M E T U
Northern Cyprus Campus

	Applied Mathematics for Engineers Midterm	
Code : *Math 210* Acad.Year: *2012-2013* Semester : *Spring* Date : *21.4.2013* Time : *14:40* Duration : *120 minutes*	Last Name: Name: Student No: Department: Section: Signature: 8 QUESTIONS ON 6 PAGES TOTAL 100 POINTS	

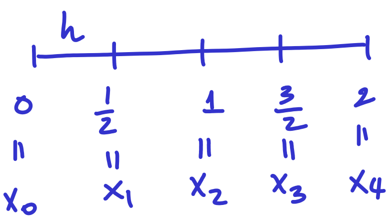
1 (12)	2 (18)	3 (18)	4 (14)	5 (13)	6 (8)	7 (5)	8 (12)		

Show your work! No calculators! Please draw a $\boxed{\textbf{box}}$ **around your answers!**
Please do not write on your desk!

DISCLAIMER: When solving matrix equations by elimination, please use the following instructions. (a) $\mathbf{cR_i} + \mathbf{R_j}$: Add c times row i to row j (b) $\mathbf{R_i} \leftrightarrow \mathbf{R_j}$: Switch row i with row j.

1.(10+2 pts) Consider the differential equation $-\frac{1}{2}u'' + 2u' = \delta(x-1) \quad u(0) = 0, \quad u(2) = 0.$
(a) Write the difference equations with $h = \frac{1}{2}$ for this differential equation and **solve** them. (For u' use centered differences.)

Setup:



$h = \frac{1}{2} \qquad h^2 = \frac{1}{4}$

(Boundary condition $u(0)=0$)

$x_1 = \frac{1}{2} \qquad -\frac{1}{2}\,\frac{\overset{0}{u(x_0)} - 2u(x_1) + u(x_2)}{h^2} + \frac{2[-\overset{0}{u(x_0)} + u(x_2)]}{2h} = 0$

$x_2 = 1 \qquad -\frac{1}{2}\,\frac{u(x_1) - 2u(x_2) + u(x_3)}{h^2} + \frac{2[-u(x_1) + u(x_3)]}{2h} = \frac{1}{h}$

(representing $\delta(x-1)$)

$x_3 = \frac{3}{2} \qquad -\frac{1}{2}\,\frac{u(x_2) - 2u(x_3) + \underset{0}{u(x_4)}}{h^2} + 2\,\frac{[-u(x_2) + \underset{0}{u(x_4)}]}{2h} = 0$

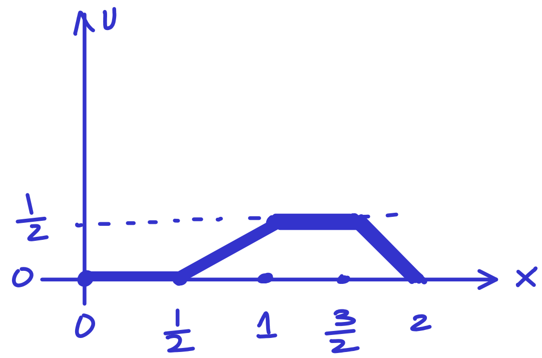
(Boundary contition $u(2)=0$)

$\Rightarrow \left(\frac{1}{2h^2}K + \frac{2}{2h}C\right)u = \frac{1}{h}\begin{bmatrix}0\\1\\0\end{bmatrix}$

$K = \begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}, \quad C = \begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 1\\ 0 & -1 & 0\end{bmatrix}$

$\Rightarrow \left(\frac{1}{2h}K + C\right)u = \begin{bmatrix}0\\1\\0\end{bmatrix} \overset{2h=1}{\Rightarrow} (K+C)u = \begin{bmatrix}2 & 0 & 0\\ -2 & 2 & 0\\ 0 & -2 & 2\end{bmatrix}u = \begin{bmatrix}0\\1\\0\end{bmatrix} \Rightarrow \boxed{u = \begin{bmatrix}0\\ \frac{1}{2}\\ \frac{1}{2}\end{bmatrix}}$

(b) Graph your answer in **(a)**.



2.(10+4+4 pts) Consider the matrix $A = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

(a) Calculate the eigenvalues and eigenvectors of A.

