Previously: EXAMPLES: $k(G)$ & $u(G)$ are Hopf Algebras

PROP: comm, cocomm Hopf alg has $S^2 = id$.

Ex: Let G be a finite group — "Most natural example"

$k(G)$: algebra of functions on G w/ values in k

- $(fg)(x) = f(x).g(x)$ (alg str)
- $(\Delta f)(x \otimes y) = f(xy)$ (coalg str)
- $(Sf)(x) = f(x^{-1})$ (antipode)

Hopf Algebra Axioms are easy to verify (Exercise)

(Note: that $k(G) \otimes k(G) \cong k(G \times G)$)

Ex: Let G be an algebraic group over $\mathbb{C}$ (or some other field) and suppose $G \subset M_n(\mathbb{C})$ (affine)

$\mathbb{C}[G]$: ring of regular functions on G $= \mathbb{C}[x^i_j] / \langle p(x) = 0 \rangle$

ideal generated by polynom. vanishing on G.

- Alg. structure is usual
- Δ defined on generators is matrix mult:

$$\Delta x^i_j = \sum x^i_k \otimes x^k_j$$

- Counit $\eta(x^i_j) = \delta^i_j$
- Antipode $S(x^i_j)$ is ij entry of inverse of matrix

$\rightsquigarrow$ you can write this w/ cofactors....

Specific example:

$$k[SL_2] = k[a,b,c,d] / \langle \det\begin{pmatrix} a & b \\ c & d\end{pmatrix} = 1 \rangle$$

$$\begin{aligned}
\Delta a &= a \otimes a + b \otimes c \\
\Delta b &= a \otimes b + b \otimes d \\
\Delta c &= c \otimes a + d \otimes c \\
\Delta d &= c \otimes b + d \otimes d
\end{aligned}$$

$\left(\begin{array}{l} a \approx x^1_1 \\ b \approx x^1_2 \\ c \approx x^2_1 \\ d \approx x^2_2 \end{array} \right)$

$\Delta x^1_1 = x^1_1 \otimes x^1_1 + x^1_2 \otimes$

$\eta a = 1$ $\quad$ $\eta b = 0$

$Sa = d$ $\quad$ $Sb = -b$

$Sc = -c$ $\quad$ $Sd = a$

Specific example:

$SL_q(2) \qquad q \in k^*$

$SL_q(2) = k\langle a,b,c,d\rangle / \sim$ ← free assoc alg gen by a, b, c, d

$$\begin{cases} ca \sim qac & db \sim qbd \\ ba \sim qab & dc \sim qcd \\ & bc \sim cb \end{cases}$$

"Quantum Commutative"

$da - ad \sim (q - q^{-1})\, bc$

q-determinant: $ad - q^{-1} bc \sim 1$

→ Δ is the same as previous ex.

→ $\boxed{\begin{matrix} Sa = d & Sb = -qb \\ Sc = -q^{-1} c & Sd = a \end{matrix}}$ ← (b & c get $q \leftrightarrow q^{-1}$)

Another way to get $SL_q(2)$:

Rewrite: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} e & f \\ 0 & g \end{bmatrix} \begin{bmatrix} h & 0 \\ i & j \end{bmatrix} \Longrightarrow \begin{cases} a = eh + fi \\ b = fj \\ c = gi \\ d = gj \end{cases}$

(Generalization to $SL_q(n)$ will use bidiagonal matrices)

Consider $k\langle e, f, g, h, i, j\rangle$ w/ induced relations:

$$\left.\begin{matrix} fe = qef & \\ gf = qfg & eg = ge \\ ih = qhi & \\ ji = qij & hj = jh \end{matrix}\right\}$$ and functions from different matrices commute

↳ EX $da - ad = (gj)(eh + fi) - (eh + fi)(gj)$

$= gjeh + gjfi - ehgj - figj$

$= \cancel{ehgj} + q^2 figj - \cancel{ehgj} - figj$

$= (q^2 - 1)\, figj$

$= (q^2 - 1)\, q^{-1} \ldots (q - q^{-1})\, b \ldots$

↳ **EX** q-determinant ⇝ Notes: ordinary determinant of

$$\begin{bmatrix} e & f \\ 0 & g \end{bmatrix}\begin{bmatrix} h & 0 \\ i & j \end{bmatrix} = eg \cdot hj$$

$$\rightsquigarrow ad - q^{-1}bc = (eh + fi)(gj) - q^{-1}(fj)(gi)$$

$$= ehgj + \cancel{figj} - \cancel{q^{-1}q\,figj}$$

$$= eg \cdot hj$$

Questions:

- Is it important that matrix mult. was used to get underlying structure?
  - → way to use this to get simplified proofs?
- What does this look like in R-matrices?
  - → Jones polynomial written like this?
  - → Can you see other invariants?
- Understand this as cover of $SL_q(2)$ + some extra info.?