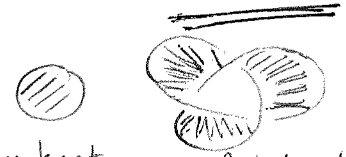
1 Ozgur (Knots: lecture 2) 1

Seifert Surfaces

- Given a knot / link is there a surface embedded in $\mathbb{R}^3$ w/ this as boundary?

$$\partial S = K$$

EX:



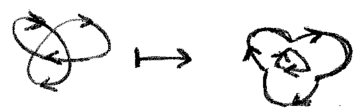
Note: This is not orientable...

Def: A Seifert Surface for a knot K is an orientable surface S w/ $\partial S = K$

Remark: Not unique.

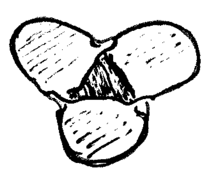
- Seifert's algorithm
  1. Choose an orientation for knot
  2. Follow along knot switching to follow orientation at crossings

EX



  3. Fill each Seifert circle w/ disk (nested disks go over/under)
  3. Insert twisted band at crossings

EX:



Recall: Genus g surface w/ boundary $\sim S^1$ has ~~genus~~ $\chi = 1-2g$

Def: Genus of a knot is minimal genus of Seifert surface

Remark: Unfortunately, sometimes no projection / diagram will have Siefert's algorithm giving surface of minimal genus.

Note: For a given diagram, $\chi = \#\text{circles} - \#\text{crossings}$

2) **Thm**: $g(J \# K) = g(J) + g(K)$

- **Invariants from "nowhere"**

  Knots $\longrightarrow$ polynomials

  $\|$

  Projections / Reidermeister Moves

* Alexander Polynomial
- Jones Polynomial
- HOMFLY Polynomial

} Projections $\longrightarrow$ polynom | invariant under Reidermeister Moves.

Jones Polynomial (1984)

- Kaufmann bracket: $\langle K \rangle$ ← Not invariant under Reidermeister Moves

  Rule 1: $\langle O \rangle = 1$

  (Note: $\langle 8 \rangle \neq 1$)

  Rule 2: $\langle \times \rangle = A \langle)(\rangle + B \langle \asymp \rangle$

  $\langle \times \rangle = A \langle \asymp \rangle + B \langle)(\rangle$

  Rule 3: $\langle \text{link} \cup O \rangle = C \langle \text{link} \rangle$

- Making $\langle\ \rangle$ invariant under Reidemeister 2 & 3 moves places requirements on A, B, C:

$$\langle \rangle = A \langle \rangle + B \langle \rangle$$

$$= A^2 \langle \rangle + AB \langle \rangle + AB \langle \rangle + B^2 \langle \rangle$$

$$= AB \langle)(\rangle + (A^2 + ABC + B^2) \langle \asymp \rangle$$

Reidemeister Move #2 $= \langle)(\rangle$

$\Longrightarrow AB = 1 \qquad B = A^{-1}$

$\Rightarrow C = (A^2 + \ldots$

③ So $\langle\ \rangle$ invariant under Reidemeister 2 $\Rightarrow$

$$B = A^{-1} \quad \text{and} \quad C = -(A^2 + A^{-2})$$

Reidemeister #3:

$$\langle\ \rangle = A\langle\ \rangle + A^{-1}\langle\ \rangle$$

Reid #2

$$= A\langle\ \rangle + A^{-1}\langle\ \rangle$$

by symmetry $= \langle\ \rangle$ OK.

Reidemeister #1:

$$\langle\ \rangle = A\langle\ \rangle + B\langle\ \rangle$$

$$= (AC + B)\langle\ \rangle$$

$$A(-A^2 - A^{-2}) + A^{-1} = -A^3 - A^{-1} + A^{-1}$$
$$= -A^3$$

$$\langle\ \rangle = \dots = (-A^{-3})\langle\ \rangle$$

Not invariant....

Idea: Find another almost-invariant w/ same defect. Add them.

- Writhe: Orient a link.
  - $+1$
  - -1

  add these over all crossings.

R1: $w(\ ) = +1$ $\quad w(\ ) = 0$

$w(\ ) = -1$

R2: $w(\ ) = 0$ $\quad w(\ ) = 0$

R3: $w(\ ) = 1$

$w(\ ) = 1$

R3 doesn't change any local crossings.

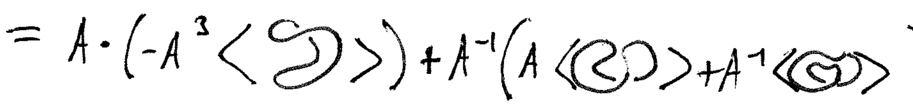
Def: The Jones polynomial for a knot/~~link~~ link is

$$X(L) = (-A^3)^{-w(L)} \langle L \rangle$$

Lemma: If L and L' are mirror images then $X(L)(A) = X(L')(A^{-1})$

EX: Kaufmann Bracket of trefoil

Reidemeister #1



$$= A\cdot(-A^3)(-A^3) + A^{-1}\cdot A\cdot(-A^3) + A^{-1}\cdot A^{-1}(-A^{-3})$$

$$= A^7 - A^3 - A^{-5}$$

Alexander Polynomial

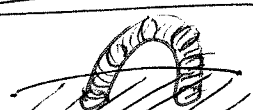
"Knot Theory and the Alexander Polynomial" Bachelor thesis of R.T. McNeill (Smith College) } good notes

→ Approach #1: Seifert Surfaces

Recall: A Seifert surface for a knot K is orientable surface S w/ $\partial S = K$ } not unique

((Idea: Given Seifert Surfaces S_1, S_2 for K ~~∃ standard operations to move from one to another.~~))

Basic Moves: • Tubing:



• Compressing:

Fact: Any two Seifert surfaces for the same knot are connected by a series of "tubing" & "compressing" operations

→ Invariants under "tubing" & "compressing":

Not $H_1(S)$, but maybe we can get something from this, still.

Seifert Graph & Seifert Matrix

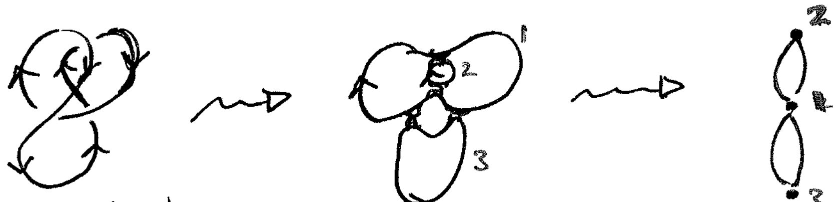


figure 8 knot

Seifert Circles w/ bands for Seifert Surface

Seifert graph

- vertex for each Seifert circle
- edge for each band connecting.

We can read off $H_1(S)$ from Seifert graph } → Basis is edges of graph which aren't in a spanning tree.

Seifert matrix is matrix of size $h_1 \times h_1$ where $h_1 = \dim (H_1)$

consider loops generating H_1 ⇝ coming from Seifert graph

(6) Thicken seifert surface by ε

$$S \rightsquigarrow S \times [0, \varepsilon]$$

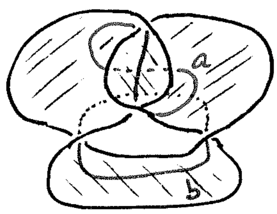
loops on surface $\rightsquigarrow$ loops on $S \times \{0\}$ a, b

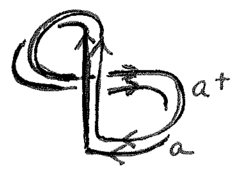
and

parallel transports on $S \times \{\varepsilon\}$ a^+, b^+

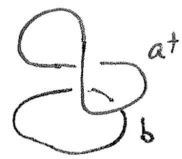
Seifert matrix: $(S_{ij}) = \text{lnk}\#(\ell_i, \ell_j^+)$ (w/ sign)

Note: Not symmetric.

