The known class of unital multinormed C^*-algebras called Op^*-algebras ([7], [6, Sec. 3.2.3]) presents a bright example of local operator systems. The central role in local operator systems is played by the concept of local positivity. In terms of local positivity, we prove the Arveson–Hahn–Banach–Webster theorem for local operator systems which involves a maximal Fréchet operator C^*-algebra instead of $\mathscr{B}(H)$. Further, we propose a locally convex version of the Stinespring theorem for a mapping between a multinormed C^*-algebra and a local operator system, which in turn invokes a representation theorem for unital multinormed C^*-algebras. Finally, we propose a local operator system version of the known result on injectivity by Choi and Effros [2].

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1. Let X be a linear space, and let seminorms $p^{(n)}\colon M_n(X) \to [0,\infty)$ be given on the $n \times n$-matrix spaces $M_n(X)$ over X. The family $p = \{p^{(n)}\colon n \in \mathbb{N}\}$ is called a *matrix seminorm* [4] on X if p possesses the following properties:

M1. $p^{(m+n)}(v \oplus w) \leqslant \max\{p^{(m)}(v), p^{(n)}(w)\}$;

M2. $p^{(n)}(\alpha v \beta) \leqslant \|\alpha\| p^{(m)}(v) \|\beta\|$

for all $v \in M_m(X)$, $w \in M_n(X)$, $\alpha \in M_{n,m}$, and $\beta \in M_{m,n}$, where $v \oplus w$ is the direct sum of v and w and $M_{n,m} = M_{n,m}(\mathbb{C})$ is the space of all scalar $n \times m$-matrices. A linear space X is called an (*abstract*) *operator space* [3, Sec. 2.1] if X is equipped with a matrix norm. A linear space X equipped with a family of matrix seminorms $\{p_\alpha\colon \alpha \in \Lambda\}$ is called an *abstract local operator space*. We say that a polynormed space X has a *bounded topology* if there exists a bounded barrel in X. In particular, a barrelled polynormed space with bounded topology is normable.

If X is a local operator space with bounded polynormed topology, then we say that X is a *locally bounded operator space*. To a linear mapping $\varphi\colon X \to Y$ between linear spaces X and Y, there corresponds a linear mapping $\varphi^{(n)}\colon M_n(X) \to M_n(Y)$, $\varphi^{(n)}([x_{ij}]) = [\varphi(x_{ij})]$, for each $n \in \mathbb{N}$. Let X and Y be local operator spaces with (saturated) families of matrix seminorms $\{p_\alpha\}_{\alpha\in\Lambda}$ and $\{q_\iota\}_{\iota\in\Omega}$, respectively. A linear mapping $\varphi\colon X \to Y$ is said to be *matrix continuous* if for each $\iota \in \Omega$ there exists an $\alpha \in \Lambda$ and a positive constant $C_{\iota\alpha}$ such that $q_\iota^{(n)} \varphi^{(n)} \leqslant C_{\iota\alpha} p_\alpha^{(n)}$ for all n. If $C_{\iota\alpha} \leqslant 1$ for all possible ι and α, then we say that φ is a *matrix contraction*. Further, an injective map φ is called a *matrix isometry* if $\Omega = \Lambda$ and $q_\alpha^{(n)} \varphi^{(n)} = p_\alpha^{(n)}$ for all α and n.

2. Let us introduce local nets for a subspace of unbounded operators. Fix a Hilbert space H, and let $\mathfrak{L}(H)$ be the set of all unbounded operators on H. Consider a subset $V \subseteq \mathfrak{L}(H)$ whose elements have the same domain E that is a dense subspace in H and satisfy $T(E) \subseteq E$ for all $T \in V$. We say that V is a *subspace* (respectively, an *associative subalgebra*) *of unbounded operators on* H if V is closed with respect to the usual algebraic operations defined for the elements of V. The subspace E is called the *support* of V, and we use the notation $\operatorname{supp}(V)$ for E. If $\operatorname{supp}(V)$ is closed, then V is just a linear subspace (respectively, associative subalgebra) in the space $L(H)$ of all linear transformations on H.