Eigenvalues: $\det\begin{bmatrix} \lambda-4 & +1 & 0 \\ 1 & \lambda-4 & 0 \\ 0 & 0 & \lambda-3 \end{bmatrix} = [(\lambda-4)(\lambda-4)-1](\lambda-3)=0$

$\boxed{\Rightarrow \lambda = 5, 3, 3}$

Eigenvectors:

Eigenvalue $\lambda=3$:

$A-\lambda I = A-3I = \left[\begin{array}{ccc|} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{1R_1+R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{matrix} x_1 = x_2 \\ x_2 = \text{free} \Rightarrow x_2 = s \\ x_3 = \text{free} \Rightarrow x_3 = t \end{matrix}$ $\quad x_1 = s$ $\quad x = \begin{bmatrix} s \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Pick as eigenvalues!

Eigenvalue $\lambda=5$

$A-\lambda I = A-5I = \left[\begin{array}{ccc} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 2 \end{array}\right] \xrightarrow{-R_1+R_2} \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \begin{matrix} x_1 = -x_2 \\ x_2 = \text{free} \Rightarrow x_2 = v \\ x_3 = 0 \end{matrix}$ $\quad x_1 = -v$ $\quad x = \begin{bmatrix} -v \\ v \\ 0 \end{bmatrix} = v\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

(b) Produce a matrix decompostion $Q\Lambda Q^T$ for A where Q is an orthogonal matrix, Λ is a diagonal matrix.

I order eigenvalues as 5, 3, 3 with corresponding eigenvectors

$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow A = \underbrace{\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Normalized Eigenvectors} \,=\, Q} \underbrace{\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{\text{Eigenvalues} \,=\, \Lambda} \underbrace{\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{Q^T}$

$A = Q \cdot \Lambda \cdot Q^T$

(c) For which values of x does the limit $\lim_{k\to\infty}(\frac{1}{x}A)^k$ exist? Provide the limiting matrix whenever it exists.

1 (a) When $|x|>5$, then $\left(\frac{1}{x}A\right)^k = Q\left(\frac{1}{x}\Lambda\right)^k Q^T \longrightarrow Q(\text{zero matrix}_{3\times3})Q^T = 0$ as $k \to \infty$

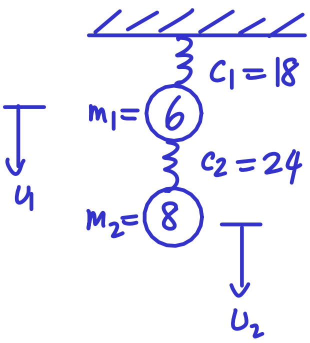
Because $\left|\frac{3}{x}\right|$ and $\left|\frac{5}{x}\right| < 1$.

(b) If $x=5$, then $\left(\frac{3}{5}\right)^k \to 0$ and $\left(\frac{5}{5}\right)^k = 1 \longrightarrow 1$.

So $\left(\frac{1}{5}A\right)^k = Q\left(\frac{1}{5}\Lambda\right)^k Q^T \longrightarrow Q\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}Q^T = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

3.(14+4 pts) Consider the fixed/free spring-mass system consisting of two masses $m_1 = 6$ and $m_2 = 8$ and two springs with constants $c_1 = 18$ and $c_2 = 24$ with no external forces present.

(a) Find $u_1(t)$ and $u_2(t)$ (the position of the two masses as functions of time), if the masses start at position $u_1(0) = 0$, $u_2(0) = 4$ and velocity $u_1'(0) = 2$, $u_2'(0) = 3$.



We want to solve

$$Mu'' + Ku = 0 \qquad u = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \quad M = \begin{bmatrix} 6 & 0 \\ 0 & 8 \end{bmatrix} \quad K = \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} = \begin{bmatrix} 42 & -24 \\ -24 & 24 \end{bmatrix}$$

Look for eigenvalues of

$$M^{-1}Kx = \omega^2 x$$

$$M^{-1}K = \underbrace{\begin{bmatrix} 1/6 & 0 \\ 0 & 1/8 \end{bmatrix}}_{M^{-1}} \underbrace{\begin{bmatrix} 42 & -24 \\ -24 & 24 \end{bmatrix}}_{K} = \begin{bmatrix} 7 & -4 \\ -3 & 3 \end{bmatrix}$$

$$\det\begin{pmatrix} 7-\lambda & -4 \\ -3 & 3-\lambda \end{pmatrix} = (7-\lambda)(3-\lambda) - 12 = \lambda^2 - 10\lambda + 21 - 12 = \lambda^2 - 10\lambda + 9 = 0 \Rightarrow \lambda = 1, 9$$