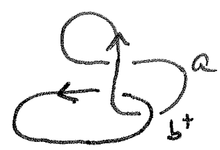




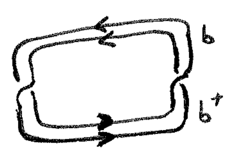
$\text{lnk}(a, a^+) = -1$



$\text{lnk}(b, a^+) = 0$



$\text{lnk}(a, b^+) = 1$



$\text{lnk}(b, b^+) = 1$

$$\boxed{\text{Seifert Matrix } (S_{ij}) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}}$$

Invariants from M:

~~(1) |det M|~~
~~(2) Signature~~

(1) Signature: $\sigma(M + M^T)$

(2) Determinant: $|\det(M + M^T)|$

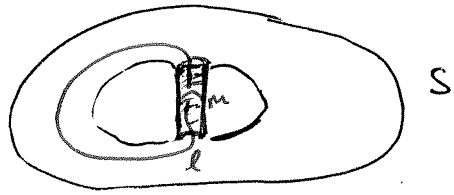
(3) Alexander Polynomial:

$$\Delta(K) = \det\left(t^{1/2} M - t^{-1/2} M^T\right)$$

⑦ **Claim:** ①, ②, ③ are invariant under "tubing" & "compression"

Tubing (w/ good choice of basis)

$$M \longmapsto \left[\begin{array}{c|cc} M & * & 0 \\ & \vdots & \vdots \\ & * & 0 \\ \hline 0 \cdots 0 & 0 & 1 \\ 0 \cdots 0 & 0 & 0 \end{array}\right]$$



adding tube adds two new homology generators m,

$\mathrm{lnk}(m, m^+) = 0$
$\mathrm{lnk}(m, \ell^+) = 0$
$\mathrm{lnk}(\ell, m^+) = 1$
$\mathrm{lnk}(a_i, m^+) = 0 = \mathrm{lnk}(m, a_i^+)$

→ If $\mathrm{lnk}(\ell, \ell^+) = \lambda$, replace ℓ by $(\ell - \lambda m)$:

- does not change linking #s above.
- now $\mathrm{lnk}(\ell-\lambda m, (\ell-\lambda m)^+) = \left.\begin{array}{l} \mathrm{lnk}(\ell,\ell^+) \leftarrow \lambda \\ - \lambda\,\mathrm{lnk}(m,\ell^+) \leftarrow 0 \\ - \lambda\,\mathrm{lnk}(\ell, m^+) \leftarrow -\lambda \\ + \lambda^2\,\mathrm{lnk}(m,m^+) \leftarrow 0 \end{array}\right\} = 0$

→ If $\mathrm{lnk}(a_i, \ell^+) = \lambda_i$, replace a_i by $a_i - \lambda_i m$

- does not change $\mathrm{lnk}(a_i, a_i^+)$
- now $\mathrm{lnk}(\ell, a_i^+) = 0$
  $\mathrm{lnk}(a_i^+, \ell) = 0$
  like above.

Compressing (w/ good choice of basis) is reverse

$$\left[\begin{array}{c|cc} M & * & 0 \\ & \vdots & \vdots \\ & * & 0 \\ \hline 0 \cdots 0 & 0 & 1 \\ 0 \cdots 0 & 0 & 0 \end{array}\right] \longrightarrow M$$

→ ① & ② invariant is obvious.

③ $$\det\left[\begin{array}{c|cc} t^{1/2}M - t^{-1/2}M^T & t^{1/2}* & 0 \\ & \vdots & \vdots \\ & t^{1/2}* & 0 \\ \hline -t^{-1/2}* \;\cdots\; -t^{-1/2}* & 0 & t^{1/2} \\ 0 \;\cdots\; 0 & -t^{-1/2} & 0 \end{array}\right] \xrightarrow{\text{type 3 row/col op}} \det\left[\begin{array}{c|cc} t^{1/2}M - t^{-1/2}M^T & 0 & 0 \\ & \vdots & \vdots \\ & 0 & 0 \\ \hline 0 \cdots 0 & 0 & t^{1/2} \\ 0 \cdots 0 & -t^{-1/2} & 0 \end{array}\right]$$

$\det(t^{1/2}M - t^{-1/2}M^T)$

Ex: Alexander Polynomial for figure 8 knot:

$$\det\begin{pmatrix} -t^{1/2}+t^{-1/2} & t^{1/2} \\ -t^{-1/2} & t^{1/2}-t^{-1/2} \end{pmatrix} = -t+2-t^{-1}+1 = \boxed{-t+3-}$$

Ex Alexander Polynomial for trefoil knot:

$$\boxed{t-1+t^{-1}}$$ ← Also mirror of trefoil.

Lemma: ① Alexander polynomial is invariant under $t \mapsto t^{-1}$

② Alexander polynomial of mirror image knot is $t \mapsto t^{-1}$

Cor: Alexander polynomial cannot tell knot from mirror image.

→ Two more classical methods compute Alexander polynom.

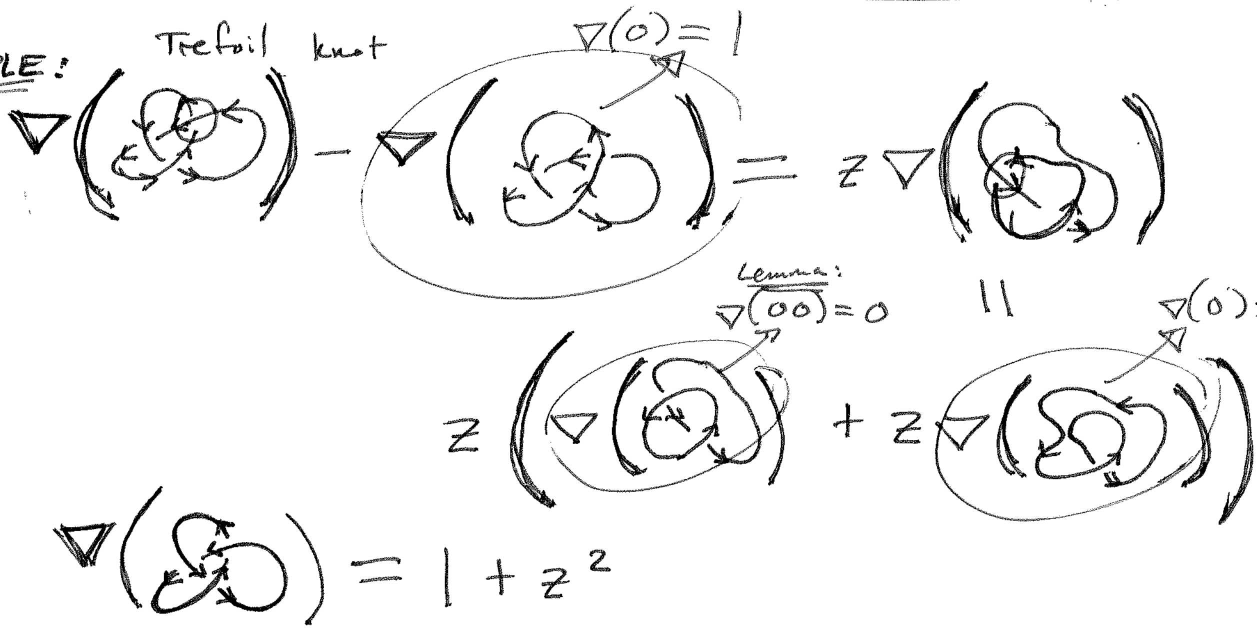
→ More ways to get Alexander polynomial.

Method 2: Skein relation (resolving trees) (Conway)

Def: Conway polynomial $\nabla(K)$

- $\nabla(\bigcirc) = 1$ (for all projections)
- $\nabla($  $) - \nabla($  $) = z\,\nabla($  $)$
- Conway polynom. of ambient isotopic links are equal.

EXAMPLE: Trefoil knot



Conway ⟼ Alexander:

$$\Delta(t) = \nabla(t^{1/2} - t^{-1/2})$$

Alexander   Conway

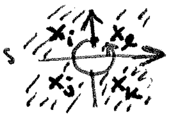
Method 3: Fundamental group: • $\pi_1(\mathbb{R}^3 \setminus K)$ is knot invariant.

→ Problem: This reduces to word problem (**undecidable**)

Dehn presentation: Label the faces of the knot projection

faces → generators

crossings → relations

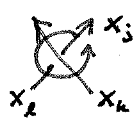


$x_i x_j^{-1} x_k x_\ell^{-1}$ ← r_s crossing

$$\pi_1(\mathbb{R}^3 \setminus K) \cong \langle x_1, x_2, \dots \mid r_1, r_2, \dots \;\&\; x_i \rangle$$

(any one of $x_1, \dots, x_f$ (face w/ basept))

Wirtinger presentation: Label the arcs of the knot projection



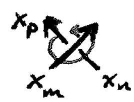
arcs → generators

crossings → relations

out = inverse

$x_j^{-1} x_k^{-1} x_\ell x_k$ ← r_s

in = no inv.



$x_p^{-1} x_m x_n x_m^{-1}$ ← r_t

$$\pi_1(\mathbb{R}^3 \setminus K) \cong \langle x_1, x_2, \dots \mid r_1, r_2, \dots, \hat{r}_i, \dots, r_n \rangle$$

(can throw out one at random.)

→ **Previously:** $\pi_1(\mathbb{R}^3 \setminus K)$ → Dehn presentation, → Wirtinger presentation $\langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$

Comparing group presentations:

Tietze Moves

$$\langle X \mid R \rangle \xmapsto{\mathrm{I}} \langle X \mid R \cup \{s\} \rangle \qquad s \in \langle\langle R \rangle\rangle$$

Normal closure

$$\langle X \mid R \rangle \xmapsto{\mathrm{II}} \langle X \cup \{y\} \mid R \cup \{yw^{-1}\} \rangle \qquad \boxed{y = w \in \langle X \rangle}$$

Thm: If $\langle X \mid R \rangle \cong \langle Y \mid S \rangle$ then $\exists$ finite seq. of Tietze moves between them.