Now let $\mathfrak{p} = \{P_\alpha\}$ be a net of orthoprojections in $\mathfrak{L}(H)$, and let $H_\alpha = \operatorname{im}(P_\alpha)$. The family $\{H_\alpha\}$ is an upward filtered family of closed subspaces in H. Therefore, their union $\bigcup_\alpha H_\alpha$ is a linear subspace in H.

Definition A. Let V be a subspace of unbounded operators on a Hilbert space H, and let $\mathfrak{p} = \{P_\alpha\}$ be a net of orthoprojections in $\mathfrak{L}(H)$ such that $\operatorname{supp}(V) = \bigcup_\alpha H_\alpha$, where $H_\alpha = \operatorname{im}(P_\alpha)$.

We say that $\mathfrak{p}$ is a *local net for* V if $P_\alpha T = T P_\alpha$ (on $\operatorname{supp}(V)$) and $P_\alpha T P_\alpha \in \mathscr{B}(H)$ for all $T \in V$ and $P_\alpha \in \mathfrak{p}$. Further, $\mathfrak{p}$ is called a *weak local net for* V if all subspaces H_α are invariant with respect to V and $V|H_\alpha \subseteq \mathscr{B}(H_\alpha)$ for all α.

Note that in this case the support space $\operatorname{supp}(V)$ can be viewed as the strict inductive limit of the directed Hilbert space family $\mathscr{E} = \{H_\alpha\colon \alpha \in \Lambda\}$; that is, $\operatorname{supp}(V) = \varinjlim \mathscr{E}$ [5, Sec. 4.3]. Moreover, each operator $T \in V$ determines a linear operator $T_0 = T|\operatorname{supp}(V)$ on the support. This operator leaves each subspace H_α invariant and satisfies $T_0|H_\alpha \in \mathscr{B}(H_\alpha)$. Therefore, T_0 is continuous on $\operatorname{supp}(V)$.

Now we fix $n \in \mathbb{N}$ and consider the nth Hilbert space power H^n of H. If V is a subspace of unbounded operators on H, then the matrix space $M_n(V)$ can be identified with a subspace of unbounded operators on H^n by means of the canonical identifications $M_n(V) \subseteq \mathfrak{L}(H^n)$, $n \in \mathbb{N}$. Moreover, $\operatorname{supp}(M_n(V)) = \operatorname{supp}(V)^n$. If $T \in \mathfrak{L}(H)$, then we write $T^{\oplus n} \in \mathfrak{L}(H^n)$ to denote the operator $(x_i)_i \mapsto (T x_i)_i$. Set $\mathfrak{p}^{\oplus n} = \{P_\alpha^{\oplus n}\}$ for an orthoprojection net $\mathfrak{p} = \{P_\alpha\} \subseteq \mathfrak{L}(H)$. Then $\mathfrak{p}^{\oplus n}$ is a (weak) local net for the subspace $M_n(V) \subseteq \mathfrak{L}(H^n)$ whenever so is $\mathfrak{p}$ for V. A *concrete operator space* V is defined as a subspace of $\mathscr{B}(H)$ for some Hilbert space H (see H [3, Sec. 2.1]). The inclusions $M_n(V) \subseteq M_n(\mathscr{B}(H)) = \mathscr{B}(H^n)$, $n \in \mathbb{N}$, determine a matrix norm on V. In the general case, let $V \subseteq \mathfrak{L}(H)$ be a subspace of unbounded operators, and let $\mathfrak{p} = \{P_\alpha\}_{\alpha\in\Lambda}$ be a (weak) local net for V. Fix $\alpha \in \Lambda$. Then each $T \in M_n(V)$ leaves the space $H_\alpha^n (= \operatorname{im}(P_\alpha^{\oplus n}))$ invariant.

