METU - NCC

CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES MIDTERM 2	
Code : *MAT 120* Acad.Year: *2014-2015* Semester : *SPRING* Date : *02.05.2015* Time : *09:40* Duration : *120 min*	Last Name: Name : Student # : KEY Signature : 5 QUESTIONS ON 5 PAGES TOTAL 100 POINTS

1. (20)	2. (20)	3. (20)	4. (20)	5. (20)	

Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

1. *(8+12=20 pts)* Consider the function

$$f(x,y) = \frac{1}{1+x^2+5y^2}\,.$$

(A) Find the directional derivative of f at the point $K(2,1)$ in the direction indicated by the vector $\mathsf{v} = 5\mathsf{i} + 12\mathsf{j}$.

$D_u f = \vec{\nabla} f \cdot \vec{u}$ where $\vec{u}$ must be a unit vector ($|\vec{u}|=1$).

$$\vec{\nabla} f = \left\langle -\frac{2x}{(1+x^2+5y^2)^2}, -\frac{10y}{(1+x^2+5y^2)^2} \right\rangle \Rightarrow \vec{\nabla} f(2,1) = \left\langle -\frac{4}{100}, -\frac{10}{100} \right\rangle$$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 5,12\rangle}{\sqrt{5^2+12^2}} = \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle$$

$$\Rightarrow D_u f(2,1) = \left\langle -\frac{4}{100}, -\frac{10}{100} \right\rangle \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle = \frac{-4\cdot 5 - 10\cdot 12}{1300} = -\frac{140}{1300}$$

(B) Determine the direction at the point $L(3,0)$ in which the rate of change of f is the largest. Compute this rate of change.

$D_u f = \vec{\nabla} f \cdot \vec{u} = |\vec{\nabla} f||\vec{u}| \cos\theta = |\vec{\nabla} f| \cos\theta$, θ is the angle between $\vec{\nabla} f$ and $\vec{u}$.

To get the largest value, $\cos\theta$ must be 1, which means $\theta = 0$

In other words $\vec{u}$ is in the same direction of $\vec{\nabla} f$

$$\vec{\nabla} f(3,0) = \left\langle -\frac{2\cdot 3}{(1+9)^2}, -\frac{0}{(1+9)^2} \right\rangle = \left\langle -\frac{6}{100}, 0 \right\rangle$$

$$\Rightarrow \vec{u} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{\left\langle -\frac{6}{100}, 0 \right\rangle}{\sqrt{\left(-\frac{6}{100}\right)^2 + 0^2}} = \langle -1, 0 \rangle$$

$$D_u f(3,0) = \left\langle -\frac{6}{100}, 0 \right\rangle \langle -1, 0 \rangle = \frac{6}{100}$$

2.(*6+6+8=20 pts*) Consider the function $f(x,y) = (x-1)(y-1)e^{-x-2y}$.

(A) Find the critical points of f.

$$\frac{\partial f}{\partial x} = (y-1)e^{-x-2y} + (x-1)(y-1)\cdot -e^{-x-2y} = (2-x)(y-1)e^{-x-2y} = 0$$

$$\frac{\partial f}{\partial y} = (x-1)e^{-x-2y} + (x-1)(y-1)\cdot -2e^{-x-2y} = (x-1)(3-2y)e^{-x-2y} = 0$$

Both derivative becomes 0 at $(1,1)$ and $(2,\frac{3}{2})$, these are the critical points

(B) Classify the critical points if f.

$$D = \frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$\left.\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= -1(y-1)e^{-x-2y} + (2-x)(y-1)\cdot -e^{-x-2y} = (x-3)(y-1)e^{-x-2y}\\
\frac{\partial^2 f}{\partial y^2} &= -2(x-1)e^{-x-2y} + (x-1)(3-2y)\cdot -2e^{-x-2y} = (x-1)(4y-8)e^{-x-2y}\\
\frac{\partial^2 f}{\partial x \partial y} &= (2-x)e^{-x-2y} + (2-x)(y-1)\cdot -2e^{-x-2y} = (2-x)(3-2y)e^{-x-2y}
\end{aligned}\right\} \Rightarrow$$

