Özgür Kişisel — Quantum Groups VI — 20.11.2012

Diagramatics of Actions:

→ Algebra H acting on k-module V

$$\begin{tikzcd} & H\otimes H\otimes V \arrow[dl, "m\otimes id"'] \arrow[dr, "id\otimes\triangleright"] & \\ H\otimes V \arrow[dr, "\triangleright"'] & & H\otimes V \arrow[dl, "\triangleright"] \\ & V & \end{tikzcd}$$

(Notation: $\triangleright$ is action map $\triangleright : H\otimes V \to V$)

→ (action equalizes $H\otimes H\otimes V \underset{id\otimes\triangleright}{\overset{m\otimes id}{\rightrightarrows}} H\otimes V$)

$$\begin{tikzcd} H\otimes V \arrow[dr, "\triangleright"] & \\ k\otimes V \arrow[u, "\varepsilon\otimes id"] \arrow[r, phantom, "\cong"] & V \end{tikzcd}$$

→ We can add to this by asking for V to be an algebra/coalgebra & requiring action to be compatible w/ extra structure of V

Diagramatics of Coactions:

→ Coalgebra C acting on k-module V

$$\begin{tikzcd} & V \arrow[dl, "\beta"'] \arrow[dr, "\beta"] & \\ C\otimes V \arrow[dr, "\Delta\otimes id"'] & & C\otimes V \arrow[dl, "id\otimes\beta"] \\ & C\otimes C\otimes V & \end{tikzcd}$$

(Notation: β is coaction map $\beta : V \longrightarrow C\otimes V$)

→ (coaction coequalizes $C\otimes V \underset{id\otimes\beta}{\overset{\Delta\otimes id}{\rightrightarrows}} C\otimes C\otimes V$)

$$\begin{tikzcd} k\otimes V \arrow[r, phantom, "\cong"] & V \arrow[dl, "\beta"] \\ C\otimes V \arrow[u, "\eta\otimes id"] & \end{tikzcd}$$

Notation: $\beta(v) = v^{(1)} \otimes v^{(2)}$ $\begin{cases} v^{(1)} \in C \\ v^{(2)} \in V \end{cases}$

"V is a k-comodule for the C-coaction"

→ If C is a bialgebra it can act as algebra

Remark: Using this notation, coalgebra coacts if

$$v^{(1)}_{(1)} \otimes v^{(1)}_{(2)} \otimes v^{(2)} = \underbrace{v^{(1)} \otimes v^{(2)} \otimes v^{(3)}}_{\text{i.e. } v^{(1)} \otimes v^{(2)(1)} \otimes v^{(2)(2)}}$$

Def: A bialgebra H coacts on an algebra A if

① A is an H-comodule

② The coaction $\beta : A \to H \otimes A$ is an alg. homom.

(i.e.
$$\begin{tikzcd} A \otimes A \arrow[r, "\beta\otimes\beta"] \arrow[d, "m"'] \arrow[dr, phantom, "\circlearrowleft"] & H\otimes A \otimes H \otimes A \arrow[d, "m_\otimes"] \\ A \arrow[r, "\beta"] & H \otimes A \end{tikzcd}$$
)

→ bialgebra H coacting on coalgebra is similar.

(i.e.
$$\begin{tikzcd} C \otimes C \arrow[r, "\beta\otimes\beta"] \arrow[dr, phantom, "\circlearrowleft"] & H\otimes C \otimes H \otimes C \\ C \arrow[r, "\beta"] \arrow[u, "\Delta"] & H \otimes C \arrow[u, "\Delta_\otimes"'] \end{tikzcd}$$
)

Examples.

- A coalgebra coacts on itself by Δ.

- If V, W are H-comodules then $V \otimes W$ also has H-comodule structure via

$$\beta_{V\otimes W}(v \otimes w) = v^{(1)} w^{(1)} \otimes \left(v^{(2)} \otimes w^{(2)}\right)$$

(→ $v^{(1)} w^{(1)} \in H$ (use product in H)
$v^{(2)} \otimes w^{(2)} \in V \otimes W$)

⇝ H-comodules is a monoidal category.
(later we will introduce braidings & get invariants)

Def: A morphism/intertwiner $\phi : V \to W$ between comodules is map compatible w/ coaction.

i.e. $(id \otimes \phi) \circ \beta_V = \beta_W \circ \phi$

$$\left(\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta} \\ C \otimes V & \xrightarrow[id \otimes \phi]{} & C \otimes W \end{array}\right)$$

Prop: Every Hopf algebra H coacts on itself as a coalgebra by

$$Ad(h) = h_{(1)} S h_{(3)} \otimes h_{(2)}$$ "Adjoint coaction"

Proof:

- (coaction)

$$\begin{aligned}(id \otimes Ad) \circ Ad(h) &= (id \otimes Ad)\, h_{(1)} S h_{(3)} \otimes h_{(2)} \\ &= h_{(1)} S h_{(5)} \otimes \left(h_{(2)} S h_{(4)} \otimes h_{(3)}\right) \\ &= \text{``}(\Delta \otimes id)\, h_{(1)} h_{(2)} S(h_{(4)} h_{(5)}) \otimes h_{(3)}\text{''} \\ &= (\Delta \otimes id)\left(h_{(1)} S h_{(3)} \otimes h_{(2)}\right) \\ &= (\Delta \otimes id) \circ Ad(h)\end{aligned}$$

(Note: S reverses order) $\rightsquigarrow$

- (counit)

$$\begin{aligned}(id \otimes \eta) \circ Ad(h) &= (id \otimes \eta)\; h_{(1)} S h_{(3)} \otimes h_{(2)} \\ &= h_{(1)} S h_{(3)}\, \eta(h_{(2)}) \\ &= \underbrace{h_{(1)} \eta(h_{(2)})}\, S h_{(3)} \\ &= h_{(1)} S h_{(2)} = \eta(h)\end{aligned}$$

- (respects coalg str.)

$$\begin{array}{ccc} H \otimes H & \xrightarrow{Ad \otimes Ad} & (H \otimes H) \otimes (H \otimes H) \\ {\scriptstyle\Delta}\uparrow & & \uparrow{\scriptstyle\Delta_\otimes} \\ H & \xrightarrow[Ad]{} & H \otimes H \end{array}$$

Homework (notation will be terrible!)

$\blacksquare$

Example: Quantum Plane.

The algebra $\mathbb{A}_q^2$ is $\dfrac{k\langle x,y\rangle}{(yx = qxy)}$

$\hookrightarrow$ Right $SL_q(2)$-comodule algebra under

$$\beta : \mathbb{A}_q^2 \longrightarrow SL_q(2) \otimes \mathbb{A}_q^2$$

$$[x \;\; y] \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{cases} \beta(x) = x \otimes a + y \otimes c \\ \beta(y) = x \otimes b + y \otimes d \end{cases}$$

Recall: $SL_q(2)$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

w/

$ba = qab$ $\quad bc = c$

$ca = qac$ $\quad \cancel{}$

$db = qbd$

$dc = qcd$

$ad - da = (q^{-1} - q$

$ad - q^{-1}bc = 1$

Remark: It is not immediately clear that this is well-defined

Ex $\beta(yx) = q\,\beta(xy)$