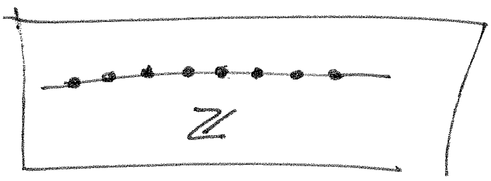
Def: Let $S \subset \mathbb{Z}^n$ be a semi-group.
Then $G(S)$ is the smallest subgroup of $\mathbb{Z}^n$ containing S.

"Group generated by S"

Thm:
- If X is n-dim'l affine toric variety, then S_X is a saturated, finitely generated subsemigroup of $\mathbb{Z}^n$ w/ $G(S_X) = \mathbb{Z}^n$.
- If S is a saturated, f.g subsemigroup of $\mathbb{Z}^n$ w/ $G(S) = \mathbb{Z}^n$, then $\exists\, X$ (unique up to toric isom) w/ $S_X = S$.

EX
- Saturated subsemigroups of $\mathbb{Z}$

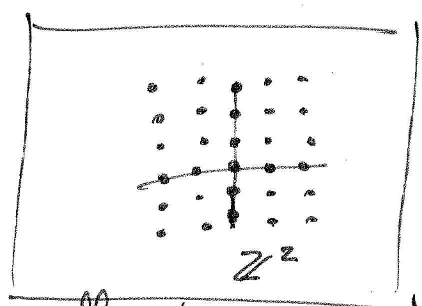


$\left.\begin{array}{l}\mathbb{Z}_{\geq 0}\\ \mathbb{Z}_{\leq 0}\end{array}\right\} \longrightarrow$ affine line $\mathbb{A}^1$

~~$\{0\}$~~

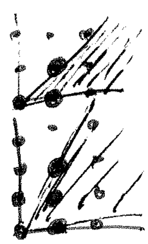
$\mathbb{Z} \longrightarrow$ torus $\mathbb{A}^1 \setminus \{0\}$

- Saturated subsemigroups of $\mathbb{Z}^2$



$\mathbb{Z}^2 \longrightarrow$ torus $(k^*)^2$

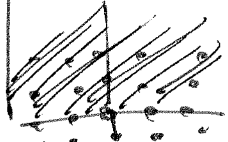
$(\mathbb{Z}_{\geq 0})^2 \longrightarrow$ affine plane $\mathbb{A}^2$



$k[S_\tau] = k[x, x y] \cong k[x, y] \longrightarrow$ affine plane

$k[S_\tau] = k[x, xy, xy^2]$

$= k[x,y,z]/(y^2 - xz)$



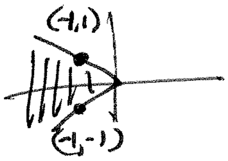
$\mathbb{Z} \times \mathbb{Z}_{\geq 0} \rightsquigarrow k^* \times k$

(2) **Remark**: It was enough to look at the cone instead of just the semi-group.

Def: Let $M = \mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$.

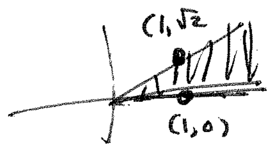
- The <u>cone generated by</u> $\underline{\bar{v}_1, \dots, \bar{v}_m}$ is $\sigma = \{c_1\bar{v}_1 + \dots + c_m v_m \text{ w/ } c_i \geq 0, c_i \in \mathbb{R}\}$
  (non-negative hull of $\bar{v}_1, \dots, \bar{v}_m$)
- If σ can be generated by vectors w/ entries in $\mathbb{Z}$ it is called a <u>lattice cone</u>

EX



lattice cone

EX



not a lattice cone

Gordon's Lemma: σ is a <u>lattice cone</u> $\Longleftrightarrow$ $\sigma \cap \mathbb{Z}^n$ is <u>finitely gen. semigroup</u>

(n-dim'l toric varieties X) $\longleftrightarrow$ (f.g saturated semigroups $S \subset \mathbb{Z}^n$ w/ $G(S) = \mathbb{Z}^n$) $\longleftrightarrow$ (n-dim'l lattice cone σ)

Fig 1. Story so far.

Singularities

Thm: Let X be n-dim'l toric variety, $\sigma_X \subset \mathbb{R}^n$ corresp. lattice cone.

Suppose $\bar{v} \in \sigma_X, -\bar{v} \in \sigma_X \Rightarrow \bar{v} = \bar{0}$. ("strictly convex" — no lines)

$\boxed{X \text{ nonsingular}} \Longleftrightarrow \boxed{\sigma_X \cap \mathbb{Z}^n \text{ is generated by } n \text{ lattice vectors}}$ ← Must be on boundary.

(**Lemma**: Furthermore, $X \cong \mathbb{A}^n$.)