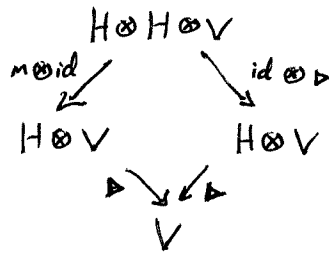


Diagrammatics of Actions:

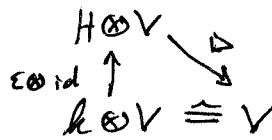
→ Algebra H acting on k -module V



(Notation: \triangleright is action map)
 $\triangleright : H \otimes V \rightarrow V$

→ action equalizes

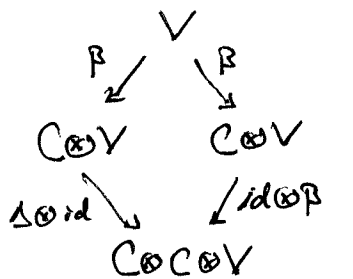
$$\begin{array}{ccc}
 H \otimes H \otimes V & \xrightarrow{m \otimes id} & H \otimes V \\
 & \xrightarrow{id \otimes \triangleright} &
 \end{array}$$



→ We can add to this by asking for V to be an algebra/coalgebra & requiring action to be compatible w/ extra structure of V

Diagrammatics of Coactions:

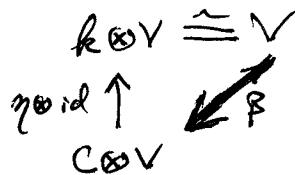
→ Coalgebra C acting on k -module V



(Notation: β is coaction map)
 $\beta : V \rightarrow C \otimes V$

→ coaction coequalizes

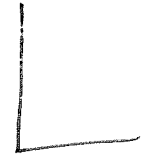
$$\begin{array}{ccc}
 C \otimes V & \xrightarrow{\Delta \otimes id} & C \otimes C \otimes V \\
 & \xrightarrow{id \otimes \beta} &
 \end{array}$$



Notation: $\beta(v) = v^{(1)} \otimes v^{(2)}$ $\begin{cases} v^{(1)} \in C \\ v^{(2)} \in V \end{cases}$
 "V is a k -comodule for the C -coaction"

→ If C is a bialgebra it can act as algebra

Remark: Using this notation, coalgebra coacts if



$$v^{(1)}_{(1)} \otimes v^{(1)}_{(2)} \otimes v^{(2)} = v^{(1)}_{(1)} \otimes v^{(2)}_{(1)} \otimes v^{(2)}_{(2)}$$

i.e. $v^{(1)}_{(1)} \otimes v^{(2)}_{(1)} \otimes v^{(2)}_{(2)}$

Def: A bialgebra H coacts on an algebra A if

- ① A is an H -comodule
- ② The coaction $\beta: A \rightarrow H \otimes A$ is an alg. homom.

(i.e.
$$\begin{array}{ccc} A \otimes A & \xrightarrow{\beta \otimes \beta} & H \otimes A \otimes H \otimes A \\ m \downarrow & \cong & \downarrow m \\ A & \xrightarrow{\beta} & H \otimes A \end{array}$$
)

\rightarrow bialgebra H coacting on coalgebra is similar.

(i.e.
$$\begin{array}{ccc} C \otimes C & \xrightarrow{\beta \otimes \beta} & H \otimes C \otimes H \otimes C \\ \Delta \uparrow & \cong & \uparrow \Delta \\ C & \xrightarrow{\beta} & H \otimes C \end{array}$$
)

Examples.

- A coalgebra coacts on itself by Δ .
- If V, W are H -comodules then $V \otimes W$ also has H -comodule structure via

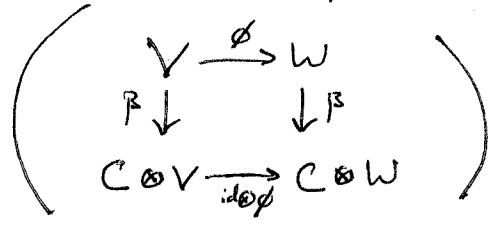
$$\beta_{V \otimes W}(v \otimes w) = v^{(1)} w^{(1)} \otimes (v^{(2)} \otimes w^{(2)})$$

($\rightarrow v^{(1)} w^{(1)} \in H$ (use product in H)
 $v^{(2)} \otimes w^{(2)} \in V \otimes W$)

\rightarrow H -comodules is a monoidal category.
 (later we will introduce braidings & get invariants)

Def: A morphism/intertwiner $\phi: V \rightarrow W$ between comodules is map compatible w/ coaction.

i.e. $(id \otimes \phi) \circ \beta_V = \beta_W \circ \phi$



Prop: Every Hopf algebra H coacts on itself as a coalgebra by $Ad(h) = h_{(1)} S h_{(3)} \otimes h_{(2)}$ "Adjoint coaction"

Proof:

• (Coaction) $(id \otimes Ad) \circ Ad(h) = (id \otimes Ad) h_{(1)} S h_{(3)} \otimes h_{(2)}$
 $= h_{(1)} S h_{(3)} \otimes (h_{(2)} S h_{(4)} \otimes h_{(3)})$
 (Note: S reverses order) $\rightsquigarrow = (\Delta \otimes id) h_{(1)} h_{(2)} (S h_{(4)} S h_{(5)}) \otimes h_{(3)}$
 $= (\Delta \otimes id) (h_{(1)} S h_{(3)} \otimes h_{(2)})$
 $= (\Delta \otimes id) \circ Ad(h)$

• (Co-unit) $(id \otimes \eta) \circ Ad(h) = (id \otimes \eta) h_{(1)} S h_{(3)} \otimes h_{(2)}$
 $= h_{(1)} S h_{(3)} \eta(h_{(2)})$
 $= \underbrace{h_{(1)} \eta(h_{(2)})}_{= \epsilon(h)} S h_{(3)}$
 $= h_{(1)} S h_{(3)} = \eta(h)$

• (respects coalg str.)

$$\begin{array}{ccc} H \otimes H & \xrightarrow{Ad \otimes Ad} & (H \otimes H) \otimes (H \otimes H) \\ \Delta \uparrow & & \uparrow \Delta \otimes \\ H & \xrightarrow{Ad} & H \otimes H \end{array}$$

Homework (notation will be terrible!)



Example: Quantum Plane.

The algebra A_q^2 is $\frac{k\langle x, y \rangle}{(yx = qxy)}$

\hookrightarrow Right $SL_q(2)$ -comodule algebra under

$$\beta: A_q^2 \longrightarrow SL_q(2) \otimes A_q^2$$

$$[x \ y] \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{cases} \beta(x) = x \otimes a + y \otimes c \\ \beta(y) = x \otimes b + y \otimes d \end{cases}$$

Remark: It is not immediately clear that this is well-defined

ex $\beta(yx) = q \beta(xy)$

Recall: $SL_q(2)$
 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$
w/
 $ba = qab$ $bc = 0$
 $ca = qac$ $cd = 0$
 $db = qbd$
 $dc = qcd$
 $ad - da = (q^{-1} - q)$
 $ad - q^{-1}bc = 1$