<u>Recall</u> $u_{q,n}(sl_2) = \mathbb{C}\langle E, F, g, g^{-1}\rangle / \sim$

$(q \in \mathbb{C}^* \text{ w/ } q^n = 1)$

$$\underbrace{\begin{array}{l} gEg^{-1} \sim q^2 E \\ gFg^{-1} \sim q^{-2} F \\ [E,F] \sim \dfrac{g-g^{-1}}{q-q^{-1}} \end{array}}_{\text{stuff w/ } q} \qquad \underbrace{\begin{array}{l} g^n \sim 1 \\ E^n \sim 0 \\ F^n \sim 0 \end{array}}_{\text{stuff w/}}$$

<u>coalgebra structure</u>:

$$\Delta g = g \otimes g \qquad\qquad \eta(g) = 1$$
$$\Delta E = E \otimes g + 1 \otimes E \qquad\qquad \eta(E) = \eta(F) = 0$$
$$\Delta F = F \otimes 1 + g^{-1} \otimes F$$

<u>Hopf structure</u>:

$$Sg = g^{-1}$$
$$SE = -Eg^{-1}$$
$$SF = -gF$$

We should check that Δ, η, S ~~give Hopf alg. structures~~ are well-defined.

$\to$ must respect the relations

- $$\begin{aligned} \Delta(gEg^{-1}) &= \Delta g \; \Delta E \; \Delta g^{-1} \\ &= (g \otimes g)(E \otimes g + 1 \otimes E)(g^{-1} \otimes g^{-1}) \\ &= gEg^{-1} \otimes g + 1 \otimes gEg^{-1} \\ &= q^2 E \otimes g + 1 \otimes q^2 E \\ &= \Delta(q^2 E) \end{aligned}$$

- $\Delta(gFg^{-1}) = \Delta(q^{-2}F)$ is similar

- $$\begin{aligned} \Delta[E,F] &= [\Delta E, \Delta F] \\ &= [E \otimes g + 1 \otimes E, F \otimes 1 + g^{-1} \otimes F] \\ &= [E,F] \otimes g + (\underbrace{Eg^{-1}}_{q^2 g^{-1}E} \otimes \underbrace{gF}_{q^{-2}Fg} - g^{-1}E \otimes Fg) + g^{-1} \otimes [E,F] \\ &= \frac{g-g^{-1}}{q-q^{-1}} \otimes g + \quad 0 \quad + g^{-1} \otimes \frac{g-g^{-1}}{q-q^{-1}} \end{aligned}$$

$= \Delta\left(\dfrac{g-g^{-1}}{q-q^{-1}}\right)$

(2)

- $\Delta g^n = (\Delta g)^n = g^n \otimes g^n = 1 \otimes 1 = \Delta(1)$

- $\Delta E^n = (\Delta E)^n = (E \otimes g + 1 \otimes E)^n$

$$\left[\text{Note: } (E\otimes g)(1\otimes E) = E \otimes gE = E \otimes q^2 Eg = q^2 (1\otimes E)(E\otimes g) \right]$$

Recall: q-binomial lemma said if $AB = q^2 BA$ then

$$(A+B)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-2}} A^k B^{n-k}$$

so $$(E\otimes g + 1 \otimes E)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-2}} E^k \otimes g^k E^{n-k}$$

$k=n$: $E^n \otimes g^n = 0 \otimes 1 = 0$

$k=0$: $E^0 \otimes g^0 E^n = 1 \otimes 0 = 0$

$0<k<n$: $\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-2}} = \dfrac{[n]_{q^{-2}}!}{[k]_{q^{-2}}!\,[n-k]_{q^{-2}}!}$

but $[n]_{q^{-2}}! = [n]_{q^{-2}} [n-1]_{q^{-2}} \cdots [1]_{q^{-2}}$

& $[n]_{q^{-2}} = 1 + q^{-2} + q^{-4} + \dots + q^{-2n}$
$= 0$ if q is $2n^{\text{th}}$ root of 1.

- $\Delta F^n = 0$ similarly.

