Recall from last time:

- Hopf algebras are <u>dually paired</u> if $\exists$
$$\langle -,- \rangle : H \otimes H' \longrightarrow k$$
w/
(1) $\langle \phi\psi, h \rangle = \langle \phi \otimes \psi, \Delta h \rangle$
(2) $\langle 1, h \rangle = \eta(h)$
(3) $\langle \Delta\phi, h \otimes g \rangle = \langle \phi, hg \rangle$
(4) $\langle \phi, 1 \rangle = \eta(\phi)$
(5) $\langle S\phi, h \rangle = \langle \phi, Sh \rangle$

- A <u>bialgebra</u> <u>acts</u> on a <u>k-module</u> if
$\Leftrightarrow$ underlying algebra acts on V

$\rightarrow$ <u>Note:</u> H acting on V & W $\Rightarrow$ interesting H-action on $V \otimes W$ (via Δ)

- A <u>bialgebra</u> <u>acts</u> on an algebra if
(1) underlying algebra acts on underlying k-module
(2) $m : A \otimes A \to A$ commutes w/ action of H (compatible w/ Δ)
$$h \triangleright (ab) = (h_{(1)} \triangleright a) \cdot (h_{(2)} \triangleright b)$$
(3) unit of A is compatible w/ action
$$h \triangleright 1_A = \eta(h) \cdot 1$$

Prop: (Left coregular action)
Suppose H' dually paired w/ H.
Then H' acts on H by

$$R^*_\phi(h) = h_{(1)} \langle \phi, h_{(2)} \rangle$$

for $\phi \in H'$, $h \in H$.

Proof:

We will show ~~same stuff~~ these properties

(1) (algebra action) $\quad R^*_{\phi_1}(R^*_{\phi_2}(h)) \overset{??}{=} R^*_{\phi_1\phi_2}(h)$

$$\begin{aligned} R^*_{\phi_1} R^*_{\phi_2} h &= R^*_{\phi_1} h_{(1)} \langle \phi_2, h_{(2)} \rangle \\ &= h_{(1)} \langle \phi_1, h_{(2)} \rangle \langle \phi_2, h_{(3)} \rangle \\ &= h_{(1)} \langle \phi_1 \phi_2, h_{(2)} \rangle \\ &= R^*_{\phi_1\phi_2}(h) \end{aligned}$$

$$R^*_1(h) \overset{??}{=} h$$

$$\begin{aligned} R_1^* h &= h_{(1)} \langle 1, h_{(2)} \rangle \\ &= h_{(1)} \varepsilon(h_{(2)}) \\ &= h \end{aligned}$$

(2) (multiplication commutes w/ Δ)

$$R^*_\phi(hg) \overset{??}{=} R^*_{\phi_{(1)}}(h) R^*_{\phi_{(2)}}(g)$$

$$\begin{aligned} R_\phi^*(hg) &= (hg)_{(1)} \langle \phi, (hg)_{(2)} \rangle \\ &= h_{(1)} g_{(1)} \langle \phi, h_{(2)} g_{(2)} \rangle \\ &= h_{(1)} g_{(1)} \langle \phi_{(1)}, h_{(2)} \rangle \langle \phi_{(2)}, g_{(2)} \rangle \\ &= h_{(1)} \langle \phi_{(1)}, h_{(2)} \rangle \; g_{(1)} \langle \phi_{(2)}, g_{(2)} \rangle \\ &= R^*_{\phi_{(1)}}(h) \cdot R^*_{\phi_{(2)}}(g) \end{aligned}$$

(3) (unit is compatible) $\quad R^*(1) = 1 \langle \phi, 1 \rangle = \varepsilon(\phi)$

Examples of left coregular actions:

(1) $k(G)$ acts on kG (G a finite group)

(functions on G) (group algebra of G)

Recall: on kG, $\Delta g = g \otimes g$ so $g_{(1)} = g_{(2)} = g$

$$\hookrightarrow R^*_\phi(g) = g\langle \phi, g\rangle$$
$$= \phi(g)\cdot g$$

(2) $\mathbb{C}[G]$ acts on $\mathcal{U}(\mathfrak{g})$

(function algebra on group) (univ. enveloping alg of Lie alg.)

Recall: on $\mathcal{U}(\mathfrak{g})$, $\Delta\xi = \xi\otimes 1 + 1\otimes\xi$

$$R^*_{x^i_j}(\xi) = \xi_{(1)}\langle x^i_j, \xi_{(2)}\rangle$$

$$\hookrightarrow R^*_{x^i_j}(\xi) = \xi\langle x^i_j, 1\rangle + 1\langle x^i_j, \xi\rangle$$

$$= \xi\cdot\delta^i_j + (\xi)^i_j\cdot 1$$

(3) kG acting on $k(G)$ (G a finite group)

$$R^*_g(\phi) = \phi_{(1)}\langle g, \phi_{(2)}\rangle \in k(G)$$

$$R^*_g(\phi)(h) = \phi_{(1)}(h)\cdot\langle g, \phi_{(2)}\rangle$$
$$= \phi_{(1)}(h)\cdot\phi_{(2)}(g)$$
$$= \langle\phi, hg\rangle$$

(so $R^*_g(\phi) = \langle\phi, -g\rangle$)

(4) $\mathcal{U}(\mathfrak{g})$ acting on $\mathbb{C}[G]$

$\hookrightarrow$ EXERCISE

Dual to actions are <u>coactions</u>: (first we act on coalgebras)

Def: A <u>bialgebra</u> <u>acts</u> on a <u>coalgebra</u> if

(1) coalgebra is an H-module.

(2) $\Delta : C \to C \otimes C$ compatible w/ action of H & counit of C compat w/ H.

$$\Delta(h \triangleright c) = (h_{(1)} \triangleright c_{(1)}) \otimes (h_{(2)} \triangleright c_{(2)})$$

Prop: (Coadjoint action)

Suppose H' is dually paired w/ H.

Then H' <u>acts</u> on H <u>as a coalgebra</u> by

$$Ad^*_\phi(h) = h_{(2)} \langle \phi, (Sh_{(1)}) h_{(3)} \rangle$$

Proof: (Partial)

(1) Coalgebra action $\quad Ad^*_\phi Ad^*_\psi h \overset{??}{=} Ad^*_{\phi\psi}(h) \overset{??}{=} \cancel{\Delta Ad^*_\phi(h_{(1)}) Ad^*_\psi(h_{(2)})}$

$$Ad^*_\phi Ad^*_\psi h = h_{(3)} \langle \phi, (Sh_{(2)}) h_{(4)} \rangle \langle \psi, (Sh_{(1)}) h_{(5)} \rangle$$

$$= h_{(3)} \langle \phi_{(1)}, Sh_{(2)} \rangle \langle \phi_{(2)}, h_{(4)} \rangle \langle \psi_{(1)}, Sh_{(1)} \rangle \langle \psi_{(2)}$$

$$= \langle (\phi\psi)_{(1)}, Sh_{(1)} \rangle h_{(2)} \langle (\phi\psi)_{(2)}, h_{(3)} \rangle$$

$$= h_{(2)} \langle \phi\psi, (Sh_{(1)}) h_{(3)} \rangle$$

$$= Ad^*_{\phi\psi}(h)$$

etc...

→ <u>Next Time</u>: Coactions !!