Lecture Notes about Solving Differential Equations by MatLab

We will solve the following differential equation by finding an analogue difference equation

$$-\frac{d^2u}{dx^2} = x, \qquad u(0) = 0, u(1) = 0$$

As we know that the discrete version of the second derivative, which is the second difference is given by the equation

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = x_i$$

Hence, our difference equation becomes

If we write this for n=4, i.e. *i* varies from 1 to 4, then h=1/(4+1)=1/5, and we get the following linear system as given a matrix equation.

$$\frac{1}{\left(\frac{1}{5}\right)^2} \cdot \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \\ \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$$

To put this equation into MatLab, and let it solve, we need to write the matrices. In our case, the coefficient matrix, called also K, is symmetric, and constant entries on the diagonal, below and above the diagonal. MatLab has a special command for these matrices, **toeplitz.**

In the symmetric case, we just write the first column of the matrix in the **toeplitz** command. The following command will generate the matrix above.

>>K=toeplitz([2 -1 0 0])

To write the result column, we just write the following;

>>b= 1/5*[1 2 3 4]'
(Note:' is used for transpose which makes the result row a column.)
Then,

>>A= 1/(1/5)^2* toeplitz([2 -1 0 0]) ;

To solve the linear system, we use the following command:
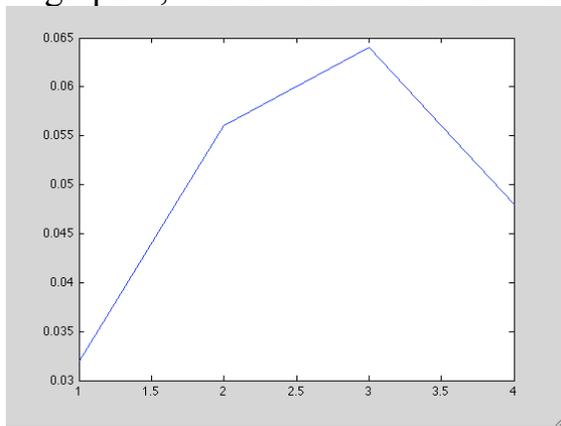
>>u=A\b

Then we get the following result:

u =

  0.032000000000000
  0.056000000000000
  0.064000000000000
  0.048000000000000

We can use

>>plot(u)

to graph it, which will be shown as the following;

As you see, the x-axis doesn't match the sample points we got from discretization.
To do this, we have to define x-values as the following;

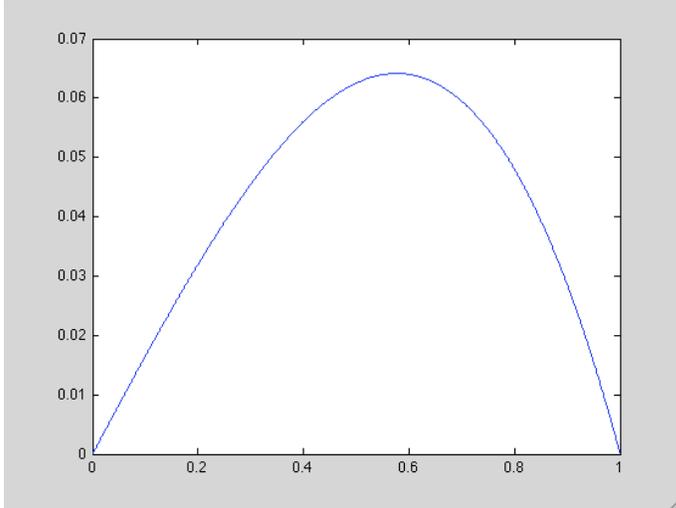**>>x=linspace(1/5,4/5,4);**

This will generate the following:

x =

  0.2 0.4 0.6 0.8

Now by using

>>plot(x,u)

We get the following graph:

Now, to do the same thing for n=1000, we have to write the K matrix with size of 1000x1000.
Our difference equation becomes;

$$\frac{1}{(\frac{1}{1001})^2} \cdot \begin{bmatrix} 2 & -1 & 0 & . & . & & \\ & . & . & . & & & \\ 0 & . & -1 & 2 & -1 & . & \\ 0 & . & 0 & -1 & 2 & -1 & \\ & . & . & . & & & \\ & & & & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ . \\ . \\ . \\ . \\ . \\ u_{1000} \end{bmatrix} = \begin{bmatrix} \frac{1}{1001} \\ \frac{2}{1001} \\ . \\ . \\ . \\ \frac{1000}{1001} \end{bmatrix}$$

Writing this is also very easy with toeplitz command.