Recall: G_{ab} = abelianization of G

$\mathbb{Z}G$ = free $\mathbb{Z}$-module on G (group ring)

Fox Calculus

Def: $D : \mathbb{Z}G \to \mathbb{Z}G$ w/

(i) $D(w_1 + w_2) = D(w_1) + D(w_2)$

(ii) $D(w_1 \cdot w_2) = D(w_1)\varepsilon(w_2) + w_1 D(w_2)$

$\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ augmentation

Note: $D(g_1 g_2) = D(g_1) \cdot \overset{\varepsilon(g_2)}{1} + g_1 D(g_2)$

$D(e) = D(e \cdot e) = D(e) + D(e)$

$\Rightarrow D(e) = 0$

Consider $F(x_1, \dots, x_n) = F(n) = F$ ← free group

w/ maps $\partial/\partial x_i : \mathbb{Z}F \to \mathbb{Z}F$

by $x_j \longmapsto \delta_{ij}$

→ **Elementary ideals:**

$$G \cong F/\langle\langle R \rangle\rangle = \langle x_1, \dots, x_n \mid R \rangle$$

$$\mathbb{Z}F \xrightarrow{\partial/\partial x_i} \mathbb{Z}F \xrightarrow{\gamma} \mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}G_{ab}$$

γ: quotient, α: abelianization

$\mathbb{Z}G_{ab}$: $\langle X \mid R \cup \{[x_i, x_j]\}_{i,j} \rangle$

Def The Alexander matrix $= \gamma\left[\frac{\partial}{\partial x_j} r_i\right]$ ← matrix in $M(\mathbb{Z}G_{ab})$

relation i, generator j

Ex $G = D_3 = \langle x_1, x_2 \mid x_1^2,\ x_2^3,\ x_1x_2x_1x_2 \rangle$

$$\begin{cases} \partial/\partial x_1\, x_1^2 = \partial/\partial x_1\, x_1 \cdot \varepsilon x_1 + x_1\, \partial/\partial x_1\, x_1 = 1 + x_1 \\ \partial/\partial x_2\, x_2^2 = 0 \end{cases}$$

$$\begin{cases} \partial/\partial x_1\, x_2^3 = 0 \\ \partial/\partial x_2\, x_2^3 = \partial/\partial x_2\, x_2^2 \cdot \varepsilon x_2 + x_2^2\, \partial/\partial x_2\, x_2 = (1 + x_2)\cdot 1 + x_2^2 \\ \qquad = 1 + x_2 + x_2^2 \end{cases}$$

$$\begin{cases} \partial/\partial x_1\, (x_1x_2x_1x_2) = 1 + x_1x_2 \\ \partial/\partial x_2\, (x_1x_2x_1x_2) = x_1 + x_1x_2x_1 \end{cases}$$

Alexander matrix: $\begin{bmatrix} 1+x_1 & 0 \\ 0 & 1+x_2+x_2^2 \\ 1+x_1x_2 & x_1+x_1x_2x_1 \end{bmatrix}$

in $\mathbb{Z}G_{ab}$: $x_1(1+x_1x_2)$ ~~$x_1 + x_1^2x_2$~~

Note: $\frac{\partial}{\partial x_j}(g_1 \cdots g_m) = \left(\frac{\partial}{\partial x_j} g_1\right) + g_1\left(\frac{\partial}{\partial x_j} g_2\right) + g_1g_2\left(\frac{\partial}{\partial x_j} g_3\right) + \cdots$

$$\frac{\partial}{\partial x_j} g^m = (1 + g + g^2 + \cdots + g^{m-1})\left(\frac{\partial}{\partial x_j} g\right)$$

Def: If $A \in M_{n\times m}(R)$ the k^{th} elementary ideal of A

$$E_k(A) = \begin{cases} 0 & \text{if } (n-k) > m \\ 1 & \text{if } (n-k) \le 0 \\ \left(\begin{matrix}\text{the ideal generated} \\ \text{by determinants of} \\ (n-k)\times(n-k) \text{ minors}\end{matrix}\right) & \text{otherwise} \end{cases}$$

$\Longrightarrow$ $0 = E_0(A) \subseteq E_1(A) \subseteq \cdots \subseteq E_n(A) = E_{n+1}(A) = R$

Lemma: The sequence of elementary ideals is invariant under

1. Permuting rows / columns
2. Adding a zero row ~~/column~~
3. Adding a new row/col = linear comb. of rows/cols
4. Adding a new row + col of zero w/ intersection = 1.

13) **Prop**: Elementary ideals are invariant under group presentation equiv. ③

Last time:

Tietze $\langle X : R\rangle \overset{I}{\longleftrightarrow} \langle X : R\cup S\rangle \quad S \in \langle\langle R\rangle\rangle$

$\langle X : R\rangle \overset{II}{\longleftrightarrow} \langle X\cup y : Rw^{-1}\rangle \quad w \in F(X).$

Derivative: $D: \mathbb{Z}G \to \mathbb{Z}G$ $\qquad \varepsilon : \mathbb{Z}G \to \mathbb{Z}$ augmentation

- $D(w_1+w_2) = D(w_1) + D(w_2)$
- $D(w_1w_2) = D(w_1)\,\varepsilon(w_2) + w_1 D(w_2).$

$D(e) = 0$

$D(x^{-1}) = -x^{-1}D(x)$

$D(g^n) = (e + g + g^2 + \cdots + g^{n-1})\,D(g) = \frac{g^n-1}{g-1}D(g).$

$D(g_1\cdots g_n) = D(g_1) + g_1D(g_2) + g_1g_2D(g_3) + \cdots + g_1\cdots g_{n-1}D(g_n)$

$\frac{\partial}{\partial x_j} : \mathbb{Z}F \to \mathbb{Z}F$

$\quad x_i \mapsto \delta_{ij}$

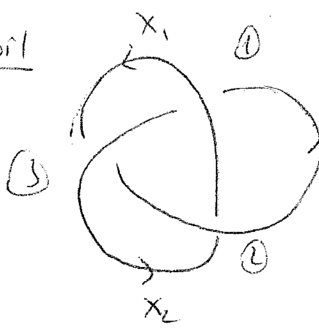
$\mathbb{Z}F \xrightarrow{\frac{\partial}{\partial x}} \mathbb{Z}F \xrightarrow{\gamma} \mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}G_{ab}$

Alexander matrix: $|a_{ij}|$ where $a_{ij} = \alpha\gamma\left(\frac{\partial r_i}{\partial x_j}\right)$

gen of $E_k(A) = \Delta_k(K).$

$$E_k(A) = \begin{cases} 0 & n-k > m \\ R & n-k \leq 0 \\ (n-k)\times(n-k) \text{ minors of } A & \text{otherwise.} \end{cases} \qquad \Delta_1(K) = \Delta(K)$$

Trefoil



$r_1 = x_2^{-1}x_1x_3x_1^{-1}$

$\cancel{r_2 = x_1^{-1}x_3x_2x_3^{-1}}$

$r_3 = x_3^{-1}x_2x_1x_2^{-1} \Rightarrow x_3 = x_2x_1x_2^{-1}$

$\pi_1(\mathbb{R}^3 \setminus 3_1) = \langle x_1, x_2, x_3 : r_1, r_3\rangle = \langle x_1, x_2 : x_2^{-1}x_1x_2x_1x_2^{-1}x_1^{-1}\rangle.$

$x_2^{-1}x_1x_2x_1x_2^{-1}x_1^{-1} = 1 \Rightarrow x_1x_2x_1 = x_2x_1x_2$

in $\mathbb{Z}G$ $\Rightarrow x_1x_2x_1 - x_2x_1x_2 = 0.$

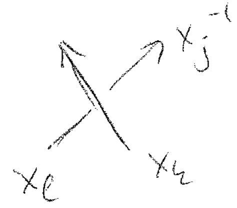
$\frac{\partial}{\partial x_1}(x_1x_2x_1 - x_2x_1x_2) = 1 + x_1x_2 - x_2 = t^2 - t + 1. \qquad \alpha(x_1), \alpha(x_2) = t$

$\frac{\partial}{\partial x_2}(x_1x_2x_1 - x_2x_1x_2) = x_1 - 1 - x_2x_1 = t - t^2 - 1. = -\frac{\partial}{\partial x_1}(r).$

$A = [\,t^2-t+1 \quad 0\,] \qquad E_k(A) = \begin{cases} 0 & k=0 \\ \langle t^2-t+1\rangle & k=1 \\ \mathbb{Z}\langle t\rangle & k\geq 1 \end{cases}$

$\Rightarrow \Delta(3_1) = t^2 - t + 1.$

15

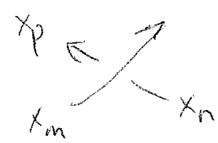


$x_j^{-1} x_k^{-1} x_l x_k = f$ $\qquad -t^2$

$\frac{\partial f}{\partial x_j} = -x_j^{-1} = -t^{-1} \equiv t$

$\frac{\partial f}{\partial x_k} = -x_j^{-1} x_k^{-1} + x_j^{-1} x_k^{-1} x_l = -t^2 + t^{-1} \equiv 1-t$

$\frac{\partial f}{\partial x_l} = x_j^{-1} x_k^{-1} = t^{-2} \equiv -1$



$x_p^{-1} x_m x_n x_m^{-1} = g$

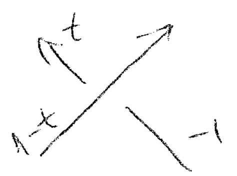
$\frac{\partial g}{\partial x_p} = -x_p^{-1} = -t^{-1} \equiv -1$ $\quad t$

