① Ibrahim Unal

"Calibrated Manifolds"

Def: (M^n, g) Riemannian mfld is calibrated if
$\exists$ p-form ϕ w/ ① Closed $d\phi = 0$
② $\phi|_\xi \leq \mathrm{vol}|_\xi$ w/ ξ p-dim'l subsp of TM

ϕ is called a "calibration"

$\Leftrightarrow$ so $\phi(e_1 \ldots e_p) \leq 1$ for $e_1, \ldots, e_p$ orthonormal

→ If $\phi|_\xi = \mathrm{vol}|_\xi$ then ξ is called a "calibrated plane"

Note: $\phi : Gr(p, n) \longrightarrow [-1, 1]$

$\Longrightarrow$ $\phi^{-1}(+1) =: \{$calibrated planes$\}$

Def: $N^p \subset M^n$ is a calibrated submanifold if $\phi_N = \mathrm{vol}|_N$

Ex: $M = \mathbb{R}^2$
$\phi = dx$

① $d(dx) = 0$ Good

② $dx\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) = a \leq 1$ for $a^2 + b^2 = 1$.

Calibrated planes: $\frac{\partial}{\partial x}$. Calibrated submanifolds: collections of horiz. line seg.

Foundations: Harvey & Lawson '82 ?

Fundamental Lemma: Let (M, ϕ) be a calibrated mfld. and $N \subset M$ calibrated submfld (closed).
Then N is volume minimizing in its homology class!!

Ex: (From Kähler geometry)

(M^n, J, g, ω) $\mathbb{C}$ - Riemannian mfld
J conjugation on TM
g metric
ω Kähler form

Kähler mfld if ω closed.
$\Updownarrow$
$\nabla J = 0$

Notes: $\omega(X, Y) = g(JX, Y)$

ω is a calibration b/c ① $d\omega = 0$ (if Kähler)
② $\omega(X, Y) = g(JX, Y)$
$\leq \|JX\| \cdot \|Y\|$
$= \|X\| \cdot \|Y\|$ $= 1$ for orthonormal basis.

Equality iff $JX \parallel Y$

$\Longrightarrow J : \xi \to \xi$ i.e. ξ is a $\mathbb{C}$-line!!!

(2) Relation between special holonomy and calibrated manifolds

Recall: M Riemannian mfld w/ ϕ p-form on M
① closed
② $\phi|_\xi \leq vol|_\xi \;\; \forall$ subsp $\xi \subset T$

Recall: Connection is $\nabla : C^\infty(TM) \times C^\infty(TM) \longrightarrow C^\infty(TM)$

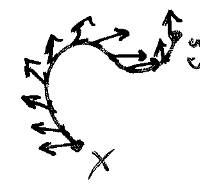
$$(X, Y) \longmapsto \nabla_X Y$$

w/ $\nabla_{fX} gY \underset{p}{=} f \, \nabla_X (gY) \underset{g}{=} f\big((X \cdot g) Y + g \nabla_X Y\big)$

"tensorial" "Leibniz"

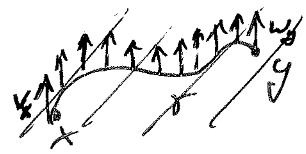
$\longrightarrow$ "Differentiate Y in the direction of X"

- Riemannian connection is the unique connection w/
  1) $\nabla g = 0$
  2) $\nabla_X Y - \nabla_Y X = [X, Y]$



- Holonomy

$x, y \in M$ w/ path between them



$\Longrightarrow$ Parallel transport

$$P_\gamma : T_x M \to T_y M$$
$$v_x \longmapsto w_y$$

w_y is value of unique v.f. V w/ $\begin{cases} \nabla_{\dot\gamma} V = 0 \text{ at } \gamma \\ V(0) = v_x \end{cases}$

Now consider γ loop in M 1-ctd.

$$P_\gamma : T_x M \longrightarrow T_x M$$

"Holonomy Group" of M

$\rightsquigarrow$ Invertible $\quad P_\gamma^{-1} = P_{\gamma^{-1}}$
$\rightsquigarrow$ Product $\quad P_\alpha \circ P_\beta = P_{\alpha \cdot \beta}$
} Group structure

Subgroup of $GL(TM)$ (in fact, $O(TM)$)

M orientable then subgroup of $SO(TM)$

Classification of Holonomy Groups:

(Berger '50s)

M "non-symmetric" 1-ctd Riem. mfld. then

	Holonomy	dimension	Type of manifold
$\mathbb{R}$	$SO(n)$	n	orientable
$\mathbb{C}$	$U(n)$	$2n$	Kähler
	$SU(n)$	$2n$	Calabi-Yau
$\mathbb{H}$	$Sp(n) \times Sp(1)$	$4n$	Quaternion-Kähler
	$Sp(n)$	$4n$	Hyper Kähler
$\mathbb{O}$	G_2	7	G_2-manifolds
	$Spin(7)$	8	$Spin(7)$-manifolds

Special Holonomy

all of these come w/ forms → calibrations!!!

$U(n) \rightsquigarrow$ Kähler form ω w/ $\frac{\omega^p}{p!}$ are calibrations

$\mathbb{C}$ curves a/ $\mathbb{C}$ submflds are calibrated submflds.