$D(1,1) = -e^{-6} < 0$

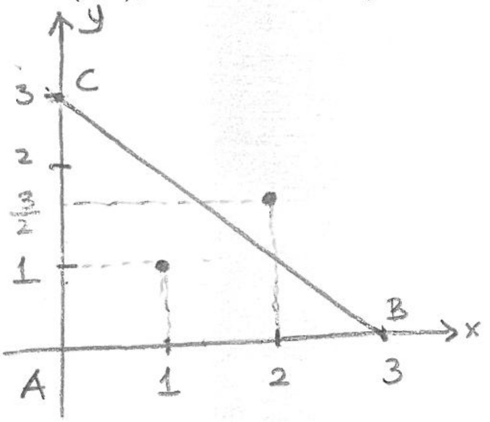
$(1,1)$ is saddle point

$D(2,\frac{3}{2}) = e^{-10} > 0$

$\frac{\partial^2 f}{\partial x^2} = -\frac{e^{-5}}{2} < 0$

$(2,\frac{3}{2})$ is local max

(C) What is the largest value of f on the closed triangular region with vertices $A(0,0)$, $B(3,0)$, $C(0,3)$, and where does f attain this value ?



On A-B: $y=0$, $0 \leq x \leq 3$

$f(x,0) = (1-x)e^{-x} \Rightarrow f' = -e^{-x} + (1-x)\cdot -e^{-x} = (x-2)e^{-x} = 0$

$\Rightarrow x = 2$

$f(2,0) = -1\cdot e^{-2}$

On B-C: $x+y=3 \Rightarrow y = 3-x$, $0 \leq x \leq 3$

$f(x,3-x) = (1-x)(2-x)e^{x-6} = (x^2-3x+2)e^{x-6}$

$\Rightarrow f' = (2x-3)e^{x-6} + (x^2-3x+2)\cdot e^{x-6} = (x^2-x-1)e^{x-6} = 0$

$\Rightarrow x = \frac{1 \mp \sqrt{1+4}}{2} \Rightarrow x = \frac{1+\sqrt{5}}{2} \Rightarrow y = \frac{5-\sqrt{5}}{2}$

$f\left(\frac{1+\sqrt{5}}{2}, \frac{5-\sqrt{5}}{2}\right) = (2-\sqrt{5})\, e^{\frac{-11+\sqrt{5}}{2}}$

On C-A: $x=0$, $0 \leq y \leq 3$

$f(0,y) = (1-y)e^{-2y} \Rightarrow f' = -1e^{-2y} + (1-y)\cdot -2e^{-2y} = (2y-3)e^{-2y} = 0$

$y = \frac{3}{2}$

$\Rightarrow f(0,\frac{3}{2}) = -e^{-3}$

End point values:

$f(0,0) = 1$

$f(3,0) = -2e^{-3}$

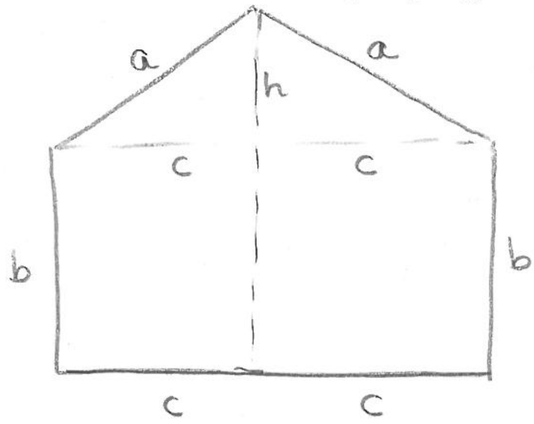
$f(0,3) = -2e^{-6}$

At the critical point:

$f(1,1) = 0$

By comparing all these values; f has absolute max at $(0,0)$.

3.(*20 pts*) A pentagon is formed by placing an isosceles triangle on a rectangle as shown in the figure. If the pentagon has 2m parameter then find the sides a, b and c by using Lagrange Multipliers to maximize its area.



Area $= 2bc + ch$

$h^2 + c^2 = a^2 \Rightarrow h = \sqrt{a^2 - c^2}$

$\Rightarrow A(a,b,c) = \left(2b + \sqrt{a^2 - c^2}\right) c$

Perimeter $= 2a + 2b + 2c$

$P(a,b,c) = 2(a+b+c) = 2$.