Eigenvalues: 1 & 9

Frequencies: 1 & 3

Eigenvectors: $\lambda = 1 \Longrightarrow \begin{bmatrix} 7-1 & -4 \\ -3 & 3-1 \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix} \to \text{eigv} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\lambda = 9 \Longrightarrow \begin{bmatrix} 7-9 & -4 \\ -3 & 3-9 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ -3 & -6 \end{bmatrix} \to \text{eigv} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$u(t) = (a_1 \cos t + b_1 \sin t)\begin{bmatrix} 2 \\ 3 \end{bmatrix} + (a_2 \cos(9t) + b_2 \sin(9t))\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$u(0) = a_1\begin{bmatrix} 2 \\ 3 \end{bmatrix} + a_2\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$u'(0) = b_1\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 3b_2\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 6 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad u(t) = (\cos t + \sin t)\begin{bmatrix} 2 \\ 3 \end{bmatrix} + (-\cos 3t)\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

(b) Find the elongation of each spring at $t = \pi$.

$$u(\pi) = (-1)\begin{bmatrix} 2 \\ 3 \end{bmatrix} + (-(-1))\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2+2 \\ -3-1 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

$$e_1^{(\pi)} = u_1(\pi) = 0$$

$$e_2(\pi) = u_2(\pi) - u_1(\pi) = -4 - 0 = -4$$

4.(3+4+3+4 pts) This problem has three unrelated parts.

(a) Calculate the matrix product

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 6 \\ 2 & 5 & 8 \\ 4 & 7 & 10 \end{pmatrix}$$

(b) For which values of c the following matrix equation written in augmented form has (i) NO solution, (ii) MORE THAN one solution, (iii) a SINGLE solution?

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 5 \\ 0 & -1 & 1 & c \end{array}\right) \xrightarrow{1R_2+R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 5+c \end{array}\right)$$

Final row tells us that

i) For $5+c \neq 0$, i.e. $c \neq -5$, there are no solutions

ii) For $c=-5$, there are many solutions

iii) This system has NEVER a single solution.

(c) Which of the vectors $t = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, $u = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $w = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$ is an eigenvector for the matrix $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$?

A vector v is an eigenvalue for matrix A if for some number λ, the equation $Av=\lambda v$ is satisfied.

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 13 \\ 11 \end{pmatrix}$$

Eigenvalues

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ -6 \end{pmatrix} = 3\begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

These vectors are Eigenvectors

(d) For which value(s) of t, does A have all eigenvalues,$\lambda > 0$?

$$A = \begin{pmatrix} t & 3 & 0 \\ 3 & t & 4 \\ 0 & 4 & t \end{pmatrix}$$

(A is clearly symmetric for all values of t.)

If all eigenvalues of a symmetric matrix are >0, then the matrix is positive-definite! Equivalently, all top left determinants (minors) must be >0.

① (1x1 top left determinant of A) $= t > 0$

② (2x2 〃) $= t^2-9>0$ $\Rightarrow t>3$

③ (3x3 〃) $= t(t^2-25)>0$

$\Rightarrow \boxed{t>5}$

5.(2+4+4+3 pts)The energy function for a matrix K is given as
$E(u_1, u_2, u_3) = u_1^2 + 2u_1u_2 + 2u_2^2 - 4u_2u_3 + 3u_3^2$

(a) Write the matrix K. K is a symmetric matrix so that $E(u) = u^T K u$.

$$\Rightarrow K = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & -2 & 3 \end{bmatrix}$$

(b) Find the LU decomposition of A.

$$K = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}}_{U}$$

$$K = L \quad U$$

(c) Use the LU decomposition from **(b)** to answer the following questions.

(i) Is K invertibe? Explain.

K is invertible because all pivots $(1, 1, -1)$ are NONZERO.

(ii) Is K positive-definite? Explain.

K is not positive-definite because it has -1 as a pivot.

(d) Find the LDL^T decomposition of A.

$$K = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{=L} \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}^{=D} \overbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & +1 \end{bmatrix}}^{=L^T} = LDL^T$$

6.(1pt each) Determine whether the following statements are True/False.Circle your answer.(No need to explain.)

