An Introduction to Potential Theory on Calibrated Manifolds (I. Unal) ①

Def: $f : X \to \mathbb{R}$ (X Riemannian mfld) w/

$\Delta f \geq 0$ then f is "subharmonic"

$\Delta f = 0$ then f is "harmonic"

$\Delta f \leq 0$ then f is "super-harmonic"

Subharmonic means

$$f(a) \leq \frac{1}{\mathrm{vol}(B(a,r))} \int_{B(a,r)} f \, dV$$

→ "Value of f is always less than average value nearby"

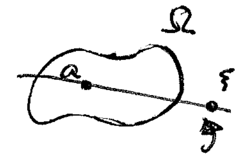
~~around~~ ⟹ Max values always occur on boundaries.

Def: $f : \Omega \subset \mathbb{C}^n \to \mathbb{R} \cup \{-\infty\}$ is plurisubharmonic (psh) if

(1) f upper semi-cont $f^{-1}(\{-\infty, c\})$ open in Ω

(2) for every complex line L

$f|_{\Omega \cap L}$ is subharmonic



$L = \{a + z\xi\}$ w/ $z \in \mathbb{C}$ }

If $f \in C^2(\Omega)$ is psh

$$\frac{\partial^2 f}{\partial z \partial \bar{z}}(a + z\xi) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} \xi_i \bar{\xi}_j \geq 0$$

$$\xi \cdot \underbrace{\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}}_{\text{Must be positive semi-def matrix}} \xi^* \geq 0$$

(→ Note: not all psh f are in C^2)

Properties: Let f, g be psh

① cf is psh for $c > 0$

② $g + f$ is psh

③ $\psi \circ f$ is psh if ψ is convex & $\psi' > 0$ ($\psi'' > 0$)

On a $\mathbb{C}$-mfld M:

There is conjugation map
$$J: TM \longrightarrow TM, \qquad J^2 = -Id$$

$\leadsto$ conjugate differential
$$d^c\phi = -J \circ d\phi$$

$$\begin{cases} d = \partial + \bar\partial \\ d^c = i(\bar\partial - \partial) \\ dd^c = 2i\,\partial\bar\partial \end{cases}$$

$\to$ In local coords: $\displaystyle dd^c f = 2i\partial\bar\partial f = \sum \frac{\partial^2 f}{\partial z_i \partial \bar z_j} dz_i \wedge d\bar z_j$ — (1,1) form

If $2i\partial\bar\partial f(v, \bar v) \geq 0$ all $v \in T^{(1,0)}M$ we say f is semi-positive def.

$\leadsto$ i.e. $dd^c f(v, Jv) \geq 0$ $\quad v \in TM$ $\qquad$ (psh!)

Note $v \wedge Jv$ is a $\mathbb{C}$-line

On a Kähler mfld M:

M $\mathbb{C}$-mfld w/ Hermetian metric $h = g + i\omega$ — (1,1) form

h Kähler $\iff$ ω is closed $\quad (d\omega = 0)$

Thm: Every Kähler mfld is calibrated w/ calibration ω.

Calibrated mfld is (X, ϕ) $\quad \begin{cases} X & \text{Riemannian} \\ \phi & p\text{-form} \end{cases}$

comass(ϕ) $\sup \{ \langle \phi, \xi_x \rangle$ w/ $\xi_x = v_1 \wedge v_2 \wedge \dots \wedge v_p$ w/ $\{v_i\}$ orthonormal a simple p-vector of $T_x M \}$

Def: ϕ a calibration if comass$(\phi) = 1$ & $d\phi = 0$.

$$\Longrightarrow \phi|_{\xi_x} \leq \mathrm{vol}|_{\xi_x} = 1$$

Def: If $\phi|_{\xi_x} = 1$ then ξ_x is called a "calibrated plane"

- $G(\phi) = \bigcup G(\phi)_x$ — collection of all calibrated planes
- $G(\phi)_x = \{ \xi_x \text{ w/ } \phi(\xi_x) = 1 \}$ — collection of calibrated planes at x

Def: $N^p \subset (M,\phi)$ is a calibrated submanifold if

$$\left\{ \text{unit tangent vectors of } N \right\} \subset \text{contact set of } M = \left\{ \text{calibrated planes of } M \right\}$$

Thm: Every calibrated submanifold is volume minimizing in its homology class.

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psh on calibrated mflds

(M,ϕ) calibrated w/ $G(\phi) =$ contact set

$$\begin{array}{ccccc} C^\infty(M) & \xrightarrow{d^\phi} & \mathcal{E}^{p-1}(M) & \xrightarrow{d} & \mathcal{E}^p(M) \\ f & \longmapsto & \phi(\nabla f, \underbrace{\ldots}_{\text{to be filled}}) & \longmapsto & d\phi(\nabla f, \ldots) \end{array}$$

needs: ϕ parallel $\Updownarrow$ $(\nabla\phi = 0)$

Def: f is ϕ-psh if $d^\phi d^\phi f(\Gamma) \geq 0$ for every Γ calibrated plane

Kähler case

ω is the calibration

$$\omega(X,Y) = g(JX, Y) \leq |JX|\cdot|Y|$$
$$|JX| = |X|\cdot|Y| \leq 1 \quad \text{if} \quad |X|, |Y| \leq 1$$

$\rightsquigarrow$ equality if $JX \parallel Y \Longrightarrow X \wedge JX$ is $\mathbb{C}$-line.

$$G(\omega) = \{\mathbb{C}\text{-lines}\}$$

.. ~~calibrated planes are tangent space of psh functions~~

.. psh definition from before matches ω-psh.