>>K=toeplitz([2 -1 zeros(1,998)]);
and
>>A= 1/(1/1001)^2*toeplitz([2 -1 zeros(1,998)]);

Here zeros(k,r) command is used, and it produces a k by r matrix whose entries are all zero. In our case, zeros(1, 998) will produce a row with 998 zeros as necessary to write K.

In this case, to write the result part of the linear system, we can use :
>>b=1/1001*linspace(1,1000,1000);

If you plot u, then we see the following graph:

Now, we will solve the following differential equation by finding an analogue difference equation.

$$-\frac{d^2u}{dx^2} + \frac{dy}{dx} = 1, \qquad u(0) = 1, u(1) = 0$$

In this case, we also have the first derivative appearing in the differential equation, for this we are going use the first difference. But, we have three choices for the first differences, backward, forward, and center. We know that center is the best among all in the sense that its error is the smallest. Hence our choice will be the center difference, which can be written as the following:

$$\frac{u_{i+1} - u_{i-1}}{2h}$$

Now, we will write the difference equation for n=1000, i.e. i varies from 1 to 1000, then h=1/(1000+1)=1/1001, and we get the following linear system as given a matrix equation.

$$\frac{1}{(\frac{1}{1001})^2} \cdot \begin{bmatrix} 2 & -1 & 0 & 0 & . & \\ -1 & 2 & -1 & 0 & . & \\ . & . & . & & & \\ . & -1 & 2 & -1 & \\ . & . & & \\ & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ . \\ . \\ . \\ . \\ u_{1000} \end{bmatrix} + \frac{1}{(2 \cdot \frac{1}{1001})} \cdot \begin{bmatrix} 0 & 1 & 0 & . & . \\ -1 & 0 & 1 & 0 & . \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ . & & & . \\ & & -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ . \\ . \\ . \\ . \\ u_{1000} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ . \\ 1 \end{bmatrix}$$

As you see from the matrix equation, the matrix for center difference is again a toeplitz matrix, but not symmetric. In this case, we use the same command, but we need to write

the first column and first row of the matrix inside the parenthesis. Hence, we write the following commands for the coefficient matrices:

>> **K=toeplitz([2 -1 zeros(1,998)]);**
>>**C=toeplitz([0 -1 zeros(1,998)],[0 1 zeros(1,998)]);**
And the coefficient matrices become:
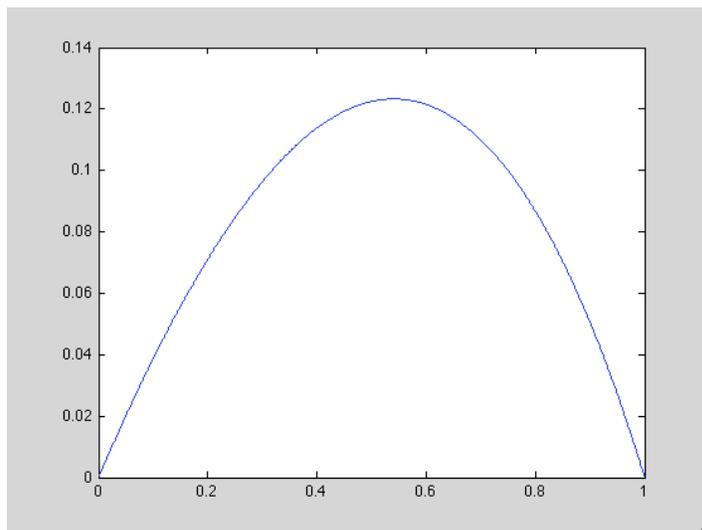>> **A=1/(1001)^2*K;**
>>**B=1/(2/1001)*C;**
and the image vector is:
>>**b=[ones(1,1000)]';**
<mark>Similar to the zeros command, **ones(r,c)** produces a matrix with all ones of size r by c.</mark>
Then again to solve it we use:
>>**u=(A+B)\b;**

and we graph it by using plot command. We get the following graph:

