M E T U
Northern Cyprus Campus

Math 260	Linear Algebra	Final Exam	11.01.2013
Last Name : KEY Name : Student No	Dept./Sec.: Time : 09: 00 Duration : 90 *minutes*	Signature	
5 QUESTIONS ON 4 PAGES		TOTAL 100 POINTS	

1	2	3	4	5				

Q1 (20p.) Diagonalize the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ -5 & 0 & -2 \\ 15 & 2 & 0 \end{bmatrix} \in M_3(\mathbb{C})$ if it is possible.

$$\Delta(t) = \begin{vmatrix} 3-t & 0 & 0 \\ -5 & -t & -2 \\ 15 & 2 & -t \end{vmatrix} = (3-t)(t^2+4) = -(t-3)(t-2i)(t+2i)$$

$$\sigma(A) = \{3, 2i, -2i\}$$

$$\lambda = 3 \Rightarrow A-3 = \begin{bmatrix} 0 & 0 & 0 \\ -5 & -3 & -2 \\ 15 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} -5 & -3 & -2 \\ 0 & 7 & 9 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 35 & 0 & -13 \\ 0 & 7 & 9 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V_{3,1} = \ker(A-3) = \text{Span}\{(\tfrac{13}{5}, -9, 7)\}$$

$$\lambda = 2i \Rightarrow A-2i = \begin{bmatrix} 3-2i & 0 & 0 \\ -5 & -2i & -2 \\ 15 & 2 & -2i \end{bmatrix} \sim \begin{bmatrix} 3-2i & 0 & 0 \\ 5 & 2i & 2 \\ 15+5i & 0 & 0 \end{bmatrix}$$

$$V_{2i,1} = \ker(A-2i) = \{x=0,\ iy+z=0\} = \text{Span}\{(0,1,-i)\}$$

$$\lambda = -2i \Rightarrow A+2i = \begin{bmatrix} 3+2i & 0 & 0 \\ -5 & 2i & -2 \\ 15 & 2 & 2i \end{bmatrix} \sim \begin{bmatrix} 3+2i & 0 & 0 \\ -5 & 2i & -2 \\ 15-5i & 0 & 0 \end{bmatrix}$$

$$V_{-2i,1} = \ker(A+2i) = \{x=0,\ iy-z=0\} = \text{Span}\{(0,1,i)\}$$

Hence $f = \left((\tfrac{13}{5}, -9, 7), (0,1,-i), (0,1,i) \right)$ and

$$M_{f,f}(A) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -2i \end{bmatrix}$$

Q2 (25 p.) Find the Jordan normal form and the related Jordan basis of the matrix

$$A = \begin{bmatrix} 3 & 2 & -3 \\ 4 & 10 & -12 \\ 3 & 6 & -7 \end{bmatrix} \in M_3(\mathbb{C}).$$

$$\Delta(t) = \begin{vmatrix} 3-t & 2 & -3 \\ 4 & 10-t & -12 \\ 3 & 6 & -7-t \end{vmatrix} = -(t-3)(t-10)(t+7) - 72 - 72 + 9(10-t) + 72(3-t) + 8(t+7) =$$

$$= -(t-3)(t-10)(t+7) + 218 - 73t =$$

$$= -t^3 + 6t^2 + 61t - 210 + 218 - 73t =$$

$$= -t^3 + 6t^2 - 12t + 8 = -(t-2)^3 \Rightarrow \sigma(A) = \{2^{③}\}.$$

$$\lambda = 2 \Rightarrow A - 2 = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 8 & -12 \\ 3 & 6 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore $V_{2,1} = \ker(A-2) = \{x + 2y - 3z = 0\}$ and $(A-2)^2 = 0$ or $V_{2,2} = \mathbb{C}^3$.

Take $f_1 = (1,0,0)$. Then $f_2 = (A-2)f_1 = (1,4,3)$ and $f_3 = (0,3,2)$. Thus

$$\begin{array}{ccc} f_1 & & \\ f_2 = (A-2)f_1, & & f_3 \\ \mathbb{C}^3 = \mathcal{U}_1 & \oplus & \mathcal{U}_2 \end{array}$$

$$f = ((1,0,0), (1,4,3), (0,3,2))$$

and

$$M_{f,f}(A) = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Q3 (30 p.) Find the Jordan normal form and the related Jordan basis of the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 & 1 \\ -1 & 2 & -1 & 1 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \in M_4(\mathbb{C})$$

$$\Delta(t) = \begin{vmatrix} -t & 1 & -1 & 1 \\ -1 & 2-t & -1 & 1 \\ -1 & 1 & 1-t & 0 \\ -1 & 1 & 0 & 1-t \end{vmatrix} = \begin{vmatrix} t & -1 & 1 & -1 \\ 1 & t-2 & 1 & -1 \\ 1 & -1 & t-1 & 0 \\ 1 & -1 & 0 & t-1 \end{vmatrix} \underset{C_1+C_2}{=} \begin{vmatrix} t-1 & -1 & 1 & -1 \\ t-1 & t-2 & 1 & -1 \\ 0 & -1 & t-1 & 0 \\ 0 & -1 & 0 & t-1 \end{vmatrix} R_2-$$