$\frac{\partial g}{\partial x_m} = x_p^{-1} - x_p^{-1} x_m x_n^{-1} x_m^{-1} = t^{-1} - 1 \equiv 1-t$

$\frac{\partial g}{\partial x_n} = x_p^{-1} x_m = 1 \equiv t.$

---

(1)



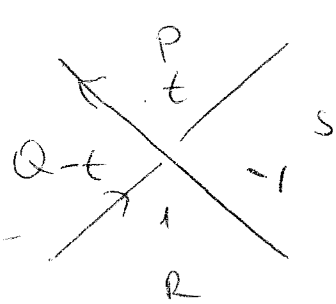
Overcrossing $\to 1-t$
left of over crossing $\to t$
right " $\to -1$

Construct a matrix whose rows are indexed by crossings and columns ~~are~~ are indexed by arcs. delete one row and one column to get the Alex. polyn

$$\begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline 1 & 1-t & t & -1 \\ 2 & t & -1 & 1-t \\ 3 & -1 & 1-t & t \end{array}$$

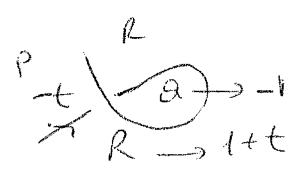
$(1-t)(-1) - t^2 = t - 1 - t^2 \equiv t^2 - t + 1.$

(2)



$$\begin{array}{cccccccccc} & & P & & Q & & R & & S & \\ i & [\,0 \ldots & t & 0 & -t & 0 & 1 & 0 & -1 &] \end{array}$$

isthmus $\to$

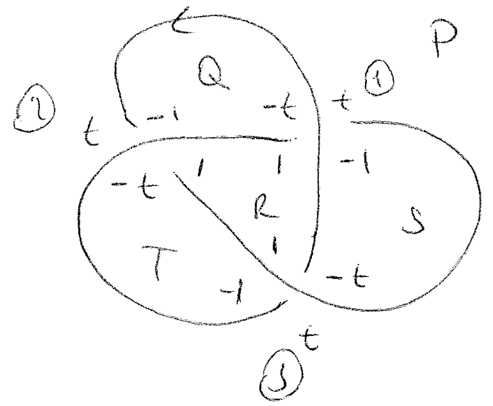


Delete any 2 columns ~~by~~ indexed by a region which have a common boundary.

Going through the over crossings

before the crossing: right -1, left 1; after the crossing: right t, left $-t$

16



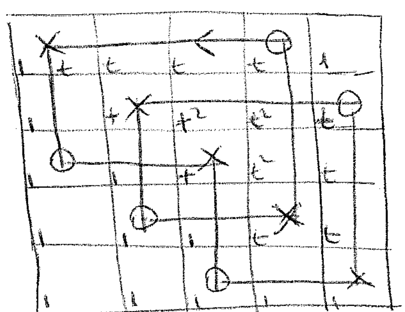
$$\begin{array}{c|ccccc} & P & Q & R & S & T \\ \hline 1 & t & -t & 1 & -1 & 0 \\ 2 & t & -1 & 1 & 0 & -t \\ 3 & t & 0 & 1 & -t & -1 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -t \\ 1 & -t & -1 \end{bmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -t \\ 0 & 1-t & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1-t \end{vmatrix}$$

$$= -1-(-t)(1-t) = -1+t-t^2 \equiv t^2-t+1.$$

Grid Diagrams

Put exactly one X and one O in each row and column. Connect each X and O ~~draws~~ by straight lines lying on the same row and column.



Vertical lines are always over crossing. Orient the diagram. To write the matrix take the lower left corner of each square. Draw a ray in the west direction. Count the # of linking

$\leftarrow\uparrow_+ \quad \leftarrow\downarrow_-$

Assign $t^{a(i,j)}$ where $a(i,j)$ is the linking #.

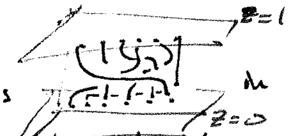
If $\tilde{\Delta}_K(t)$ is the determinant of the resulting matrix then $\Delta_K(t) = \dfrac{\tilde{\Delta}(t)}{(1-t)^{N-1}}$ where N is the size of the matrix.

$$\begin{vmatrix} 1 & t & t & t & 1 \\ 1 & t & t^2 & t^2 & t \\ 1 & 1 & t & t^2 & t \\ 1 & 1 & 1 & t & t \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & t-1 & t-1 & t-1 & 0 \\ 0 & t-1 & t^2-1 & t^2-1 & t-1 \\ 0 & 0 & t-1 & t^2-1 & t-1 \\ 0 & 0 & 0 & t-1 & t-1 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} t-1 & t-1 & t-1 & 0 \\ 0 & (t-1)(t) & t(t-1) & t- \\ 0 & t-1 & t^2-1 & t- \\ 0 & 0 & t-1 & t- \end{vmatrix}$$

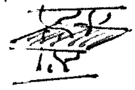
$$= -(t-1)\begin{vmatrix} (t-1)t & t(t-1)^2 & 0 \\ t-1 & t(t-1) & 0 \\ 0 & t-1 & t-1 \end{vmatrix} = -(t-1)^2\left(t^2(t-1)^2-(t-1)^3\right)$$

$$= -(t-1)^4\underbrace{(t^2-t+1)}_{\Delta_K(t).}$$

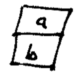
(Missing Notes)

Braid Groups: B_n = Isotopy classes of paths  in $\mathbb{R}^3$

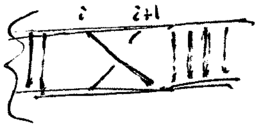
- each path w/o critical points

  i.e.  all horiz. planes intersect at exactly n points

- ~~crossings have unique height??~~

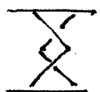
Operations on B_n: • $a \cdot b =$ 

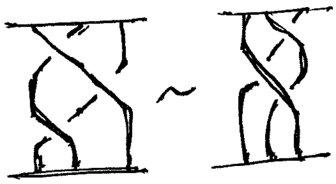
• inverse is mirror image 

Generators:  $= \sigma_i \Big\}_1^{n-1}$

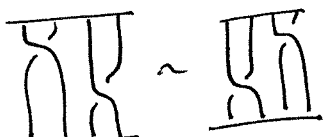
→ A word in generators ⟺ paths w/ crossings on different levels.

Artin Relations:

•  $\sigma_i \cdot \sigma_i^{-1} = 1$ (Reidemeister 2)

•  $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

$\left(\sigma_{i+1}^{-1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i^{-1}\right)$

•  $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| \geq 2$

Thm (Artin): $B_n = \langle \sigma_i \rangle / \text{Artin Relations}$

Examples: $B_1 = \{1\}$

$B_2 = \mathbb{Z}$

$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1^{-1} \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2^{-1} \rangle$

BTW this is $\pi_1(\mathbb{R}^3 \setminus \text{trefoil})$

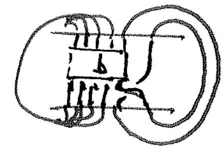
Connecting corresponding points top — bottom → Link "Closure"

Thm (Alexander): Every link can be given by connecting corresp. points on braids.
("Closure map is surjective onto links")

Q: When do braids have isotopic closure?

Thm (Markov): Braids have isotopic closure iff related by sequence of the following moves:

- $b \longleftrightarrow a b a^{-1}$ $(a, b \in B_n)$
- $b \longleftrightarrow b \cdot \sigma_n^{\pm 1}$ $(\sigma_n \in B_{n+1} \quad b \in B_n)$



Reduced Burau Representation

$$\phi : B_n \longrightarrow GL_{n-1}(\mathbb{Z}[t, t^{-1}])$$

$$\sigma_1 \longmapsto \begin{bmatrix} -t & 1 & 0 & \cdots & \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & \\ & 0 & & \ddots & 1 \end{bmatrix} \qquad \sigma_n \longmapsto \begin{bmatrix} 1 & \ddots & & 0 \\ & & 1 & 0 \\ 0 & & t & -t \end{bmatrix}$$

$$\sigma_i \longmapsto \begin{bmatrix} 1 & \ddots & & & \\ & 1 & 0 & 0 & \\ 0 & t & -t & 1 & 0 \\ & 0 & 0 & 1 & \\ & & & \ddots & 1 \end{bmatrix} \longleftarrow (\text{row } i)$$

alexander polynomial of closure of braid.

