Chapter 5

Exercise 9 (easy)

Prove: If $E \subset F$ and E is linearly dependent, then so is F.

Proof:

If E is linearly dependent then there are vectors $\underline{e}_1, \dots, \underline{e}_k \in E$ with dependence relation

$$a_1\underline{e}_1 + \cdots + a_k\underline{e}_k = \underline{0} \qquad (\text{some } a_i \neq 0)$$

However, $E \subset F$ so $\underline{e}_1, \dots, \underline{e}_k \in F$ also. Thus this is also a dependence relation among vectors of F. ▨

Exercise 10 (easy)

Prove: If $E \subset F$ and F is linearly independent, then so is E.

Proof:

Let $\underline{e}_1, \dots, \underline{e}_k$ be a set of vectors in E. Suppose that $a_1\underline{e}_1 + \cdots + a_k\underline{e}_k = \underline{0}$ for some a_i. Since $E \subset F$ $\underline{e}_1, \dots, \underline{e}_k \in E \subset F$ are also vectors of F. F is linearly independent so the a_i above must all be 0. Thus E is linearly independent. ▨

Exercise 15 (medium)

Prove: E is linearly independent if and only if for every $F \subsetneq E$, $\mathrm{Span}(F) \neq \mathrm{Span}(E)$.

Proof:

($\Rightarrow$) Suppose E is linearly independent and let $F \subsetneq E$. Pick a vector $\underline{e} \in E \setminus F$. If $\mathrm{Span}(F) = \mathrm{Span}(E)$ then $\underline{e} \in \mathrm{Span}(E) = \mathrm{Span}(F)$.

Then $\underline{e}$ would be linearly dependent on $F \subset E \setminus \underline{e}$

So E would be a linearly dependent set, a contradiction.

$\#$

(Exercise 15 continued)

($\Leftarrow$) Suppose that, for every $F \subsetneq E$, $\mathrm{Span}(F) \neq \mathrm{Span}(E)$.
If E were linearly dependent, it would contain a vector $\underline{e}$ with $\underline{e} \in \mathrm{Span}(E \setminus \underline{e})$.
Let $F = E \setminus \underline{e}$. This gives a contradiction, since $\mathrm{Span}(F) = \mathrm{Span}(E)$. ∎

<u>Exercise 17</u> (hard)

<u>Prove</u>: If E, F are independent sets then
(1) $E \cap F$ is independent.
(2) $E \cup F$ is independent (and $E \cap F = \emptyset$) if and only if $\mathrm{Span}(E) \cap \mathrm{Span}(F) = \{\underline{0}\}$

<u>Proof</u>:

(1) If $\{\underline{v}_1, \dots, \underline{v}_k\} \subset E \cap F$ with
$$a_1 \underline{v}_1 + \cdots + a_k \underline{v}_k = \underline{0}$$
then because $\{\underline{v}_1, \dots, \underline{v}_k\} \subset E$ we must have $a_1, \dots, a_k = 0$.
Thus $E \cap F$ is independent.

(2)
($\Rightarrow$) Suppose $E \cup F$ is independent, and let $\underline{v} \in \mathrm{Span}(E) \cap \mathrm{Span}(F)$. Because $\underline{v} \in \mathrm{Span}(E)$ we can write
$$\underline{v} = a_1 \underline{e}_1 + \cdots + a_n \underline{e}_n \quad \text{with } \underline{e}_i \in E$$
Similarly
$$\underline{v} = b_1 \underline{f}_1 + \cdots + b_m \underline{f}_m \quad \text{with } \underline{f}_j \in F.$$

Thus
$$\underline{0} = \underline{v} - \underline{v} = (a_1 \underline{e}_1 + \cdots + a_n \underline{e}_n) - (b_1 \underline{f}_1 + \cdots + b_m \underline{f}_m).$$
But $E \cup F$ is independent so all a_i and $b_j = 0$.
Thus $\underline{v} = \underline{0}$.

(Exercise 17 continued)

($\Leftarrow$) Suppose E, F are independent sets and $\text{Span}(E) \cap \text{Span}(F) = \{\underline{0}\}$. If $\{\underline{e}_1, \ldots, \underline{e}_n, \underline{f}_1, \ldots, \underline{f}_m\} \subset E \cup F$ (with $\underline{e}_i \in E$ and $\underline{f}_j \in F$) so that

$$a_1 \underline{e}_1 + \cdots + a_n \underline{e}_n + b_1 \underline{f}_1 + \cdots + b_m \underline{f}_m = \underline{0},$$

then

$$a_1 \underline{e}_1 + \cdots + a_n \underline{e}_n = -b_1 \underline{f}_1 - \cdots - b_m \underline{f}_m .$$

However the left side above is in $\text{Span}(E)$ and the right side is in $\text{Span}(F)$. Thus

$$a_1 \underline{e}_1 + \cdots + a_n e_n = -b_1 \underline{f}_1 - \cdots - b_m \underline{f}_m \in \text{Span}(E) \cap \text{Span}(F)$$

so, it is $\underline{0}$. But E is linearly independent so $a_1 \underline{e}_1 + \cdots + a_n \underline{e}_n = \underline{0}$ forces $a_1 = a_2 = \cdots = a_n = 0$

Similarly because F is linearly indep $b_1 = b_2 = \cdots = b_m = 0$. ∎

Chapter 4 (long)

Exercise 16 Let E, F be subsets of a vector space.