$$= \begin{vmatrix} t-1 & -1 & 1 & -1 \\ 0 & t-1 & 0 & 0 \\ 0 & -1 & t-1 & 0 \\ 0 & -1 & 0 & t-1 \end{vmatrix} = (t-1)^4 \Rightarrow \sigma(A) = \{1^{(4)}\}.$$

$$\lambda = 1 \Rightarrow A - 1 = \begin{bmatrix} -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$V_{1,1} = \ker(A-1) = \{x = y,\ z = w\} \Rightarrow m(1) = \dim(V_{1,1}) = 2.$

Note that $(A-1)^2 = 0$. Whence $V_{1,2} = \mathbb{C}^4$ and $\dim(V_{1,2}/V_{1,1}) = 2$. Choose $v_1 = (1,0,0,0)$, $v_2 = (0,0,1,0)$ a basis for $\mathbb{C}^4$ over $V_{1,1}$. Then we have

$f_1 = v_1$, $f_2 = (A-1)v_1$, $f_3 = v_2$, $f_4 = (A-1)v_2$, that is,

$$\begin{matrix} v_1 & v_2 \\ (A-1)v_1 & (A-1)v_2 \\ \mathcal{U}_1 & \oplus\ \mathcal{U}_2 = \mathbb{C}^4 \end{matrix}$$

$f = ((1,0,0,0), (-1,-1,-1,-1), (0,0,1,0), (-1,-1,0,0))$ and

$$M_{f,f}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Q4 (10 p.) Find an orthonormal basis for the complex plane $2ix - iy + z = 0$ in $\mathbb{C}^3$.

Take a basis $\vec{b}_1 = (1, 0, -2i)$, $\vec{b}_2 = (0, 1, i)$ for the plane.

Put $\vec{a}_1 = \vec{b}_1$, $\vec{a}_2 = \vec{b}_2 - \frac{\langle \vec{b}_2, \vec{a}_1 \rangle}{\langle \vec{a}_1, \vec{a}_1 \rangle} \vec{b}_1 = (0, 1, i) - \frac{-2}{5}(1, 0, -2i) =$

$= \left(\frac{2}{5}, 1, \frac{i}{5}\right) \Rightarrow \|\vec{a}_2\| = \frac{\sqrt{30}}{5}$.

Put $\vec{f}_1 = \frac{\vec{a}_1}{\|\vec{a}_1\|} = \left(\frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}} i\right)$ and $\vec{f}_2 = \frac{\vec{a}_2}{\|\vec{a}_2\|} = \left(\frac{2}{\sqrt{30}}, 1, \frac{i}{\sqrt{30}}\right)$

Then $\vec{f} = (\vec{f}_1, \vec{f}_2)$ is an orthonormal basis for the plane.

Q5 (15 p.) Find the Fourier coefficients of the vector $2 - 5x + 3x^2$ with respect to the orthogonal basis $(1, x, x^2 - 1/3)$ for the inner product space $\mathcal{P}_2(\mathbb{R})$.

First note that $\vec{f} = \left(\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\, x, \frac{3}{2}\sqrt{\frac{5}{2}}\left(x^2 - \frac{1}{3}\right)\right)$ is the related orthonormal basis for $P_2(\mathbb{R})$.

Then $2 - 5x + 3x^2 = \lambda_1 \vec{f}_1 + \lambda_2 \vec{f}_2 + \lambda_3 \vec{f}_3$ with

$$\lambda_1 = \left\langle 2 - 5x + 3x^2, \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\sqrt{2}} \int_{-1}^{1} (2 - 5x + 3x^2)\, dx = \frac{6}{\sqrt{2}}$$

$$\lambda_2 = \left\langle 2 - 5x + 3x^2, \sqrt{\frac{3}{2}}\, x \right\rangle = \sqrt{\frac{3}{2}} \int_{-1}^{1} (2x - 5x^2 + 3x^3)\, dx$$

$$= \sqrt{\frac{3}{2}}\ \frac{-5}{3} \cdot 2 = -\frac{10}{3}\sqrt{\frac{3}{2}}, \text{ and}$$

$$\lambda_3 = \left\langle 2 - 5x + 3x^2, \frac{3}{2}\sqrt{\frac{5}{2}}\left(x^2 - \frac{1}{3}\right) \right\rangle = \frac{3}{2}\sqrt{\frac{5}{2}} \int_{-1}^{1} \left(3x^4 - 5x^3 + x^2 + \frac{5}{3}x - \frac{2}{3}\right) dx = \frac{3}{2}\sqrt{\frac{5}{2}} \left(\frac{3}{5} \cdot 2 + \frac{2}{3} - \frac{4}{3}\right)$$

$$= \frac{3}{2}\sqrt{\frac{5}{2}}\ \frac{8}{15} = \frac{4}{5}\sqrt{\frac{5}{2}}.$$