$SU(n) \rightsquigarrow$ Calabi-Yau $\Omega = dz_1 \wedge \cdots \wedge dz_n$ then $\mathrm{Re}(\Omega)$ is a calibration.

"Special Lagrangian" submflds are calibrated submflds.

$\omega|_S = 0$

w/ Im part $= 0$

$Sp(n) \rightsquigarrow$ Hyper Kähler w/ multiple $\mathbb{C}$-structures each w/ Kähler forms

$\cancel{\omega^p}, \omega_I, \omega_J, \omega_k$

$\frac{\omega_I^p}{p!}, \frac{\omega_J^p}{p!}, \frac{\omega_k^p}{p!}$

$\left(\frac{\omega_I^p}{p!} + \frac{\omega_J^p}{p!} + \frac{\omega_k^p}{p!}\right)\frac{1}{3}$

Quaternion lines are calibrated ~~submflds~~ planes

$G_2 \rightsquigarrow G_2$-mflds w/ ϕ "associative" 3-form

Hodge $*\phi = \psi$ is calibration.

$Spin(7) \rightsquigarrow \Psi$ Cayley 4-form is calibration.

(4)

In a Kähler manifold

Def: f is pluri-subharmonic if

$$\frac{i}{2}\frac{\partial^2 f}{\partial z^i \partial \bar{z}^j}\, dz^i \wedge d\bar{z}^j \Big|_{\xi} \geq 0$$

ξ complex line

Idea:

$$\underset{\substack{\vee\!| \\ 0}}{g(u,u)} = g(Ju,Ju) = \omega(\underbrace{u,Ju}_{\text{complex line}})$$

$\rightsquigarrow$ What kind of function has properties like ω??

Generalization: $\mathbb{C}$-line $\longmapsto$ Calibrated ~~submanifolds~~ planes

need a $(1,1)$ form to replace $\frac{i}{2}\frac{\partial^2 f}{\partial z^i \partial \bar{z}^j}\, dz^i \wedge d\bar{z}^j$

Suppose (X,ϕ) is a calibrated manifold w/ $\nabla\phi = 0$, ϕ a p-form

(write $\mathcal{E}^n(X)$ for smooth n-forms)

$$dd^\phi : \mathcal{E}^0(X) \xrightarrow{d^\phi} \mathcal{E}^{p-1}(X) \xrightarrow{d} \mathcal{E}^p(X)$$

Notation: $\nabla f \lrcorner \phi$

$$f \longmapsto \phi(\nabla f, \ldots) \longmapsto d(\nabla f \lrcorner \phi)$$

(Previous Kähler ex is $f \mapsto \partial\bar{\partial} f$
$= dd^c f$

$d = \partial + \bar{\partial}$
$d^c = \partial - \bar{\partial}$ } approx...)

Def: If $dd^\phi f\,|_\xi \geq 0$ all calibrated planes ξ
then f is called ***ϕ-plurisubharmonic***

Note: In Kähler case $\mathbb{C}$-curves are calibrated submanifolds;
also f pluri-subharmonic $\Longleftrightarrow$ $f|_\xi$ subharmonic
ξ $\mathbb{C}$-curves

Thm: If f is ϕ-plurisubharmonic then $f\big|_{\text{Calibrated Submfld}}$ is subharmonic
(i.e. $\Delta f \geq 0$)

⑤

Def: A Stein manifold (aka. "pseudoconvex") is a $\mathbb{C}$-submfld of $\mathbb{C}^N$

(**Note:** by Louisville thm Stein manifolds are noncompact.)

$\iff \exists$ a plurisubharmonic <u>exhaustion function</u> on M

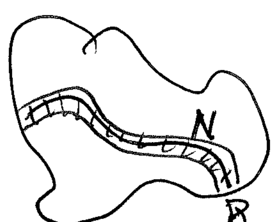
> **Def:** M is ϕ-convex if there is a ϕ-plurisubharmonic <u>exhaustion function</u> on M

Thm (Andreotti–Frankel): If M^n is Stein then $H_k(M) = 0$ if $k > n$

Def: The free dimension $fd(\phi) = \left(\begin{array}{l}\text{max dimension of a} \\ \text{subspace w/ no calibrated planes.}\end{array}\right)$

> **Thm** M is ϕ-convex then $H_k(M) = 0$ for $k > fd(\phi)$.

<u>Ibrahim's Research</u>



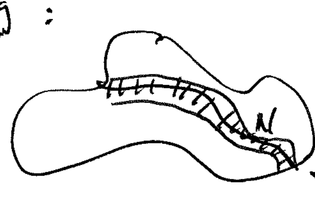
M^n $\mathbb{C}$-mfld

totally real submfld
(tangent planes don't contain any real lines)

$J T_x N \cap T_x N = \{0\}$

> **Thm** N totally real submfld $\Rightarrow \exists$ nbhood (open) that is stein

<u>Analog</u>:



M calibrated mfld

(totally) ϕ-free
(tangent planes contain no calibrated planes)

> **Thm** N ϕ-free $\Rightarrow \exists$ nbhood that is ϕ-convex

(6) Ibrahim: When can a mfld be a ϕ-free submfld of another? Obstructions?