Thm: $\det(I - \phi(B)) = (1 + t + \cdots + t^{n-1})\,\Delta_{\bar{B}}$

Jones Polynomial from Braid Group

RECALL: B_n – braid group on n strands
- → generated by $\{\sigma_i\}$ ← crosses strands i & $(i+1)$ 
- → Artin relations
  - $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j|\geq 2$
  - $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

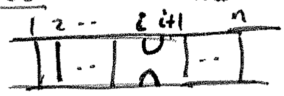
Other Representations of B_n:

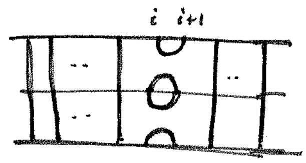
Temperley-Lieb Algebra

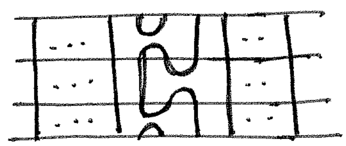
Def: Let $\tau\in\mathbb{C}$. The Temperley-Lieb Algebra $TL_{n+1}(\tau)$ is $\mathbb{C}$-algebra generated by $e_1,\dots,e_n$ w/
1. $e_i^2 = e_i$
2. $e_ie_j = e_je_i$ for $|i-j|\geq 2$
3. $e_ie_{i\pm1}e_i = \tau e_i$

(Goal: Find a representation of B_n in TL_n)

① → Diagrammatics for TL_n (Kaufmann)

E_i: 

E_i^2:  "qE_i"

$E_iE_{i+1}E_i$:  E_i

→ Let $e_i = \frac{1}{q}E_i$
- $e_i^2 = \frac{1}{q^2}E_i^2 = \frac{1}{q^2}qE_i = \frac{1}{q}E_i = e_i$
- $e_ie_{i\pm1}e_i = \frac{1}{q^3}E_iE_{i+1}E_i$
  $= \frac{1}{q^2}\cdot\frac{1}{q}E_i$ $\qquad\left(\tau = \frac{1}{q^2}\right)$

② → $\psi: B_n \longrightarrow TL_n(\tau)$ Representation.

$\sigma_i \longmapsto ae_i + b\cdot 1$

$\sigma_i\sigma_j \longmapsto (ae_i+b\cdot1)(ae_j+b\cdot1)$ if $|i-j|\geq2$

$\sigma_i^{-1} \longmapsto (ae_i+b\cdot1)^{-1} = (ce_i+d)$

$(ae_i+b)(ce_i+d)=1$

$ac\underset{\uparrow\, e_i}{e_i^2} + bce_i + ade_i + bd = 1$ $\quad\Big\langle\ \begin{matrix} ac+bc+ad=0 \\ bd=1\end{matrix}$ $\quad c = -\frac{ab^{-1}}{a+b}$, $\quad d=b^{-1}$

(→ Need $b\neq0$, $a\neq-b$)

(20)

More Ψ work:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \leftarrow \Psi \text{ well defined??}$$

$$\Psi(\sigma_i \sigma_{i+1} \sigma_i) = (ae_i + b)(ae_{i+1} + b)(ae_i + b)$$

$$= a^3 \overset{\tau e_i}{e_i e_{i+1} e_i} + a^2 b\left(\cancel{e_i e_{i+1}} + \cancel{e_{i+1} e_i} + \overset{e_i}{e_i^2}\right) + 2ab^2 e_i + \cancel{ab^2 e_{i+1}} + \ldots$$

$$\Psi(\sigma_{i+1} \sigma_i \sigma_{i+1}) = (ae_{i+1} + b)(ae_i + b)(ae_{i+1} + b)$$

$$= a^3 \underset{\tau e_{i+1}}{e_{i+1} e_i e_{i+1}} + a^2 b\left(\cancel{e_{i+1} e_i} + \underset{e_i}{e_{i+1}^2} + \cancel{e_i e_{i+1}}\right) + ab^2 (2e_{i+1} + \cancel{e_i}) + \cancel{b^3}$$

$$(a^3 \tau + a^2 b + ab^2) e_i$$

$$\|$$

$$(a^3 \tau + a^2 b + ab^2) e_{i+1}$$

These must $= 0$

$$a^3 \tau + a^2 b + ab^2 = 0$$

$$\cancel{a}(a^2 \tau + ab + b^2) = 0$$

$a \neq 0$

$$\tau = \frac{-ab - b^2}{a^2}$$

21

Cyclic equivalence on words:

Temperley–Lieb: $e_i^2 \equiv e_i$

$e_i e_j \equiv e_j e_i \quad |i-j| \geq 2$

$e_i e_{i+1} e_i \equiv \tau e_i$

$\left.\right\}$ & identities generated by these

\+ cyclic permutations:

$$e_{i_1} \cdots e_{i_j} e_{i_{j+1}} \cdots e_{i_k} \equiv e_{i_{j+1}} \cdots e_{i_k} e_{i_1} \cdots e_{i_j}$$

Goal: Trace on cyclic equivalence class is constant

(trace should satisfy: (1) Linearity
(2) $tr(ab) = tr(ba)$ $\qquad tr: TL(\tau) \longrightarrow \cdots$)

What is tr?? What are equivalence classes above??

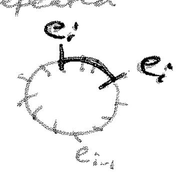
EX:

$e_i \equiv e_j$ b/c

$$\begin{aligned} e_i e_{i+1} e_i &\equiv \tau e_i \\ &\text{\rotatebox{90}{$\equiv$}} \\ e_i^2 e_{i+1} & \\ &\text{\rotatebox{90}{$\equiv$}} \\ e_i e_{i+1} & \\ &\text{\rotatebox{90}{$\equiv$}} \\ e_i e_{i+1}^2 & \\ &\text{\rotatebox{90}{$\equiv$}} \\ e_{i+1} e_i e_{i+1} &\equiv \tau e_{i+1} \end{aligned}$$

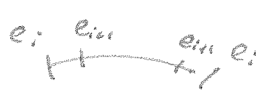
Lemma: Every word is cyclically equivalent to a word w/ no repeated letters.
(Furthermore, all minimal length words have this property)

Proof:
Suppose w has minimal length in its class ~~w~~ has repeated letters. Let e_i be repeated in w w/ minimal i.



~~Pick two "closest" e_i in cyclic word.~~

$\Rightarrow e_{i-1}$ not repeated. Pick two e_i and arc between containing no e_i or e_{i-1}



$\Rightarrow$ Must be two e_{i+1} between them
(to avoid $e_i e_{i+1} e_i \equiv \tau e_{i+1}$)

$\Rightarrow$ Must be two e_{i+2} between them

etc. $\qquad \divideontimes$.

Lemma: Every word is cyclically equivalent to a word w/ letters increasing index.

Proof:

- lowest letter commutes to next lowest
- group commutes to next lowest
- etc. ∎

Lemma: $e_{i_1}\cdots e_{i_k} \equiv \tau^a e_1 e_2 \cdots e_s \quad (i_1 < i_2 < \cdots < i_k)$

Will have $s+a=k$.

Proof:

Claim 1: $e_{i_1}\cdots e_{i_\ell}\, e_{i_{\ell+1}}\, e_{i_{\ell+2}}\cdots e_{i_k} \equiv e_{i_1}\cdots e_{i_\ell}\, e_{i_{\ell+1}-1}\, e_{i_{\ell+1}}\, e_{i_{\ell+2}}\cdots e_{i_k}\ \frac{1}{\tau}$

gap of ≥ 1

add in left neighbor.

(Ex $e_1 e_3 e_5 \equiv \frac{1}{\tau} e_1 e_2 e_3 e_5$)

$$e_{i_1}\cdots e_{i_\ell}\, e_{i_{\ell+1}}\, e_{i_{\ell+2}}\cdots e_{i_k}$$
$$\Downarrow$$

move around $\left(\frac{1}{\tau}\, e_{i_{\ell+1}}\, e_{i_{\ell+1}-1}\, e_{i_{\ell+1}}\right)$

$$\|$$
$$\frac{1}{\tau}\, e_{i_{\ell+1}-1}\, e_{i_\ell}\, e_{i_\ell}$$

$$\frac{1}{\tau}\, e_{i_1}\cdots e_{i_\ell}\, e_{i_{\ell+1}-1}\, e_{i_{\ell+1}}\, e_{i_{\ell+2}}\cdots e_{i_k}$$

Claim 2: $e_{i_1}\cdots e_{i_\ell}\, e_{i_{\ell+1}}\, e_{i_{\ell+1}+1}\, e_{i_{\ell+2}}\cdots e_{i_k} \equiv e_{i_1}\cdots e_{i_\ell}\, e_{i_{\ell+1}-1}\, e_{i_{\ell+1}}\, e_{i_{\ell+2}}\cdots e_{i_k}$

gap of ≥ 1

was: $e_{i_{\ell+1}}\, e_{i_{\ell+1}+1}$

$$e_{i_\ell}\, e_{i_{\ell+1}}\, e_{i_{\ell+1}+1}\, e_{i_{\ell+2}}$$
$$\Downarrow$$

move around $\left(\frac{1}{\tau}\, e_{i_{\ell+1}}\, e_{i_{\ell+1}-1}\, e_{i_{\ell+1}}\right)$

$$\frac{1}{\tau}\, e_{i_{\ell+1}-1}\, \underbrace{e_{i_{\ell+1}}\, e_{i_{\ell+1}+1}\, e_{i_{\ell+1}}}_{\tau e_{i_{\ell+1}}}\cdots$$

Maneuver

Recall: Jones Polynomial

$$B_n \longrightarrow TL_n$$
$$\sigma_i \longmapsto (1+t)e_i - 1$$

- $tr(\sigma_i) = tr((1+t)e_i - 1)$
- $tr(e_i) = z = \frac{t}{(1+t)^2}$
- $tr(\alpha\sigma_n) = \frac{-1}{1+t} tr(\alpha) \qquad \alpha \in B_n$ & $\sigma_n \in B_{n+1}$

and $V_{\hat{\alpha}}(t) = \left(-\frac{t+1}{\sqrt{t}}\right)^{n-1} (\sqrt{t})^{e(\alpha)} tr(\alpha)$ $\quad$ ($e(\alpha)$ = writhe of α)

Now, HOMFLY:

$P_L(\ell, m)$

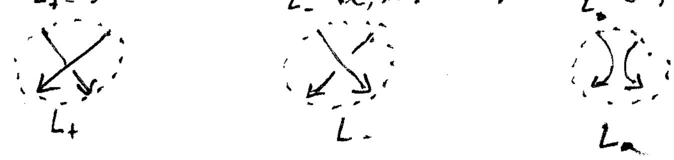
"link"

- polynomial defined on links w/
  - $P_0(\ell, m) = 1$
  - $P_{L \amalg 0}(\ell, m) = P_L(m^{-1}(\ell + \ell^{-1}))$??

