**Tensor Functor** is "functor between tensor cats respecting tensor str."

$$(\mathcal{C}, \otimes_c, I_c, a_c, l_c, r_c) \longrightarrow (\mathcal{D}, \otimes_D, I_D, a_D, l_D, r_D)$$

tensor functor is
$$\begin{cases} F: \mathcal{C} \longrightarrow \mathcal{D} & \text{functor.} \\ \phi_0 : I \xrightarrow{\cong} F(I) & \text{isomorphism.} \\ \phi_2 : F(C)\otimes F(C) \xrightarrow{\cong} F(C\otimes C) & \text{isomorphism.} \end{cases}$$

- we will work with "strict categories" where $\phi_0$ and $\phi_2$ are identity.
  → don't worry much about $\phi_0, \phi_2$ here....

→ Diagrams w/ $a, l, r$ should commute w/ $\phi_0, \phi_2$.

**Tensor Natural Transf.**
$$\begin{cases} F, F' : \mathcal{C} \longrightarrow \mathcal{D} \\ \phi_0, \phi_0' ; \phi_2, \phi_2' \end{cases}$$

$$\eta : (F, \phi_0, \phi_2) \longrightarrow (F', \phi_0', \phi_2')$$

is Natural transf $\eta : F \longrightarrow F'$ respecting $\phi_0$ & $\phi_2$

i.e.
$$\begin{tikzcd} & F(I) \arrow[d, "\eta"] \\ I \arrow[ur, "\phi_0"] \arrow[r, "\phi_0'"'] & F'(I) \end{tikzcd}$$
and
$$\begin{tikzcd} F(U)\otimes F(V) \arrow[r, "\phi_2"] \arrow[d, "\eta\otimes\eta"] & F(U\otimes V) \arrow[d, "\eta"] \\ F'(U)\otimes F'(V) \arrow[r, "\phi_2"] & F'(U\otimes V) \end{tikzcd}$$

**Tensor Isomorphism.**

Obvious.

Equivalence of Tensor Categories

Equivalence of Categories w/ tensor nat'l transf.

(& next week)

~~Next Week~~ Now: Interesting example #1 (Tangle category)

EX: Let $(\mathcal{C}, \otimes, I)$ be a strict tensor cat (i.e. $a$, $\ell$, $r$ are canonical isom.)

$\mathcal{F}$ a collection of morphisms in $\mathcal{C}$.

( → Idea: do something like writing a group as free group on generators / relations )

- Define words as follows
  - → word of rank 1 is $[f]$ where $f \in \mathcal{F}$ or $[id_V]$ where $V \in Ob(\mathcal{C})$
  - → word of rank 2 is $[f \circ g]$ or $[f \otimes g]$

  etc. ( rank $n+1$ is $[w_1 \circ w_2]$ or $[w_1 \otimes w_2]$, $w_1, w_2$ rank $n$ )

- Define subwords as follows
  - → rank 1 word itself
  - → rank >1 subwords of $w_1$ and $w_2$.

- Define corresp. morphisms by

$$\underset{\text{word}}{a} \longleftrightarrow \underset{\text{morphism in } \mathcal{C}}{\bar{a}}$$

- Define <u>basic relations</u> among words to get all equivalences

$a \sim a'$ <u>equivalent</u> if there is a sequence

$a = a_0, a_1, a_2, \ldots, a_n = a'$

where each step $a_i \to a_{i+1}$

replaces a subword $x \longleftrightarrow y$

where $x \sim y$ by <u>basic relation</u>

<u>basic relations</u> are

$([f]\circ[g])\circ[h] \sim [f]\circ([g]\circ[h])$

pure composition

$[id]\circ[f] \sim [f]$

$[f]\circ[id] \sim [f]$

$([f]\otimes[g])\otimes[h] \sim [f]\otimes([g]\otimes[h])$

pure tensor

$[id]\otimes[f] \sim [f]$

$[f]\otimes[id] \sim [f]$

mix

$([f]\otimes[g])\circ([f']\otimes[g']) \sim ([f]\circ[f'])\otimes([g]\circ[g'])$

<u>Lemma</u>: The following ~~follow~~ come from basic relations:

(a) $([f]\otimes[id])\circ([id]\otimes[g]) \sim ([id]\otimes g)\circ([f]\otimes[id])$

(b) $([id]\otimes[f_1]\otimes[id])\circ([id]\otimes[f_2]\otimes[id])\circ\cdots\circ([id]\otimes[f_n]\otimes[id])$

$$\wr$$

$[id]\otimes([f_1]\circ\cdots\circ[f_n])\otimes[id]$

(c) All words are equivalent to either something like (b) or to $[id]$.

• Let $\mathcal{M}(F)$ denote $\text{words}/\sim$

$\rightarrow$ $\mathcal{M}(F)$ gives a strict tensor category

$$\mathcal{C}(\mathcal{F}) \quad \text{has} \quad \begin{cases} \text{objects} = \text{objects of } \mathcal{C} \\ \text{morphisms} = \mathcal{M}(F) \end{cases}$$