METU - NCC

CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES FINAL EXAM	
Code : *MAT 120* Acad.Year: *2012-2013* Semester : *FALL* Date : *14.1.2013* Time : *16:00* Duration : *120 minutes*	Last Name: SOLUTIONS Name : ~~Student No.:~~ Department: Section: Signature: 8 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1. (10)	2. (10)	3. (12)	4. (20)	5. (8)	6. (12)	7. (16)	8. (12)		

Communicate your work clearly with clean handwriting!

1. *(10pts)* Find and classify the max/min values of $f(x,y) = x^3 + x^2 - 3x + 2xy + y^2$.

Critical Points:

$0 = f_x = 3x^2 + 2x - 3 + 2y$ plug in here

$0 = f_y = 2x + 2y \longrightarrow y = -x$

$0 = 3x^2 - 3$

$x = \pm 1 \longrightarrow \begin{cases} x = 1 \Rightarrow y = -1 \\ x = -1 \Rightarrow y = 1 \end{cases}$

$(1,-1)$ & $(-1,1)$

2nd Derivative Test

$f_{xx} = 6x + 2$

$f_{yy} = 2$

$f_{xy} = 2$

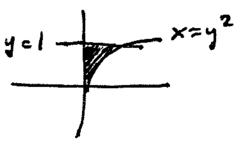
$D = (6x+2)\cdot 2 - 4 \longrightarrow \begin{cases} D(1,-1) = 12 > 0 \quad \text{Max or Min!} \\ D(-1,1) = -12 < 0 \quad \text{Not Max or Min!} \end{cases}$

$f_{yy} > 0$ at $(1,-1)$ so this is a local min

$f(1,-1) = 1 + 1 - 3 - 2 + 1 = -2$

2. *(10pts)* Compute the triple integral $\displaystyle\iiint_R 6xy\, dV$ where R is the region $0 \leqslant z \leqslant 1+x+y$, and x, y are bounded by $y=\sqrt{x}$, $y=1$ and $x=0$.

Region in xy-plane:



$$\int_0^1\int_0^{y^2}\int_0^{1+x+y} 6xy\, dz\, dx\, dy = \int_0^1\int_0^{y^2} 6xy(1+x+y)\, dx\, dy$$

$$= \int_0^1 6(y+y^2)\cdot\tfrac{1}{2}x^2 + 6y\cdot\tfrac{1}{3}x^3 \Big|_{x=0}^{x=y^2} dy$$

$$= \int_0^1 3y^5 + 3y^6 + 2y^7\, dy = \boxed{\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{3}{7}}$$

Alternate solution:

$$\int_0^1\int_{\sqrt{x}}^1\int_0^{1+x+y} 6xy\, dz\, dy\, dx = \int_0^1\int_{\sqrt{x}}^1 6xy(1+x+y)\, dy\, dx$$

$$= \int_0^1 6(x+x^2)\cdot\tfrac{1}{2}y^2 + 6x\cdot\tfrac{1}{3}y^3 \Big|_{y=\sqrt{x}}^{y=1} dx$$

$$= \int_0^1 (3x+\cancel{3x^2}+2x) - (\cancel{3x^2}+3x^3+2x^{5/2})\, dx = \boxed{\tfrac{5}{2}-\tfrac{3}{4}-\tfrac{4}{7}}$$

3. *(3×4pts)* State whether (and why) the following sequences a_n converge or diverge.

(A) $a_n = \ln(n^2+1) - \ln(n^2+3n+5)$.

$$= \ln\left(\frac{n^2+1}{n^2+3n+5}\right) \qquad \lim_{x\to\infty}\ln\left(\frac{x^2+1}{x^2+3x+5}\right) = \ln\left(\lim_{x\to\infty}\frac{1+1/x^2}{1+3/x+5/x^2}\right)$$

$$= \ln(1) = 0$$

So $a_n \to 0$ $\boxed{\text{convergent}}$

(B) $a_1 = 1$ and $a_{n+1} = 4 - a_n$ for $n \geqslant 2$.