We wont to maximize $A(a,b,c)$ subject to $a+b+c=1$.

Using Lagrange Multipliers;

$$\vec{\nabla} A = \lambda \vec{\nabla} P \Rightarrow \left\langle \frac{ac}{\sqrt{a^2-c^2}},\ 2c,\ \frac{-c^2}{\sqrt{a^2-c^2}} + 2b + \sqrt{a^2-c^2} \right\rangle = \lambda \langle 1,1,1 \rangle$$

$$\Rightarrow \frac{ac}{\sqrt{a^2-c^2}} = 2c = 2b + \frac{a^2 - 2c^2}{\sqrt{a^2-c^2}} = \lambda$$

So, $c = \frac{\lambda}{2} \Rightarrow \dfrac{a \cdot \frac{\lambda}{2}}{\sqrt{a^2 - \left(\frac{\lambda}{2}\right)^2}} = \lambda \Rightarrow \frac{a}{2} = \sqrt{a^2 - \left(\frac{\lambda}{2}\right)^2}$ ($\lambda = 0 \Rightarrow c = 0$ nonsense)

$\frac{a^2}{4} = a^2 - \frac{\lambda^2}{4} \Rightarrow \lambda^2 = 3a^2 \Rightarrow \lambda = a\sqrt{3}$ (or $\lambda = -a\sqrt{3}$ but $c = \lambda/2$ can not be $-$)

Also, $2b + \dfrac{\frac{\lambda^2}{3} - \frac{2 \cdot \lambda^2}{4}}{\sqrt{\frac{\lambda^2}{3} - \frac{\lambda^2}{4}}} = \lambda \Rightarrow 2b - \frac{\lambda}{\sqrt{3}} = \lambda$

$\Rightarrow b = \frac{\lambda}{2} + \frac{\lambda}{2\sqrt{3}}$

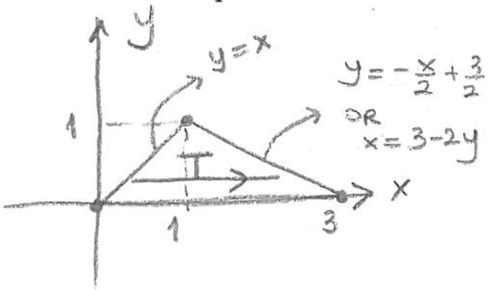
$a+b+c=1 \Rightarrow \frac{\lambda}{\sqrt{3}} + \frac{\lambda}{2} + \frac{\lambda}{2\sqrt{3}} + \frac{\lambda}{2} = 1 \Rightarrow \lambda + \frac{\lambda\sqrt{3}}{2} = 1$

$\Rightarrow \lambda = \frac{2}{2+\sqrt{3}}$

$$\Rightarrow a = \frac{2}{3+2\sqrt{3}};\quad b = \frac{1+\sqrt{3}}{3+2\sqrt{3}};\quad c = \frac{\sqrt{3}}{3+2\sqrt{3}}$$

4.(*6+6+8=20 pts*) This problem has three **unrelated** parts about double integrals.

(A) Compute $\iint_T xy^2 dA$ where T is the triangle with vertices (0,0), (1,1) and (3,0).



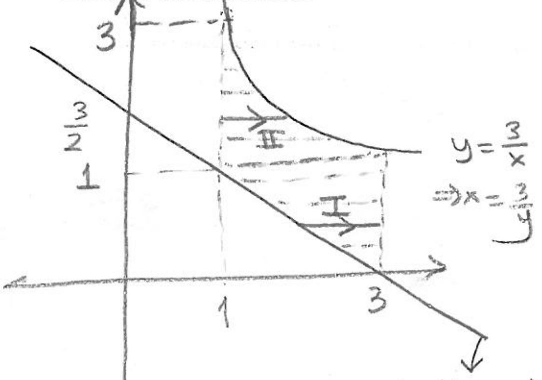
$$\int_0^1\int_y^{3-2y} xy^2\,dx\,dy = \int_0^1 \frac{x^2y^2}{2}\Big|_y^{(3-2y)} dy$$

$$= \int_0^1 \left[\frac{y^2(3-2y)^2}{2} - \frac{y^4}{2}\right] dy$$

$$= \frac{1}{2}\int_0^1 9y^2 - 12y^3 + 4y^4 - y^4\,dy = \frac{1}{2}\left(\frac{3y^5}{5} - 3y^4 + 3y^3\right)\Big|_0^1$$

$$= \frac{3}{10}$$

(B) Sketch the region of integration for $\int_1^3 \int_{-\frac{1}{2}x+\frac{3}{2}}^{\frac{3}{x}} f(x,y)\, dy\, dx$ and rewrite the integral with reverse orientation.