&

- Skein relation:

$$\ell P_{L_+}(\ell, m) + \ell^{-1} P_{L_-}(\ell, m) = m P_{L_0}(\ell, m)$$



Hecke Algebra: $H_n(\ell, m)$ $\quad$ "n strings" $\quad \ell \in \mathbb{C}$

generators: $g_1, \dots, g_{n-1}$

relations:
- $g_i g_j = g_j g_i \qquad |i-j| \geq 2$
- $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$
- $\ell g_i + \ell^{-1} g_i^{-1} = m$

$\mathbb{C}B_n \big/ \ell g_i + \ell^{-1} g_i^{-1} - m$

(Note: algebra is generated by braids) ⊕

- Homfly w/ $\ell = i t^{-1}$ & $m = i(t^{1/2} - t^{-1/2})$ $\rightsquigarrow$ Jones

To compute Homfly, use trace function from Hecke alg:

$\tau : H_n \longrightarrow \mathbb{C}$ $\quad$ w/

(1) $\tau(a+b) = \tau(a) + \tau(b)$

(2) $\tau(ab) = \tau(ba)$

(3) $\tau(1) = 1$

(4) $\tau(w g_n) = (m^{-1}(\ell + \ell^{-1}))^{-1} \tau(w)$

all $w \in H_n \subset H_{n+1}$

Compute:

$$\tau(g_i) = \tau(1\cdot g_i) = \left(m^{-1}(\ell+\ell^{-1})\right)^{-1}\cdot \cancel{\tau(1)}$$

$$\tau(g_i^{-1}) = \tau\left(\ell(m-\ell g_i)\right)$$
$$= \ell m - \ell^2\tau(g_i)$$
$$= \ell m - \ell^2 \frac{m}{\ell+\ell^{-1}} = \left(m^{-1}(\ell+\ell^{-1})\right)^{-1} = \tau(g_i) \qquad (!!)$$

FACT:

- $$P_L(\ell,m) = \left(m^{-1}(\ell+\ell^{-1})\right)^{n-1}\tau(\phi(B))$$

where $\phi : B_n \longrightarrow H_n(\ell,m)$

$\sigma_i \longmapsto g_i$

EX $\sigma_1^3 \in B_2$ (trefoil)

$$P_L(\ell,m) = \left(m^{-1}(\ell+\ell^{-1})\right)^{2-1}\tau(\phi(\sigma_1^3))$$
$$= \big(\underbrace{m^{-1}(\ell+\ell^{-1})}_{z}\big)\left(\ell^{-2}m^2z^{-1} - \ell^{-3}m - \ell^{-2}z^{-1}\right)$$
$$= \ell^{-2}m^2 - \ell^{-2} - \ell^{-3}(\ell+\ell^{-1})$$

$$g_i^2 = \ell^{-1}(mg_i - \ell^{-1})$$
$$= \ell^{-1}mg_i - \ell^{-2}$$
$$g_i^3 = \ell^{-1}mg_i^2 - \ell^{-2}g_i$$
$$= \ell^{-2}m^2 g_i - \ell^{-3}m - \ell^{-2}g_i$$

Kaufman's L-polynomial:

- $\hat{K}_{\times}(\ell,m) + \hat{K}_{\times'}(\ell,m) = m\left(\hat{K}_{)(}(\ell,m) + \hat{K}_{\asymp}(\ell,m)\right)$
- $\hat{K}_{\gamma}(\ell,m) = \ell\,\hat{K}_{\smile}(\ell,m)$ • $\hat{K}_{\gamma'}(\ell,m) = \ell^{-1}\hat{K}_{\smile}(\ell,m)$
- $\hat{K}_{O}(\ell,m) = 1$

→ Invariant under regular isotopy.

$$K_L(\ell,m) = \ell^{-w(L)}\hat{K}_{|L|}(\ell,m)$$ $|L|$ ← unoriented link

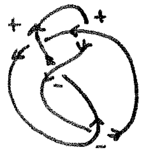
→ Invariant under isotopy

"Kaufman Polynomial"

(Jones Polynomial is recovered by using special ℓ & m)

(25) **EX** Figure 8 knot:

abuse notation $K_L \to K(L)$

L = 

$w(L) = 0$

So $K(L) = \hat{K}(L)$

K(L)

ℓ^{-2}

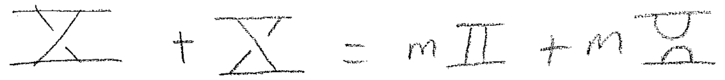
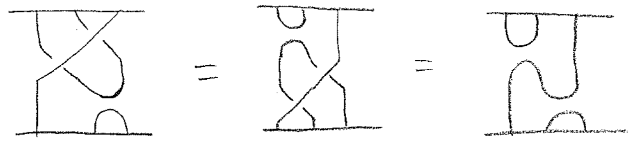
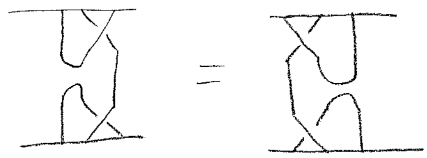
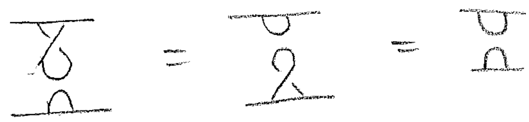
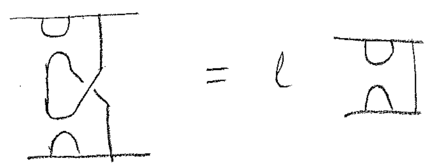
EXERCISE: $\hat{K}$  $+ \hat{K}$  $= m\big(\hat{K}$  $+ \hat{K}$ 

etc

~~Sol~~ **ANSWER:**

$$= -\ell^{-2} - \ell m - (\ell + \ell^{-1}) + m + m^2(\ell + \ell^{-1}) + m\ell$$
$$- \ell(\ell + \ell^{-1}) + \ell m + \ell m^2(\ell + \ell^{-1})$$

(26)

1. $G_i G_j = G_j G_i$ if $|i-j| \geq 2$
2. $G_i G_{i+1} G_i = G_{i+1} G_i G_{i+1}$
3. $G_i + G_i^{-1} = m(1 + E_i)$ 
4. $E_i E_{i\mp 1} E_i = E_i$
5. $G_{i\mp 1} G_i E_{i\mp 1} = E_i G_{i\mp 1} G_i = E_i E_{i\mp 1}$ 
6. $G_{i\mp 1} E_i G_{i\mp 1} = G_i^{-1} E_{i\mp 1} G_i^{-1}$ 
7. $G_{i\mp 1} E_i E_{i\mp 1} = G_i^{-1} E_{i\mp 1}$ 
8. $E_{i\mp 1} E_i G_{i\mp 1} = E_{i\mp 1} G_i^{-1}$ 
9. $G_i E_i = E_i G_i = \ell^{-1} E_i$ 
10. $E_i G_{i\mp 1} E_i = \ell E_i$ 

* Using (3) $E_i = m^{-1}(G_i + G_i^{-1}) - 1$ and (1)

$E_i E_j = E_j E_i$ if $|i-j| \geq 2$

* Using (3), (9)

$$E_i^2 = (m^{-1}(G_i + G_i^{-1}) - 1) E_i = m^{-1}(G_i E_i + G_i^{-1} E_i) - E_i$$
$$= (m^{-1}(\ell^{-1} + \ell) - 1) E_i$$

* Using (3), (9)

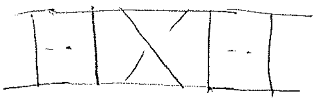
$$G_i^2 = (m(1+E_i) - G_i^{-1}) G_i = m(G_i + E_i G_i) - 1$$
$$= m(G_i + \ell^{-1} E_i) - 1$$

Özgür Kişisel

<u>R-matrices</u> (Reference: Boris Kupershmidt "What a classical r-matrix really is"). 01.03.2012

1

Artin relations for the braid group:



σ_i

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

Find representations of B_n on tensor products of a single vector space V.

