METU - NCC

LINEAR ALGEBRA
FINAL EXAM

Solutions

Code : *MAT 260*	Last Name:
Acad.Year : *2012-2013*	Name : Student No.:
Semester : *Spring*	Department : Section:
Date : *07.06.2013*	Signature :
Time : *9:00*	6 QUESTIONS ON 6 PAGES
Duration : *120 min*	TOTAL 100 POINTS

1. (12)	2. (18)	3. (15)	4. (21)	5. (18)	6. (16)		

1. *(12pts)* Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ -1 & 0 & 2 & 4 & 2 \\ 0 & 2 & 5 & 8 & 8 \\ 2 & 2 & 1 & 2 & 3 \\ 1 & 2 & 4 & 0 & 0 \end{bmatrix}$

(Hint: Use elementary row operations).

$$\det\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ -1 & 0 & 2 & 4 & 2 \\ 0 & 2 & 5 & 8 & 8 \\ 2 & 2 & 1 & 2 & 3 \\ 1 & 2 & 4 & 0 & 0 \end{bmatrix} = \det\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 5 & 8 & 7 \\ 0 & 2 & 5 & 8 & 8 \\ 0 & -2 & -5 & -6 & -7 \\ 0 & 0 & 1 & -4 & -5 \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 5 & 8 & 7 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & -4 & -5 \end{bmatrix}$$

$$= -\det\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 5 & 8 & 7 \\ 0 & 0 & 1 & -4 & -5 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= -1 \cdot 2 \cdot 1 \cdot 2 \cdot 1$$

$$= \boxed{-4}$$

2. *(6+6+6pts)* Find all values of a and b where the system

$$\begin{aligned} y - 2z &= b \\ x - y + z &= 2 \\ x + ay &= 3 \end{aligned}$$

(A) ... has one solution.

$$\left[\begin{array}{ccc|c} 0 & 1 & -2 & b \\ 1 & -1 & 1 & 2 \\ 1 & a & 0 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & b \\ 1 & a & 0 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & b \\ 0 & a+1 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & b \\ 0 & 0 & -1+2(a+1) & 1-(a+1)b \end{array}\right]$$

$$\parallel$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & b \\ 0 & 0 & 2a+1 & 1-(a+1)b \end{array}\right]$$

Only one solution if $2a+1 \neq 0$

$\boxed{a \neq -\frac{1}{2}}$

(B) ... has many solutions.

Many solutions if $2a+1=0$ and $1-(a+1)b=0$

$a=-\frac{1}{2}$ and $1-(-\frac{1}{2}+1)b=0$

$b=2$

$\boxed{a=-\frac{1}{2} \text{ and } b=2}$

(C) ... has no solutions.

No solutions if $\boxed{a=-\frac{1}{2} \text{ and } b\neq 2}$

(so $1-(a+1)b \neq 0$)

3. *(15pts)* Let $\mathbf{T} : \mathcal{P}_3 \to \mathbb{R}^3$ by $\mathbf{T}(p) = \big(p(1),\ p(0),\ p(-1)\big)$.
Compute the matrix for $\mathbf{T}$ with respect to the bases:

$\mathscr{A} = \{1 + x,\ x + x^2,\ x^2 + x^3,\ x^3\}$ on $\mathcal{P}_3$, and

$\mathscr{B} = \{(1,1,0),\ (0,1,1),\ (0,0,1)\}$ on $\mathbb{R}^3$.

- $T(1+x) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow T(1+x)_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

- $T(x+x^2) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow T(x+x^2)_{\mathcal{B}} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$

- $T(x^2+x^3) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow T(x^2+x^3)_{\mathcal{B}} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$

- $T(x^3) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow T(x^3)_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$$[T]_{\mathcal{B}}^{\mathcal{A}} = \begin{bmatrix} | & | & | & | \\ T(1+x)_{\mathcal{B}} & T(x+x^2)_{\mathcal{B}} & T(x^2+x^3)_{\mathcal{B}} & T(x^3)_{\mathcal{B}} \\ | & | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 2 & 1 \\ -1 & -2 & -2 & -1 \\ 1 & 2 & 2 & 0 \end{bmatrix}$$

4. *(6+6+6pts)* Give short proofs of the following.

(A) Prove: The sum of two nilpotent transformations might not be nilpotent.

- $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\leadsto$ so $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is nilpotent.
- $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\leadsto$ so $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is nilpotent.

BUT $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is not nilpotent!