$a_1 = 1$

$a_2 = 4-1 = 3$

$a_3 = 4-3 = 1$

$a_4 = 4-1 = 3$

$$\begin{cases} a_{2n} = 3 \\ a_{2n+1} = 1 \end{cases}$$

Sequence is $\boxed{\text{divergent}}$

(C) $a_n = \dfrac{n}{\sin(n)}$

$-1 < \sin(n) < 1$ so $1/\sin(n) < -1$ or $1/\sin(n) > 1$

in particular $\frac{n}{\sin n} < -n$ or $n/\sin(n) > n$

a_n oscillates between numbers $< -n$ and $> n$

so it is $\boxed{\text{divergent}}$

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4. *(4×5pts)* State whether (and why) the following series converge or diverge. To receive credit, your answer must be supported with work.

(A) $\sum_{n=1}^{\infty} \left(\frac{1}{n^2} + e^{1/n}\right)$.

$\sum_1 1/n^2$ is convergent b/c it is a <u>p-series</u> with $p = 2 > 1$.

$\sum_1 e^{1/n}$ is divergent by <u>n^{th} term test</u>: $e^{1/n} \to 1 \neq 0$.

So $\sum_1 (1/n^2 + e^{1/n})$ is $\boxed{\text{divergent}}$

(B) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$.

<u>Limit comparison test</u> w/ convergent p-series $\sum_1 1/n^2$: $(p = 2 > 1)$

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n^2}\right)}{1/n^2} = \lim_{x\to 0^+} \frac{\sin(x)}{x} = 1 \begin{matrix} \neq 0 \\ \neq \infty \end{matrix}$$

So $\sum_1 \sin(1/n^2)$ is $\boxed{\text{convergent}}$

(C) $\sum_{n=1}^{\infty} \frac{n\ln(n)}{(2n+1)!}$

<u>Ratio Test</u>:

Note: $\frac{(2n+1)!}{(2n+3)!} = \frac{1}{(2n+3)(2n+2)}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(n+1)\ln(n+1)}{(2n+3)!}}{\frac{n\ln n}{(2n+1)!}}\right| = \left|\underbrace{\frac{n+1}{n}}_{\to 1} \cdot \underbrace{\frac{\ln(n+1)}{\ln n}}_{\to 1} \cdot \underbrace{\frac{(2n+1)!}{(2n+3)!}}_{\to 0}\right| \longrightarrow 0 < 1$$

So $\sum_1 \frac{n\ln n}{(2n+1)!}$ is $\boxed{\text{convergent}}$

(D) $\sum_{n=1}^{\infty} (-1)^n \frac{(2n+1)^n}{4^{2n+1}}$

<u>Root Test</u>:

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{(2n+1)^n}{4^{2n+1}}} = \frac{2n+1}{4^2 \cdot \sqrt[n]{4}} \longrightarrow \infty > 1$$

so $\sum_1 (-1)^n \frac{(2n+1)^n}{4^{2n+1}}$ is $\boxed{\text{divergent}}$

5. *(3+5pts)* Give **power series** for the following functions.

(A) $f(x) = e^x$ around $a = 0$.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(B) $f(x) = 3x^2 e^{7x^3}$ around $a = 0$.

$$3x^2 e^{7x^3} = 3x^2 \left(\sum_{n=0}^{\infty} \frac{(7x^3)^n}{n!} \right)$$

$$= \sum_{n=0}^{\infty} \frac{3 \cdot 7^n}{n!} x^{3n+2}$$

6. *(12pts)* Give the **power series** for $f(x) = \frac{1}{x^2 - 2x + 5}$ around $a = 1$. Also find the **radius of convergence** and **interval of convergence** of the power series.

$$\frac{1}{x^2 - 2x + 5} = \frac{1}{(x-1)^2 + 4} = \frac{1}{4} \frac{1}{1 + \frac{(x-1)^2}{4}}$$

$$= \frac{1}{4} \frac{1}{1 - \left(-\frac{(x-1)^2}{4}\right)}$$

$$= \frac{1}{4} \sum_{n=0}^{\infty} \left(-\frac{(x-1)^2}{4}\right)^n$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{2n}}{4^{n+1}}}$$

Convergent for

$$\left| -\frac{(x-1)^2}{4} \right| < 1$$

$$|(x-1)^2| < 4$$

$$|x-1| < 2$$

Note: On endpoints of interval, series is

$$\sum_{n=0}^{\infty} (-1)^n \frac{(\pm 2)^{2n}}{4^{n+1}} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{4}$$

Divergent.