Similar (but simpler) work shows η and S well-defined

Note: S is anti-homom: so for example,

$$\begin{aligned} S(gEg^{-1}) &= S(g^{-1})\,S(E)\,S(g) \\ &= g\,(-Eg^{-1})\,g^{-1} \\ &= -gEg^{-2} \\ &= -q^2 E g^{-1} \\ &= q^2 S(E) \\ &= S(q^2 E) \end{aligned}$$

③

Quasi-Triangular Structure

Recall: $R \in H \otimes H$ is called an R-matrix if

- R is invertible
- $R \Delta R^{-1} = \Delta^{op}$
- $\left.\begin{aligned} (\Delta \otimes id) R &= R_{13} R_{23} \\ (id \otimes \Delta) R &= R_{13} R_{12} \end{aligned}\right\}$

Claim: $u_{q,n}(sl_2)$ has R-matrix which can be written explicitly!

$$R = R_g \, e_{q^{-2}}^{(q-q^{-1}) E \otimes F} \quad \text{where} \quad \begin{cases} R_g = \frac{1}{n} \sum\limits_{a,b=0}^{n-1} q^{-2ab} g^a \otimes g^b \\ e_{q^{-2}} \text{ is "} q\text{-exponential"} \\ \quad = \sum\limits_{r=0}^{n-1 \; \cancel{\infty}} \frac{(q-q^{-1})^r E^r \otimes F^r}{[r]_{q^{-2}}!} \end{cases}$$

R-matrix for cyclic grp

finite b/c $E^n = F^n = 0$

Check: $R \Delta R^{-1} = \Delta^{op}$ on generators

- $R(\Delta E) \overset{?}{=} (\Delta^{op} E) R$

$$\Updownarrow$$

$$R(E \otimes g + 1 \otimes E) = (g \otimes E + E \otimes 1) R$$

Claim: $R_g (1 \otimes E) = (g \otimes E) R_g$

$$\left(\frac{1}{n} \sum' q^{-2ab} g^a \otimes g^b\right)(1 \otimes E) = \frac{1}{n} \sum' q^{-2ab} g^a \otimes g^b E$$

$$= \frac{1}{n} \sum' q^{-2ab} g^a \otimes q^{2b} E g^b$$

$$= \frac{1}{n} \sum' q^{-2(a-1)b} g^a \otimes E g^b$$

cyclic sum

$$= \frac{1}{n} \sum' q^{-2ab} g^{a+1} \otimes E g^b$$

$$= (g \otimes E) R_g$$

R_g is symmetric so this implies

$$R_g (E \otimes g) = (E \otimes g) R_g$$

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So

$$R_g(E\otimes g^{-1} + 1\otimes E) = (E\otimes 1 + g\otimes E) R_g$$

$\to$ What about $e_{q^{-2}}^{\cdots}$??

Use previous technical lemma about moving quantum exponentials.

....

- $R(\Delta F) = (\Delta^{op} F) R$ similar

- $R(\Delta_g) = (\Delta^{op}_g) R$

Need that $(g\otimes g)(E^r \otimes F^r) = (E^r\otimes F^r)(g\otimes g)$

$$\left(\text{b/c } \begin{array}{l} gE^r = q^{2r} E^r g \\ gF^r = q^{-2r} F^r g \end{array} \right)$$

- $(\Delta \otimes id) R = R_{13} R_{23}$

This holds for R_g b/c it is R-matrix for $\mathbb{C}_{q^2}\mathbb{Z}/n$ (from before)

On exponentials,

$$(\Delta\otimes id)\, e_{q^{-2}}^{(q-q^{-1})E\otimes F} = e_{q^{-2}}^{(q-q^{-1})(\Delta\otimes id)(E\otimes F)}$$

$$= e_{q^{-2}}^{(q-q^{-1})(E\otimes g\otimes F + 1\otimes E\otimes F)}$$

$$= e_{q^{-2}}^{(q-q^{-1})E\otimes g\otimes F} \cdot e_{q^{-2}}^{(q-q^{-1})1\otimes E\otimes F}$$

(because $(E\otimes g\otimes F)(1\otimes E\otimes F) = q\,(1\otimes E\otimes F)(E\otimes g\otimes F)$

(b/c $gE = qEg$)

R_{13} the g is there b/c of R_g

R_{23}

- $R^{-1} = e_{q^2}^{-(q-q^{-1})E\otimes F} R_g^{-1}$