Set $p_\alpha^{(n)}(T) = \|T|H_\alpha^n\|$, $\alpha \in \Lambda$. Thus, $\mathfrak{p}^{(n)} = \{p_\alpha^{(n)}\}_{\alpha\in\Lambda}$ is an increasing family of seminorms on $M_n(V)$. Note that if $\operatorname{supp}(V) = H$ and $\mathfrak{p}$ is a local net, then $V \subseteq \mathscr{B}(H)$ and $\mathfrak{p}^{(n)}$ is bounded. Moreover, $\sup \mathfrak{p}^{(n)}$ is reduced to the usual operator norm in $M_n(V) \subseteq \mathscr{B}(H^n)$. Set $p_\alpha = \{p_\alpha^{(n)}\}_{n\in\mathbb{N}}$ and $\mathfrak{P} = \{p_\alpha\}_{\alpha\in\Lambda}$. Then p_α is a matrix seminorm on V. Thus, if a (weak) local net for a subspace $V \subseteq \mathfrak{L}(H)$ is given, then we automatically equip V with the matrix topology generated by the matrix seminorm set $\mathfrak{P}$. In this case, we say that V is a *concrete local operator space*. If $\operatorname{supp}(V) = H$ and $\mathfrak{p}$ is a local net for V, then we have a well-defined matrix norm $p = \sup_\alpha p_\alpha = \sup \mathfrak{P}$ on V, so that V is a locally bounded operator space. This space is called a *concrete locally bounded operator space*. By a *realization of a local* (or *locally bounded*) *operator space V on a Hilbert space H* we mean a mapping $\varphi\colon V \to \mathfrak{L}(H)$ such that $\varphi(V)$ is a concrete local operator space on H and φ is a matrix isometry onto $\varphi(V)$.

Theorem 1. *Let V be a* (*Hausdorff*) *local operator space. Then there exists a realization $\varphi\colon V \to \mathfrak{L}(H)$ of V on a Hilbert space H. Moreover, if V is a locally bounded operator space, then there exists a realization $\varphi\colon V \to \mathscr{B}(H)$ such that $\varphi(V)$ is a concrete locally bounded operator space.*

Note that if, for example, V is a Fréchet local operator space, then the weak local net for the subspace $\varphi(V)$ turns out to be a local net.

3. Let $V \subseteq \mathfrak{L}(H)$ be a concrete local operator space with a local net $\mathfrak{p} = \{P_\alpha\}_{\alpha\in\Lambda}$. Then to each $T \in V$ there corresponds a dual unbounded operator $T' \in \mathfrak{L}(H)$ such that $\operatorname{supp}(V) \subset \operatorname{dom}(T')$ and $T'|H_\alpha = (T|H_\alpha)^*$ for all α. Set $T^* = T'|_{\operatorname{supp}(V)}$. Thus, $V^* = \{T^*\colon T \in V\}$ is a subspace of unbounded operators on H with the same support $\operatorname{supp}(V)$, and $\mathfrak{p}$ is also a local net for V^*.

Definition B. A (concrete) local operator space $V \subseteq \mathfrak{L}(H)$ with a local net $\mathfrak{p}$ is called a *local operator system on H* if $V = V^*$ and $I_H \in V$, where I_H is the identity operator on $\operatorname{supp}(V)$. A subspace $W \subseteq V$ is called an *operator system subspace* if $W = W^*$ and $I_H \in W$. Further, by a local operator C^*-algebra A on H with a net $\mathscr{E} = \{H_\alpha\}_{\alpha\in\Lambda}$ we mean an associative subalgebra $A \subseteq \mathfrak{L}(H)$ that is a complete local operator system on H with a local net $\mathfrak{p}$.

Note that the topological completion $\widetilde{V}$ of a local operator system remains a local operator system on H with the same local net as V has. Moreover, if, in addition, V is an associative subalgebra in $\mathfrak{L}(H)$, then so is $\widetilde{V}$. Further, each local operator C^*-algebra is a unital multinormed C^*-algebra with respect to its local operator system topology. A *Fréchet operator C^*-algebra* is defined as an operator C^*-algebra on a Hilbert space H with a countable net $\mathscr{E} = \{H_n\}_{n\in\mathbb{N}}$. By a *maximal operator C^*-algebra on H with a net $\mathscr{E}$* we mean the set of all unbounded operators $T \in \mathfrak{L}(H)$ such that $\bigcup \mathscr{E} = \operatorname{dom}(T)$, $P_\alpha T \subset TP_\alpha$, and $P_\alpha TP_\alpha \in \mathscr{B}(H)$ for all $\alpha \in \Lambda$. For example, so is $\mathscr{B}(H)$ with the net $\mathscr{E} = \{H\}$.