Radius of convergence $= \boxed{2}$

Interval of convergence $= \boxed{(-1, 3)}$

7. *(3+3+5+5pts)* In this problem you will compute the power series for $f(x) = \arcsin(x)$ using the binomial power series.

(A) Write the binomial power series formula for $(1+x)^k$ around $a = 0$.

$$(1+x)^k = \boxed{\sum_{n=0}^{\infty} \binom{k}{n} x^n}$$

$$= 1 + kx + \frac{k(k-1)}{2} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \cdots$$

(B) Write the power series for $\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$ around $a = 0$.

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} = (1 + (-x^2))^{-1/2}$$

$$= \boxed{\sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n x^{2n}}$$

(C) Integrate to get the power series for $\arcsin(x)$ around $a = 0$.
(Include a computation of the constant of integration.)

$$\arcsin x = \int \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \int \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n x^{2n}\, dx$$

$$= \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

$$0 = \arcsin 0 = C$$

$$\arcsin x = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n \frac{x^{2n+1}}{2n+1}$$

(D) Write the first three nonzero terms explicitly (i.e. without using "choose notation" $\binom{a}{n}$).

$$\arcsin x = \binom{-1/2}{0}(-1)^0 \frac{x}{1} + \binom{-1/2}{1}(-1)^1 \frac{x^3}{3} + \binom{-1/2}{2}(-1)^2 \frac{x^5}{5} + \cdots$$

$$= x + (-\tfrac{1}{2})(-1)\frac{x^3}{3} + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2} \frac{x^5}{5} + \cdots$$

$$= \boxed{x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots}$$

8. *(12pts)* Compute the first three nonzero terms of the power series of $f(x) = \arcsin(x)$ around $a = 0$ using Taylor's theorem.

(This is why we like binomial series...)

$f(x) = \arcsin x$ $\quad f(0) = 0$

$f'(x) = (1-x^2)^{-1/2}$ $\quad f'(0) = 1$

$f''(x) = -1/2(1-x^2)^{-3/2} \cdot (-2x)$ $\quad f''(0) = 0$

$f'''(x) = 3/4(1-x^2)^{-5/2}(-2x)^2 + (-1/2)(1-x^2)^{-3/2}(-2)$ $\quad f'''(0) = 1$

$$f''''(x) = -\frac{15}{8}(1-x^2)^{-7/2}(-2x)^3 + 3/4(1-x^2)^{-5/2} \cdot 2(-2x)(-2) + 3/4(1-x^2)^{-5/2} \cdot (-2)(-2x)$$

$f''''(0) = 0$

$$f'''''(x) = \frac{15}{8} \cdot 7/2 (1-x^2)^{-9/2}(-2x)^4 + (-15/8)(1-x^2)^{-7/2} \cdot 3 \cdot (-2x)^2(-2) + (9) \cdot (-5/2)(1-x^2)^{-7/2} \cdot x + (9)(1-x^2)^{-5/2}$$

$f'''''(0) = 9$

$$\arcsin x = \boxed{x + \frac{x^3}{3!} + \frac{9 \cdot x^5}{5!} + \cdots}$$

Bonus. Use power series to compute $f^{(100)}(0)$ where $f(x) = \dfrac{x}{1+2x^3}$.

Recall: $f^{(100)}(0)$ *is the* 100^{th} *derivative of* f *evaluated at* $x = 0$.

$$f(x) = \frac{x}{1+2x^3} = x \cdot \frac{1}{1+2x^3} = x \cdot \frac{1}{1-(-2x^3)}$$

$$= x \sum_{n=0}^{\infty} (-2x^3)^n = \sum_{n=0}^{\infty} (-1)^n 2^n x^{3n+1}$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} (-1)^n 2^n x^{3n+1}$$

$$\frac{f^{(100)}(0)}{100!} x^{100} = (-1)^n 2^n x^{3n+1} \longrightarrow \begin{cases} x^{100} = x^{3n+1} \\ 100 = 3n+1 \\ 99 = 3n \\ 33 = n \end{cases}$$

$$f^{(100)}(0) = (100!) \cdot (-1)^{33} 2^{33}$$

$$= \boxed{-2^{33} \cdot (100!)}$$