(b/c $\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 \neq 0$)

(B) Prove: No set of five polynomials can be orthogonal in $\mathcal{P}_3(\mathbb{R})$ with the inner product $\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)dx$.

Recall that $\mathcal{P}_3(\mathbb{R})$ has dimension $\underline{4}$.

If a set of vectors is orthogonal, then it is linearly independent. But, no set of $\underline{5}$ elements can be linearly independent in a dimension $\underline{4}$ vector space! $\blacksquare$

(C) Prove: If $\mathcal{V}$ is a vector space with orthogonal basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathbf{v} \in \mathcal{V}$ with $\langle \mathbf{v}, \mathbf{b}_i \rangle = 0$ for all $\mathbf{b}_i$, then $\mathbf{v} = \mathbf{0}$.

If $\{\underline{b}_1, \ldots, \underline{b}_n\}$ is a basis, then

$$\underline{v} = a_1 \underline{b}_1 + \cdots + a_n \underline{b}_n \quad \text{for some } a_i.$$

$$0 = \langle \underline{v}, \underline{b}_i \rangle = \langle a_1 \underline{b}_1 + \cdots + a_n \underline{b}_n, \underline{b}_i \rangle$$
$$= a_1 \cancelto{0}{\langle \underline{b}_1, \underline{b}_i \rangle} + \cdots + a_n \cancelto{0}{\langle \underline{b}_n, \underline{b}_i \rangle}$$
$$= a_i \langle \underline{b}_i, \underline{b}_i \rangle \quad (\text{because } \{\underline{b}_i\}_1^n \text{ orthogonal})$$

So $a_i = 0$ for all i. Thus $\underline{v} = \underline{0}$. $\blacksquare$

(BONUS) Prove: If A and B are $n \times n$ matrices with the same eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ then there is a linear transformation $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ and bases $\mathscr{A}$, $\mathscr{B}$ with $A = [\mathbf{T}]_{\mathscr{A}}^{\mathscr{A}}$ and $B = [\mathbf{T}]_{\mathscr{B}}^{\mathscr{B}}$.

Diagonalize!

$$A = P \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} P^{-1}$$
$$B = Q \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} Q^{-1}$$

Let $T(x_1, \ldots, x_n) = (\lambda_1 x_1, \ldots, \lambda_n x_n)$,
$\mathcal{A} = \{\text{columns of } P^{-1}\}$,
$\mathcal{B} = \{\text{columns of } Q^{-1}\}$.

then $P^{-1}AP = [T]$
so $A = [T]_{\mathcal{A}}^{\mathcal{A}}$
Similarly $B = [T]_{\mathcal{B}}^{\mathcal{B}}$ $\blacksquare$

5. *(10+3+4+4pts)* Let $\mathbf{T} : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\mathbf{T}(x, y, z) = (x, 3x + 2y + z, x + y + 2z)$.

(A) Find the characteristic polynomial, eigenvalues, and eigenvectors of $\mathbf{T}$.

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Characteristic equation:

$$\det \begin{bmatrix} 1-t & 0 & 0 \\ 3 & 2-t & 1 \\ 1 & 1 & 2-t \end{bmatrix} = (1-t)\cdot\left((2-t)^2 - 1\right) = (1-t)\left((t-1)(t-3)\right)$$

eigenvalues: $t = 1, 1, 3$

1-eigenspace:

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow k\boxed{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}$$

3-eigenspace:

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 3 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow k\boxed{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$$

Check:

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

(B) What are the algebraic and geometric multiplicities of the eigenvalues?

Algebraic mult of 1 is 2. Geom. mult of 1 is 1.

Algebraic mult. of 3 is 1. Geom mult of 3 is 1.

(C) What are the eigenvalues and eigenvectors of $\mathbf{T}^{260}$?
(Include a short proof that your answer is correct.)

Eigenvalues are 1 and 3^{260}

$$\begin{cases} 1\text{-eigensp} = k\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \\ 3^{260}\text{-eigensp} = k\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \end{cases}$$

Proof: If $T\underline{v} = \lambda\underline{v}$ then

$$T^{260}\underline{v} = T^{259}(T\underline{v}) = T^{259}(\lambda\underline{v}) = \lambda T^{259}\underline{v} = \cdots = \lambda^{260}\underline{v}$$

So eigenvectors remain the same, eigenvalues go $\lambda \mapsto \lambda^{260}$. ∎

(D) What are the eigenvalues and eigenvectors of $(260\,\mathbf{T} + \mathbf{I})$?
(Include a short proof that your answer is correct.)