<u>Idea:</u> On $\underset{(1)}{V} \otimes \underset{(2)}{V} \otimes \cdots \otimes \underset{(n)}{V} \otimes \cdots$

σ_i acts on $\underset{(i)}{V} \otimes \underset{(i+1)}{V}$ $\qquad \sigma_i \longmapsto \mathrm{End}(\underset{(i)}{V} \otimes \underset{(i+1)}{V})$

<u>Simplest case:</u> Send all σ_i's to the same endomorphism S of $V \otimes V$.

<u>Need notation:</u>

S^{ij} denotes $S : \underset{(i)}{V} \otimes \underset{(j)}{V} \to \underset{(i)}{V} \otimes \underset{(j)}{V}$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \iff S^{i(i+1)} S^{(i+1)(i+2)} S^{i(i+1)} = S^{(i+1)(i+2)} S^{i(i+1)} S^{(i+1)(i+2)}$$

as elts of $\mathrm{End}(V \otimes V \otimes V)$

$$\iff \boxed{S^{12} S^{23} S^{12} = S^{23} S^{12} S^{23}}$$

<u>Ex:</u> ① $S = \mathrm{id}_{V \otimes V}$

② $S = P$ (switch operator)

$P : V \otimes V \to V \otimes V$

$v_1 \otimes v_2 \mapsto v_2 \otimes v_1$

$$P^{12} P^{23} P^{12} (v_1 \otimes v_2 \otimes v_3) = P^{12} P^{23} (v_2 \otimes v_1 \otimes v_3) = P^{12} (v_2 \otimes v_3 \otimes v_1) = v_3 \otimes v_2 \otimes v_1$$

$$P^{23} P^{12} P^{23} (v_1 \otimes v_2 \otimes v_3) = P^{23} P^{12} (v_1 \otimes v_3 \otimes v_2) = P^{23} (v_3 \otimes v_1 \otimes v_2) = v_3 \otimes v_2 \otimes v_1 .$$

$$\Rightarrow P^{12} P^{23} P^{12} = P^{23} P^{12} P^{23}$$

Group theoretically,

$P^2 = \mathrm{id}$.

$B_n \twoheadrightarrow S_n$.

$\sigma_i \mapsto (i\ i+1)$ transposition.

S_n acts on $V \otimes \cdots \otimes V$ by permutations.

Composition is the reps abo[ve]

Can we perturb P in a Taylor series and find new S-matrices as above?

Instead of perturbing P, perturb I

$$\boxed{S = PR} \quad \left(\Leftrightarrow \boxed{PS = R} \right)$$

$$S^{12} S^{23} S^{12} = S^{23} S^{12} S^{23}$$

$$\Leftrightarrow P^{12} R^{12} P^{23} R^{23} P^{12} R^{12} = P^{23} R^{23} P^{12} R^{12} P^{23} R^{23}$$

Lemma: A any endomorphism

(1) $A^{12} P^{23} = P^{23} A^{13}$

(2) $A^{13} P^{23} = P^{23} A^{12}$

(3) $A^{23} P^{12} = P^{12} A^{13}$

(4) $A^{13} P^{12} = P^{12} A^{23}$

Pf: (1) $A^{12} P^{23} (v_1 \otimes v_2 \otimes v_3) = A^{12} (v_1 \otimes v_3 \otimes v_2) = A(v_1 \otimes v_3) \otimes v_2$.

$P^{23} A^{13} (v_1 \otimes v_2 \otimes v_3) = P^{23} (A(v_1 \otimes v_3)_{13} \otimes (v_2)_2) = A(v_1 \otimes v_3) \otimes v_2$.

(2), (3), (4) are similar.

$$P^{12} \underline{R^{12} P^{23}} \underline{R^{23} P^{12}} R^{12} = P^{12} P^{23} \underline{R^{13} P^{12}} R^{13} R^{12} = \cancel{P^{12} R^{23} P^{12}},$$

$$= P^{12} P^{23} P^{12} R^{23} R^{13} R^{12}$$

$$P^{23} \underline{R^{23} P^{12}} \underline{R^{12} P^{23}} R^{23} = P^{23} P^{12} \underline{R^{13} P^{23}} R^{13} R^{23} = P^{23} P^{12} P^{23} R^{12} R^{13} R^{23}.$$

Since $P^{12} P^{23} P^{12} = P^{23} P^{12} P^{23} \Rightarrow \boxed{R^{23} R^{13} R^{12} = R^{12} R^{13} R^{23}}$

Quantum Yang-Baxter Eqn. (QYBE).

solutions: R-matrices.

Perturbations of R, and classical r-matrices

Suppose $R = I + hr + h^2 \rho + O(h^3)$.

$R \in End(V \otimes V)$ so are r and ρ.

$$R^{12}R^{13}R^{23} = (I + hr^{12} + h^2\rho^{12})(I + hr^{13} + h^2\rho^{13})(I + hr^{23} + h^2\rho^{23}) + O($$

$$= I + h(r^{12} + r^{13} + r^{23}) + h^2(\rho^{12} + \rho^{13} + \rho^{23} + r^{12}r^{13} + r^{12}r^{23} + r^{13}r^{23}) + O($$

$$R^{23}R^{13}R^{12} = I + h(r^{23} + r^{13} + r^{12}) + h^2(\rho^{12} + \rho^{13} + \rho^{23} + r^{23}r^{13} + r^{23}r^{12} + r^{13}r^{12}) + O(h^3)$$

$$\boxed{r^{12}r^{13} + r^{12}r^{23} + r^{13}r^{23} = r^{23}r^{13} + r^{23}r^{12} + r^{13}r^{12}}$$

Classical Yang-Baxter Eqn (CYBE) Solutions: classical r-mat

CYBE can be rewritten as:

$$c(r) = [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0.$$

Only commutators are involved.

This eqn. holds in $End(V \otimes V \otimes V) = \underbrace{End(V)}_{\mathfrak{g} = \mathfrak{gl}(V)} \otimes End(V) \otimes E$

Generalize to any ~~$\mathfrak{g}$~~ Lie alg. $\mathfrak{g}$. Then $r^{ij} \in \mathfrak{g} \otimes \mathfrak{g}$

CYBE is an eqn. in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$.

Note: Suppose $r = \sum a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$.

$$[r^{12}, r^{13}] = [\sum_i a_i \otimes b_i \otimes 1, \sum_j a_j \otimes 1 \otimes b_j] = \sum_{i,j} [a_i, a_j] \otimes b_i \otimes b_j$$

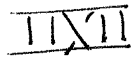
$$(a_i \otimes b_i \otimes 1)(a_j \otimes 1 \otimes b_j) - (a_j \otimes 1 \otimes b_j)(a_i \otimes b_i \otimes 1)$$

$$= a_i a_j \otimes b_i \otimes b_j - a_j a_i \otimes b_i \otimes b_j = [a_i, a_j] \otimes b_i \otimes b_j$$

$$[r^{12}, r^{23}] = \sum_{i,j} a_i \otimes [b_i, a_j] \otimes b_j$$

$$[r^{13}, r^{23}] = \sum_{i,j} a_i \otimes a_j \otimes [b_i, b_j].$$

$$\text{CYBE} \iff \sum_{i,j}\Big([a_i, a_j] \otimes b_i \otimes b_j + a_i \otimes [b_i, a_j] \otimes b_j + a_i \otimes a_j \otimes [b_i, b_j]\Big) =$$

Recall: • Braid group B_n generated by  σ_i

w/ relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

- Closing braids gives links / knots
  - $\Rightarrow$ braids related by standard basic moves gives same knots / links.

- Look at particular representations of B_n on $\underset{\text{v.s.}}{V^{\otimes n}}$

- Basic idea: $\sigma_i \longmapsto S \in \mathrm{End}(V_{(i)} \otimes V_{(i+1)})$ for some fixed matrix S.

$\Rightarrow$ Braid relation $\rightsquigarrow$ relation on $\mathrm{End}(V_{(i)} \otimes V_{(i+1)} \otimes V_{(i+2)})$

$$S^{12} S^{23} S^{12} = S^{23} S^{12} S^{23}$$

$\Rightarrow$ R-matrices

Let P be flip operator $P(v_1 \otimes v_2) = v_2 \otimes v_1$

– P satisfies Artin relations, but its too simple (it sees S_n, not B_n)

Let $S = PR$

– Artin relations for S become

$$\boxed{R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12}}$$

for R — Quantum Yang-Baxter Eqn

Let $R = I + \varepsilon r + \varepsilon^2 \rho + \varepsilon^3 (\cdots)$

– Yields "classical" Yang-Baxter Eqn for r

Yang-Baxter Eqn $\Rightarrow$ $$\boxed{[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0}$$

New Tool: Poisson-Lie Groups (ref: Chari-Pressley: "Quantum Groups")

$\Rightarrow$ Motivation: Let (M, ω) be a symplectic mfld Eg

ω is a 2-form
$d\omega = 0$
ω nondegenerate

Given ~~continuous~~ smooth function $H \in C^\infty(M) \rightsquigarrow$ vector field X_H by

$$\omega(X_H, \cdot) = dH$$

Poisson manifold:

" (M, w) where w is a bivector field
(section of $\Lambda^2 TM$ ← linear dual of 2-forms) "

Def: Let M be C^∞ mfld w/ $\dim(M) = m$

Poisson structure on M is bilinear map

$$\{\,,\}_M : C^\infty(M) \times C^\infty(M) \to C^\infty(M) \quad \text{w/}$$

(skew-symm) ① $\{f, g\} = -\{g, f\}$

(Leibniz) ② $\{fg, h\} = f \cdot \{g, h\} + g\{f, h\}$

(Jacobi) ③ $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

Remarks: ① $\{\,,\}$ is a Lie-bracket on $C^\infty(M)$

② $\forall f$, $\cancel{X_f}: g \mapsto \{g, f\}$ is a derivation

→ Define the Hamiltonian vector field for f to be $X_f = \{\cdot, f\}$

③ $\{f, g\}$ only depends on $\overline{df \text{ \& } dg}$

→ $\{f, g\} = w(df, dg)$ bivector field defined using df & dg.