Prove: If $E \subseteq F$ then $\text{Span}(E) \subseteq \text{Span}(F)$

Proof:

Suppose $E \subseteq F$ and let $\underline{v} \in \text{Span}(E)$. Then $\underline{v} = a_1 \underline{e}_1 + \cdots + a_n \underline{e}_n$ where $\underline{e}_i \in E$. But $\underline{e}_i \in F$ as well, so $\underline{v} \in \text{Span}(F)$. ∎

Prove: $\text{Span}(E \cup F) = \text{Span}(E) + \text{Span}(F)$

Proof:

($\subseteq$) If $\underline{v} \in \text{Span}(E \cup F)$ then

$$\underline{v} = a_1 \underline{e}_1 + \cdots + a_k \underline{e}_k + b_1 \underline{f}_1 + \cdots + b_\ell \underline{f}_\ell$$

where $\underline{e}_i \in E$ and $\underline{f}_j \in F$. Thus

$$\underline{v} = (a_1 \underline{e}_1 + \cdots + a_k \underline{e}_k) + (b_1 \underline{f}_1 + \cdots + b_\ell \underline{f}_\ell)$$

a sum of something in $\text{Span}(E)$ and something in $\text{Span}(F)$. So $\underline{v} \in \text{Span}(E) + \text{Span}(F)$

(Exercise 16 continued)

($\supseteq$) If $\underline{v} \in Span(E) + Span(F)$ then

$$\underline{v} = (a_1 \underline{e}_1 + \cdots + a_k \underline{e}_k) + (b_1 \underline{f}_1 + \cdots + b_\ell \underline{f}_\ell)$$
$$= a_1 \underline{e}_1 + \cdots + a_k \underline{e}_k + b_1 \underline{f}_1 + \cdots + b_\ell \underline{f}_\ell$$

So $\underline{v} \in Span(E \cup F)$. ☑

Prove: $Span(E \cap F) \subseteq Span(E) \cap Span(F)$.

Proof:

If $\underline{v} \in Span(E \cap F)$ then $\underline{v} = a_1 \underline{v}_1 + \cdots + a_k \underline{v}_n$ where $\underline{v}_i \in E \cap F$. Since each $\underline{v}_i \in E$, $\underline{v} \in Span(E)$. Similarly since each $\underline{v}_i \in F$, $\underline{v} \in Span(F)$.

Thus $Span(E \cap F) \subseteq Span(E) \cap Span(F)$. ☑

Exercise 28 (long)

Prove: $S + T = Span(S \cup T)$

Proof:

($\subseteq$) Let $\underline{v} \in S + T$. Then $\underline{v} = \underline{s} + \underline{t}$ where $\underline{s} \in S$ and $\underline{t} \in T$.

So $\underline{v} = 1 \cdot \underline{s} + 1 \cdot \underline{t} \in Span(S \cup T)$.

($\supseteq$) Let $\underline{v} \in Span(S \cup T)$. Then

$$\underline{v} = a_1 \underline{s}_1 + \cdots + a_n \underline{s}_n + b_1 \underline{t}_1 + \cdots b_m \underline{t}_m$$

where $\underline{s}_i \in S$ and $t_i \in T$.

However, S and T are subspaces, so

$a_1 \underline{s}_1 + \cdots + a_n \underline{s}_n \in S$

$b_1 t_1 + \cdots + b_m \underline{t}_m \in T$.

Thus $\underline{v} = (a_1 \underline{s}_1 + \cdots + a_n \underline{s}_n) + (b_1 \underline{t}_1 + \cdots + b_m \underline{t}_m) \in S + T$ ☑

(Exercise 28 continued)

Prove: $S \cap (S+T) = S$

Proof:

($\subseteq$) This is immediate: $S \cap (S+T) \subseteq S$.

($\supseteq$) Let $\underline{s} \in S$. Then $\underline{s} = \underline{s} + 0 \cdot \underline{t} \in S+T$. So $\underline{s} \in S \cap (S+T)$ ▨

Prove: $S+T = T+S$

Proof:

It is enough to show $\subseteq$ because the same argument works for $\supseteq$

($\subseteq$) If $\underline{v} \in S+T$ then $\underline{v} = \underline{s} + \underline{t}$ where $\underline{s} \in S$, $\underline{t} \in T$.
Addition is commutative, so $\underline{v} = \underline{s} + \underline{t} = \underline{t} + \underline{s} \in T+S$.

▨

Prove: If $S \subseteq T$ then $S+T = T$.

Proof:

Note that $S+T = \operatorname{Span}(S \cup T) \supseteq T$, so we only need to show $S+T \subseteq T$. Let $\underline{v} \in S+T$.
Then $\underline{v} = \underline{s} + \underline{t}$ where $\underline{s} \in S$ and $\underline{t} \in T$. But $S \subset T$ so $\underline{s} \in T$ also. Thus $\underline{s} + \underline{t} \in T$ (because T is a subspace)
Therefore $\underline{v} = \underline{s} + \underline{t} \in T$.

▨

Chapter 3

Exercise 12

Prove: If V and W are vector spaces, then so is $V \times W$.

$$\begin{cases} \bullet\ V \times W = \{(\underline{v}, \underline{w}) \text{ where } \underline{v} \in V,\ \underline{w} \in W\} \\ \bullet\ (\underline{v}_1, \underline{w}_1) + (\underline{v}_2, \underline{w}_2) = (\underline{v}_1 + \underline{v}_2,\ \underline{w}_1 + \underline{w}_2) \\ \bullet\ r(\underline{v}, \underline{w}) = (r\underline{v}, r\underline{w}) \end{cases}$$

Proof:

We must check the axioms:

Axiom 1. $(\underline{v}_1, \underline{w}_1) + (\underline{v}_2, \underline{w}_2) = (\underline{v}_1 + \underline{v}_2,\ \underline{w}_1 + \underline{w}_2)$

$= (\underline{v}_2 + \underline{v}_1,\ \underline{w}_2 + \underline{w}_1)$ because $+$ is commutative in V and W

$= (\underline{v}_2, \underline{w}_2) + (\underline{v}_1, \underline{w}_1)$