Now, our last example for solving difference equations will be the following:

$$-\frac{d^2u}{dx^2} = \delta\left(x - \frac{2}{5}\right) \qquad u'(0) = 0, u(1) = 0$$

In this case, our problem is a Free-Fixed Boundary problem, but the previous problems were all Fixed-Fixed Boundary problems.

Again, we will solve the discrete version of the problem. In this case, we will use the matrix T, not K.

Before we start solving, I want to make a remark about delta function. Delta function **δ(x-a)** is actually not a function. It is zero everywhere except at a. There is a spike at a and the area under the spike, which is the total impulse is equal to 1. That is the reason, if we write the Free-Fixed problem as difference equation, the correct formulation will be as:

$$\frac{T}{h^2} u = \frac{\delta}{h}$$

Here, h is the step size and δ is the column vector **(0,....0,1,0,....0)** where 1 is at the at the position of a. We need to divide by h since delta function doesn't take value 1 at a, just the area under a is 1.

By just canceling h from both sides, and substituting (u/h) as U, i.e. U=u/h, we just need to solve, TU= δ , then to get u, we will multiply the result with h.

Now let's write the difference equation for n=4, and solve it.

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Matrix T is exactly same as K, but just different at the position (1,1). So, to write T, we will first write K, and change the entry (1,1) to 1.
>>T=toeplitz([2 -1 0 0]);
>>T(1,1)=1;
>>b=[0 1 0 0]';
By writing T(i,j)=A, you can change the entry at the position (i,j) of the matrix T to A.
We put 1 in the column b to the second position, since our function $\delta(x-2/5)$ has impulse at 2/5 and 2/5 is the second point when we divide interval [0,1] into 5 parts.
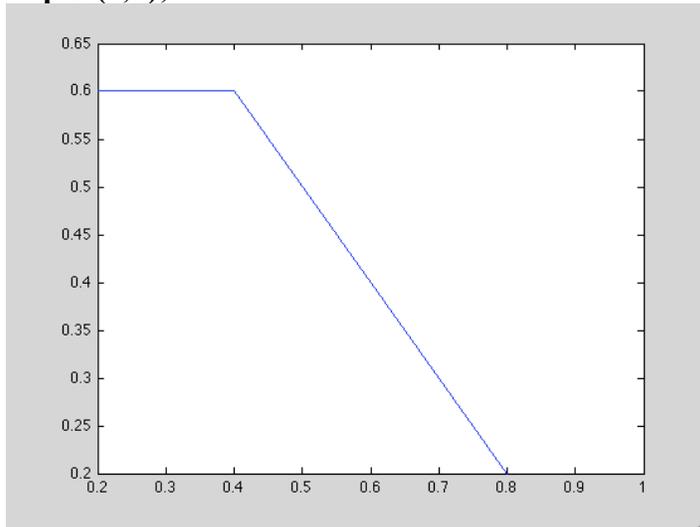When we solve, and graph it, we get the following graph:
>>U=T\b;
>>u=1/5*U;
>>x=linspace(1/5,4/5,4);
>>plot(x,u);

Note that this graph is drawn on the interval [1/5,4/5].

We saw the same graph when we solved the continuous version of this problem.

Now, I want to write a couple of commands, which are very helpful for matrix operations.

>>[L,U]=lu(A);

This command will find the LU-decomposition of the matrix A.

>>inv(A)

This command will find the inverse of matrix A.

>>size(A)

This command will tell you the size of the matrix A.

>>[n:k]

This will give you a vector whose entries are the integers from n to k.

>>diag(a1,a2,….,an)

This command will give you a diagonal matrix whose diagonal entries are a1,a2,….,an.

>>eye(n)

This command will give you the identity matrix of size n by n.

>>eig(A)

This command will calculate the eigenvalues of a matrix A.