EX On $\mathbb{R}^{2n}$ w/ coords $\{x_1, \dots, x_n, y_1, \dots, y_n\}$

use $\{x_i, y_j\} = \delta_{ij} = -\{y_j, x_i\}$

$\{x_i, x_j\} = 0 = \{y_i, y_j\}$

→ $X_{x_1+y_2} = \frac{\partial}{\partial x_2} - \frac{\partial}{\partial y_1}$ (by checking values on x_i & y_i)

Quasi-triangular Hopf Algebras

Rough idea:

Comm. Hopf Alg $\longleftrightarrow$ $F(G)$ for some G

co Comm. Hopf Alg $\longleftrightarrow$ $U(\mathfrak{g})$ for some $\mathfrak{g}$

→ Need almost cocomm. Hopf algebras

Def: A Hopf alg A/k (k comm. ring) is almost cocomm. if
$$\tau\circ\Delta(a) = R\cdot\Delta(a)\cdot R^{-1}$$
for $R\in A\otimes A$.

Remark: Comm. & almost cocomm $\Longrightarrow$ $R\cdot\Delta(a)\cdot R^{-1} = R\cdot R^{-1}\cdot\Delta(a) = \Delta(a)$

$\Longrightarrow$ cocomm.

- Let C be centralizer of $\Delta(A)$ in $A\otimes A$

(Note: $Z = \text{center}(A)$, $Z\otimes Z\subset C$)

Prop: Let $\phi: A\otimes A\longrightarrow A$ by
$$a_1\otimes a_2\longmapsto a_1 S(a_2)$$
Then $\phi(C)\subseteq Z$

(S is antipode map in Hopf algebra)

" Want to connect R and S "

R: almost co-comm

S: antipode

Proof:

Given ~~almost cocomm.~~ Hopf alg A.

$(a_1\otimes a_2)*a = a_1\cdot a\cdot S(a_2)$ — like $a_1 a a_1^{-1}$ (conjugation)

is $A\otimes A$-module structure on A.

Note $\phi(x) = x*1$.

Claim: $\Delta(a)*1 = \eta(a)\cdot 1$ — counit in Hopf alg

Claim: $\Delta(a)*\phi(c) = (\Delta(a)\cdot c)*1$

$= (c\cdot\Delta(a))*1$ if ~~$c\in Z\otimes Z$~~ $c\in C$

$= c*\Delta a*1$

$= c*\eta(a)\cdot 1$

Define $\Psi : A\otimes A\otimes A \longrightarrow A$ by

$$a_1\otimes a_2\otimes a_3 \longmapsto a_1\cdot\phi(c)\cdot S(a_2)\cdot a_3 = [(a_1\otimes a_2)*\phi(c)]\cdot a_3$$

Note: $\Psi((\Delta\otimes id)\circ\Delta) = \Psi((id\otimes\Delta)\circ\Delta)$ (coassociativity)

$$\Psi(\Delta(\Delta_1)\otimes\Delta_2) = \Psi(\Delta_1\otimes\Delta(\Delta_2))$$

$$\begin{array}{cc} \Psi(\Delta(\Delta_1)\otimes\Delta_2) & \Psi(\Delta_1\otimes\Delta(\Delta_2)) \\ \| & \| \\ (\Delta(\Delta_1)*\phi(c))\,\Delta_2 & \Delta_1\cdot\phi(c)\cdot S(\Delta_1\Delta_2)\cdot\Delta_2\Delta_2 \\ \| & \| \\ \phi(c)\cdot(\eta\otimes id)\,\Delta & \Delta_1\cdot\phi(c)\cdot\eta(\Delta_2) \\ \| & \| \\ \underline{\underline{\phi(c)\cdot id}} & (id\otimes\eta)\circ\Delta\cdot\phi(c) = \underline{\underline{id\cdot\phi(c)}} \end{array}$$

So $\phi(c) \subset Z$. ∎

Let A be almost cocomm. Hopf alg.

$$(\tau\circ\Delta)(a) = R\cdot\Delta(a)\cdot R^{-1} \qquad (R = R_1\otimes R_2)$$

(neglect $\sum$)

$$\Longrightarrow \Delta_2\otimes\Delta_1 = R_1\cdot\Delta_1\cdot R_1^{-1}\otimes R_2\cdot\Delta_2\cdot R_2^{-1}$$

So
$$\Delta_1 = R_2\cdot\Delta_2\cdot R_2^{-1} = R_2\cdot R_1\cdot\Delta_1\cdot(R_2\cdot R_1)^{-1}$$
$$\Delta_2 = R_1\cdot\Delta_1\cdot R_1^{-1} = R_1\cdot R_2\cdot\Delta_2\cdot(R_1R_2)^{-1}$$

$\Longrightarrow$ $R_2R_1\otimes R_1R_2$ commutes w/ $\Delta(a)$ all a

$$\Longrightarrow \underline{\underline{\phi(R_2R_1\otimes R_1R_2)\in \text{center}(A)}}$$

<u>Relation between R and S:</u>

$$\left(\begin{array}{l} R_{21} = \tau(R) \\ \Rightarrow R_{21} = R_2\otimes R_1 \end{array}\right)$$

$$\boxed{\begin{array}{l}\text{Prop: Let } u = S(R_2)\otimes R_1 \in A. \\ \text{then ① } u \text{ is invertible in } A \\ \qquad ② \ S^2(a) = uau^{-1}. \end{array}}$$

<u>Proof:</u>

① Show $ua = S^2(a)\,u$

(Neglect Σ)

almost cocomm $\Longrightarrow \Delta_2\otimes\Delta_1 = R_1\Delta_1R_1^{-1}\otimes R_2\Delta_2R_2^{-1}$

i.e. $(\Delta_2R_1)\otimes(\Delta_1R_2) = (R_1\Delta_1)\otimes(R_2\Delta_2)$

$$(R_1\Delta_1 \otimes R_2\Delta_2)\Delta_1 \otimes \Delta_2 = (\Delta_2 R_1 \otimes \Delta_1 R_2)\Delta_1 \otimes \Delta_2$$

$$(R_1\Delta_1\Delta_1) \otimes (R_2\Delta_2\Delta_1) \otimes \Delta_2 \qquad \Delta_2 R_1 \Delta_1 \otimes \Delta_1 R_2 \Delta_1 \otimes \Delta_2$$

$\cancel{(R \otimes 1)(\Delta \otimes id)\circ\Delta}$ $\qquad$ $\cancel{((\tau\circ\Delta)\otimes id)\circ(R\otimes 1)}\cdot\Delta$

Apply $\underline{id \otimes S \otimes S^2}$

$\cancel{R_1\Delta_1\Delta_1 \otimes R_2\Delta_2\Delta_1}$

$$(R_1\Delta_1\Delta_1) \otimes S(R_2\Delta_2\Delta_1) \otimes (S^2\Delta_2) = (\Delta_2 R_1 \Delta_1) \otimes (S(\Delta_1 R_2 \Delta_1)) \otimes (S^2(\Delta_2))$$

multiply in reverse order

$$S^2(\Delta_2)\cdot S(R_2\Delta_2\Delta_1)\cdot(R_1\Delta_1\Delta_1) = S^2(\Delta_2)\cdot S(\Delta_1 R_2 \Delta_1)\cdot(\Delta_2 R_1 \Delta_1)$$

$$S(\Delta_2\cdot S(\Delta_2\Delta_1))\cdot S(R_2)\cdot(R_1\Delta_1\Delta_1) \qquad S^2(\Delta_2)\, S(\Delta_1\Delta_1)\cdot S(R_2)\cdot(\Delta_2\Delta_1 R_1)$$

$\parallel \leftarrow$ $\underline{\text{coass} \;\&\; S}$ $\qquad$ $\parallel \leftarrow$ $\underline{\text{coass and } S}$

$$S(R_2)\cdot R_1\; id \qquad S^2(id)\; S(R_2)\cdot R_1$$

$$ua \qquad S^2(a)\,u$$

② Show u is invertible.

(inverse of R gives inverse of u)

$\underline{\text{claim}}$: $v = \mu(S^{-1}\otimes id)(R_{21}^{-1})$ is the inverse of u.

$$\left(\text{"} \; v = S^{-1}R_2^{-1} \otimes R_1^{-1} \text{"}\right)$$

$$(S^{-1}R_2^{-1}\otimes R_1^{-1})\cdot(SR_2\otimes R_1) = 1\otimes 1$$

only hard if you do not neglect Σ. $\blacksquare$