True/**False** There is a matrix A so that $Ax = 0$ has no solution.

True/**False** For any matrix A, the sum $A + A^T$ is a symmetric matrix.

True/**False** Eigenvalues of a symmetric matrix can be complex numbers.

True/**False** Eigenvalues of a positive-semi-definite matrix can be negative.

True/**False** If $AS = SD$, D a diagonal matrix, then the nonzero columns of S are eigenvectors for A.

True/**False** If A is positive-definite, then so are $A^3, A^{-1}$and A^T.

True/**False** If A and B are positive-definite, then so is $A \cdot B$.

True/**False** If $A^3 = A$, then eigenvalues of A are 1 and -1.

7.(5 pts) Use the method of least squares to find the best fitting horizontal line to the data below, and compute the error, e^Te.

x	y
0	4
1	1
2	2
3	1

$$\underline{y=a} \qquad \Rightarrow \begin{array}{l} a=4 \\ a=1 \\ a=2 \\ a=1 \end{array} \Rightarrow A=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

Equation of a horizontal line is of the form $\boxed{y=a}$.

$$\Rightarrow A^TA=[4],\ A^Tb=[8] \Rightarrow \hat{u}=\left[\tfrac{8}{4}\right]=[2] \Rightarrow$$

Best horizontal line $\boxed{y=2}$

$$A^TA\,\hat{u}=A^Tb$$

Error: $p=A\hat{u}=\begin{bmatrix}2\\2\\2\\2\end{bmatrix}$, $e=b-p=\begin{bmatrix}2\\-1\\0\\-1\end{bmatrix} \Rightarrow e^Te=4+1+1=6$

8.(10+2 pts) Use the method of least squares to find the best fitting function of the form $y=a\cos(x)+b\sin(x)$ to the data below, and compute the error, e^Te.

x	y
0	0
$\frac{\pi}{4}$	$\sqrt{2}$
$\frac{\pi}{2}$	-2

$$\Rightarrow \begin{array}{l} \cos(0)\,a+\sin(0)\,b=0 \\ \cos(\tfrac{\pi}{4})\,a+\sin(\tfrac{\pi}{4})\,b=\sqrt{2} \\ \cos(\tfrac{\pi}{2})\,a+\sin(\tfrac{\pi}{2})\,b=-2 \end{array} \Rightarrow A=\begin{bmatrix}1&0\\ \frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\ 0&1\end{bmatrix} \quad b=\begin{bmatrix}0\\ \sqrt{2}\\ -2\end{bmatrix}$$

$$A^TA=\begin{bmatrix}1&1/\sqrt{2}&0\\0&1/\sqrt{2}&1\end{bmatrix}\begin{bmatrix}1&0\\ \frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\ 0&1\end{bmatrix}=\begin{bmatrix}\frac{3}{2}&\frac{1}{2}\\ \frac{1}{2}&\frac{3}{2}\end{bmatrix}$$

$$A^Tb=\begin{bmatrix}1&1/\sqrt{2}&0\\0&1/\sqrt{2}&1\end{bmatrix}\begin{bmatrix}0\\ \sqrt{2}\\ -2\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\det(A^TA)=\left(\tfrac{3}{2}\right)^2-\left(\tfrac{1}{2}\right)^2=\tfrac{9}{4}-\tfrac{1}{4}=\tfrac{8}{4}=2$$

$$A^TA\hat{u}=A^Tb \Rightarrow \hat{u}=(A^TA)^{-1}A^Tb=\frac{1}{2}\begin{bmatrix}\frac{3}{2}&-\frac{1}{2}\\ -\frac{1}{2}&\frac{3}{2}\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=\frac{1}{4}\begin{bmatrix}3&-1\\-1&3\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}4\\-4\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}=\hat{u}$$

$$\boxed{y=\cos x-\sin x}$$

$$p=\begin{bmatrix}1&0\\ \frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\ 0&1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

$$e=b-p=\begin{bmatrix}0\\ \sqrt{2}\\ -2\end{bmatrix}-\begin{bmatrix}1\\0\\-1\end{bmatrix}=\begin{bmatrix}-1\\ \sqrt{2}\\ -1\end{bmatrix} \Rightarrow e^Te=1+2+1=4$$

(b) Plot the data and the best fit function $y=a\cos(x)+b\sin(x)$ over the interval $[0,\frac{\pi}{2}]$.