Eigenvalues are 261 and $(3\cdot 260 + 1)$

$$\begin{cases} 261\text{-eigensp} = k\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \\ 781\text{-eigensp} = k\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \end{cases}$$

Proof: If $T\underline{v} = \lambda\underline{v}$ then

$$(260T + I)\underline{v} = 260\cdot T\underline{v} + \underline{v} = 260\cdot\lambda\underline{v} + \underline{v} = (260\cdot\lambda + 1)\underline{v}$$

So eigenvectors remain the same, eigenvalues go $\lambda \mapsto (260\cdot\lambda + 1)$ ∎

6. *(7+3+6pts)* Let $\langle -, - \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by $\langle (x_1, y_1), (x_2, y_2) \rangle = 8x_1x_2 - 6(x_1y_2 + y_1x_2) + 5y_1y_2$.

(A) Prove: This is an inner product.

① Symmetric: $\langle (x_1,y_1), (x_2,y_2) \rangle = 8x_1x_2 - 6(x_1y_2 + y_1x_2) + 5y_1y_2$

$= 8x_2x_1 - 6(x_2y_1 + y_2x_1) + 5y_2y_1$

$= \langle (x_2,y_2), (x_1,y_1) \rangle$

② Linear: $\langle r(x_1,y_1), (x_2,y_2) \rangle = 8(rx_1)x_2 - 6((rx_1)y_2 + (ry_1)x_2) + 5(ry_1)y_2$

$= r\langle (x_1,y_1), (x_2,y_2) \rangle$

$\langle (x_1,y_1) + (x_2,y_2), (x_3,y_3) \rangle = 8(x_1+x_2)x_3 - 6((x_1+x_2)y_3 + (y_1+y_2)x_3)$
$+ 5(y_1+y_2)y_3$

$= \langle (x_1,y_1), (x_3,y_3) \rangle + \langle (x_2,y_2), (x_3,y_3) \rangle$

③ Positive Definite: $\langle (x,y), (x,y) \rangle = 8x^2 - 12xy + 5y^2$

$\left.\begin{array}{l} \partial/\partial x\,(8x^2 - 12xy + 5y^2) = 16x - 12y \\ \partial/\partial y\,(8x^2 - 12xy + 5y^2) = -12x + 10y \end{array}\right\}$ $(0,0)$ is critical point.

2nd derivative test: Minimum!

$D = 16 \cdot 10 - (12)^2 > 0$

(B) Compute the length of $(1,2)$ using this inner product.

$|(1,2)| = \sqrt{\langle (1,2), (1,2) \rangle}$

$= \sqrt{8\cdot1\cdot1 - 6(1\cdot2 + 2\cdot1) + 5\cdot2\cdot2} = \sqrt{4} = \underline{2}$

(C) Find a vector which is orthogonal to $(1,2)$ using this inner product.

$0 = \langle (1,2), (x,y) \rangle = 8\cdot1\cdot x - 6(1\cdot y + 2\cdot x) + 5\cdot2\cdot y$

$= -4x + 4y$

$x = y$ So $(1,1) \perp (1,2)$

Alternate Solution: $(x,y) = (1,0) - \dfrac{\langle (1,0), (1,2) \rangle}{\langle (1,2), (1,2) \rangle}(1,2)$ (G-S on $\{(1,2), (1,0)\}$)

$= (1,0) - \dfrac{8-12}{4}(1,2) = (2,2)$ So $(2,2) \perp (1,2)$

(BONUS) Find a basis $\mathscr{B}$ of $\mathbb{R}^2$ so that this inner product is the dot product of $\mathscr{B}$-coordinates.
(We don't want anyone to be bored during the exam.)

$\mathscr{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \right\}$ So $\begin{bmatrix} x \\ y \end{bmatrix}_{\mathscr{B}} = 2\begin{bmatrix} 1 & -1/2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$= \begin{bmatrix} 2x - y \\ -2x + 2y \end{bmatrix}$

$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}_{\mathscr{B}} \cdot \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}_{\mathscr{B}} = \begin{bmatrix} 2x_1 - y_1 \\ -2x_1 + 2y_1 \end{bmatrix} \cdot \begin{bmatrix} 2x_2 - y_2 \\ -2x_2 + 2y_2 \end{bmatrix} = (4x_1x_2 - 2(x_1y_2 + y_1x_2) + y_1y_2)$
$+ (x_1x_2 - 4(x_1y_2 + y_1x_2) + 4y_1y_2)$