Let V be a local operator system on H with a local net $\mathfrak{p} = \{P_\alpha\}_{\alpha\in\Lambda}$. An element $T \in V$ is said to be *locally positive* if $T|H_\alpha \geqslant 0$ (in $\mathscr{B}(H_\alpha)$) for some $\alpha \in \Lambda$. Likewise, if A is a unital multinormed C^*-algebra with a fixed defining family of C^*-seminorms $\{p_\alpha\}_{\alpha\in\Lambda}$, then an element $a \in A$ is *locally positive* if $a = b^*b + x$ for some $x \in A$, $p_\alpha(x) = 0$. Let $V \subseteq \mathfrak{L}(H)$ and $W \subseteq \mathfrak{L}(K)$ be local operator systems with local nets $\mathfrak{p} = \{P_\alpha\}_{\alpha\in\Lambda}$ and $\mathfrak{q} = \{Q_\iota\}_{\iota\in\Omega}$, respectively. A linear mapping $\varphi\colon V \to W$ is said to be *locally completely positive* if for each $\iota \in \Omega$ there exists an $\alpha \in \Lambda$ such that $\varphi^{(n)}(v)|K_\iota^n \geqslant 0$ whenever $v \in M_n(V)$, $v|H_\alpha^n \geqslant 0$, and $\varphi^{(n)}(v)|K_\iota^n = 0$ if $v|H_\alpha^n = 0$ for all n, where $H_\alpha = \operatorname{im}(P_\alpha)$ and $K_\iota = \operatorname{im}(Q_\iota)$. We say that $\varphi\colon V \to W$ is a *morphism* if it is unital and locally completely positive. A local operator space (respectively, system) V is said to be *injective* if, for each operator space (respectively, system) subspace $W_0 \subseteq W$ of a local operator space (respectively, system) W, every matrix contraction (respectively, morphism) $\psi\colon W_0 \to V$ can be extended to a matrix contraction (respectively, morphism) $\overline{\psi}\colon W \to V$. By the Arveson–Hahn–Banach theorem [8, Sec. 2.31], $\mathscr{B}(H)$ is an injective local operator space.

Now let us state the Arveson–Hahn–Banach–Webster theorem for local operator systems.

Theorem 2. *Let V be a local operator system on H, let W be an operator system subspace, and let $\mathfrak{B}_{\mathscr{E}}$ be a maximal Fréchet operator C^*-algebra. Then any locally completely positive mapping*

$\varphi\colon W \to \mathfrak{B}_{\mathscr{E}}$ has a locally completely positive extension $\Phi\colon V \to \mathfrak{B}_{\mathscr{E}}$. In particular, $\mathfrak{B}_{\mathscr{E}}$ is an injective local operator system.

Using Theorems 1 and 2, one can prove that for Fréchet operator systems injectivity in the sense of local operator spaces is equivalent to injectivity in the sense of local operator systems.

The locally convex version of the Stinespring theorem has the following form.

Theorem 3. *Let A be a unital multinormed C^*-algebra, let V be a local operator system on a Hilbert space H, and let $\varphi\colon A \to V$ be a matrix contraction. If φ is locally completely positive, then there exists a local operator C^*-algebra B on a Hilbert space K, a contraction $T\colon H \to K$, and a unital contractive $*$-homomorphism $\pi\colon A \to B$ such that $T(\operatorname{supp}(V)) \subseteq \operatorname{supp}(B)$ and*

$$\varphi(a) = T^*\pi(a)T \quad \text{on } \operatorname{supp}(V)$$

for all $a \in A$. Moreover, if φ is unital, then T is an isometry.

Corollary 4. *Let A be a multinormed C^*-algebra. Then A is a local operator C^*-algebra on a Hilbert space up to an isometric $*$-isomorphism.*

Corollary 5. *Let A and B be unital multinormed C^*-algebras, and let $\varphi\colon A \to B$ be a unital linear isomorphism between them. If both φ and φ^{-1} are locally completely positive, then φ is a topological $*$-isomorphism. In particular, if φ is a matrix isometry, then φ is an isometric $*$-isomorphism.*

A locally convex version of Paulsen's off-diagonal trick based on Theorems 2 and 3 permits us to prove the following result.

Theorem 6. *Let A be a unital multinormed C^*-algebra, let $\mathfrak{B}_{\mathscr{E}}$ be a maximal Fréchet operator C^*-algebra on a Hilbert space H, and let $\varphi\colon A \to \mathfrak{B}_{\mathscr{E}}$ be a matrix contraction. There exists a unital contractive $*$-homomorphism $\pi\colon A \to D$ of A into an operator C^*-algebra D on a Hilbert space X and contractions $H \xrightarrow{T} X \xrightarrow{S} H$ such that $T(\operatorname{supp}(B)) \subseteq \operatorname{supp}(D)$ and*

$$\varphi(a) = S\pi(a)T \quad \text{on } \operatorname{supp}(\mathfrak{B}_{\mathscr{E}}),$$

for all $a \in A$.

Now assume that V is an injective local operator system on H with a local net $\mathfrak{p} = \{P_\alpha\}_{\alpha\in\Lambda}$, and let $H_\alpha = \operatorname{im}(P_\alpha)$, $\alpha \in \Lambda$. Then V is an operator system subspace of a maximal local operator C^*-algebra $\mathfrak{B}_{\mathscr{E}}$ on the same space H and with the net $\mathscr{E} = \{H_\alpha\}_{\alpha\in\Lambda}$. The matrix topology on V (respectively, on $\mathfrak{B}_{\mathscr{E}}$) is determined by the seminorm family $\{p_\alpha\}_{\alpha\in\Lambda}$, where $p_\alpha^{(1)}(b) = \|b \,|\, H_\alpha\|_{\mathfrak{B}_{\mathscr{E}}(H_\alpha)}$, $b \in \mathfrak{B}_{\mathscr{E}}$. Note that $\operatorname{supp}(V) = \operatorname{supp}(\mathfrak{B}_{\mathscr{E}}) = \bigcup_{\alpha\in\Lambda} H_\alpha$. The identity mapping $V \to V$ has an extension to a morphism-projection $\Phi\colon \mathfrak{B}_{\mathscr{E}} \to \mathfrak{B}_{\mathscr{E}}$ onto V. For $T, S \in V$, we set $T \cdot S = \Phi(TS)$. The space V equipped with this multiplication proves to be a $\widehat{\otimes}^*$-algebra (that is, a locally convex $*$-algebra with jointly continuous multiplication). More precisely, for each $\alpha \in \Lambda$ there exists a $\beta \in \Lambda$ such that $\alpha \leqslant \beta$, $p_\alpha^{(n)}(a) \leqslant \sqrt{p_\alpha^{(n)}(a^* \cdot a)} \leqslant p_\beta^{(n)}(a)$, and $p_\alpha^{(n)}(a \cdot b) \leqslant p_\beta^{(n)}(a)p_\beta^{(n)}(b)$ for all $a, b \in M_n(V)$. It readily follows from these inequalities that the $\widehat{\otimes}^*$-algebra structure on V is a C^*-algebra structure provided that V is an operator system (in this case, $p_\alpha = p_\beta$ for all α, β) [2]. On the basis of Theorem 3 and Corollary 5, we arrive at the following assertion in the local operator system case.

Theorem 7. *Let V be an injective local operator system. Then V with its $\widehat{\otimes}^*$-algebra structure proves to be a unital multinormed C^*-algebra. This is the unique multiplication on V for which V, together with the given $*$-operation and matrix topology, is a local operator C^*-algebra. The local operator system structure induced by the multinormed C^*-algebra V is reduced to the original one.*

References

[1] A. A. Dosiev, J. Funct. Anal., **230** (2006), 446–493.

[2] M. D. Choi and E. G. Efros, J. Funct. Anal., **24**:2 (1977), 156–209.

[3] E. G. Effros and Z.-J. Ruan, Operator Spaces, London Math. Soc., Clarendon Press, Oxford, 2000.
[4] E. G. Effros and C. Webster, in: Operator Algebras and Applications (Samos 1996), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 495, Kluwer, Dordrecht, 1997, 163–207.
[5] A. Grothendieck, Topological Vector Spaces, Notes on Math. and Its Appl., Gordon and Breach, London, 1972.
[6] A. Ya. Helemskii, Banach and Polynormed Algebras: General Theory, Representations, Homology [in Russian], Nauka, Moscow, 1989.
[7] G. Lassner, Math. Nachr., **52** (1972), 161–166.
[8] C. Webster, Local Operator Spaces and Applications, Ph. D., University of California, Los Angeles, 1997.

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