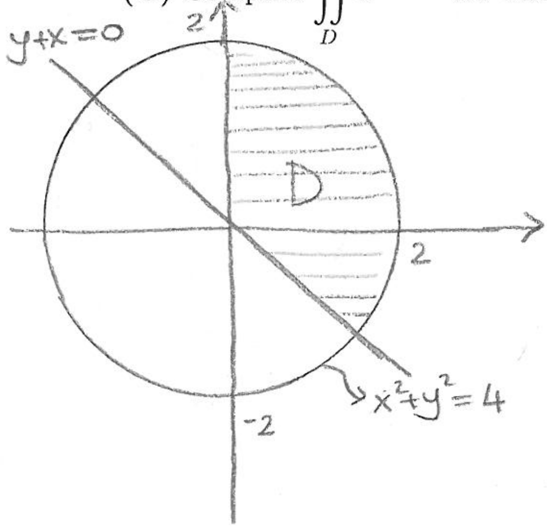


$x = 3-2y \Leftarrow y = -\frac{x}{2} + \frac{3}{2}$

Integral can be rewritten as two integrals;

$$\underbrace{\int_0^1\int_{3-2y}^{3} f(x,y)\,dx\,dy}_{\mathrm{I}} + \underbrace{\int_1^3\int_1^{\frac{3}{y}} f(x,y)\,dx\,dy}_{\mathrm{II}}$$

(C) Compute $\iint_D e^{-x^2-y^2} dA$ where $D = \{(x,y) | 0 \leqslant x \leqslant \sqrt{4-y^2}$ and $0 \leqslant y+x\}$.



In polar coordinates, integral become;

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^2 e^{-r^2} r\,dr\,d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} -\frac{e^{-r^2}}{2}\Big|_0^2 d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} -\frac{e^{-4}-1}{2}\,d\theta$$

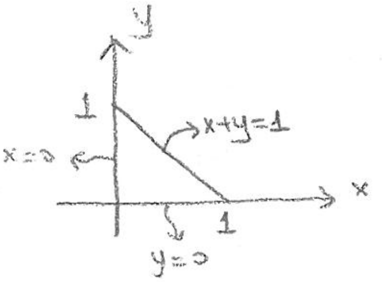
$$= -\frac{e^{-4}-1}{2}\cdot\theta\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{(1-e^{-4})\cdot 3\pi}{8}$$

5.(*8+12=20 pts*) This problem has two **unrelated** parts.

(A) Use substitution $u = x - y,\quad v = x + y$ to evaluate $\displaystyle\iint_T e^{\frac{x-y}{x+y}}\,dA$ where T is the triangle with vertices (0,0), (1,0) and (0,1).

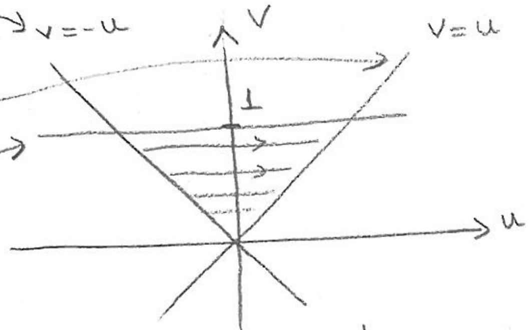
$$\left.\begin{array}{l} u = x-y \\ v = x+y \end{array}\right\} \Rightarrow \left.\begin{array}{l} x = \frac{v+u}{2} \\ y = \frac{v-u}{2} \end{array}\right\} \Rightarrow J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$



$x=0 \Rightarrow \frac{v+u}{2} = 0 \Rightarrow v = -u$ → $v=-u$

$y=0 \Rightarrow \frac{v-u}{2} = 0 \Rightarrow v = u$

$x+y=1 \Rightarrow v = 1$



Integral becomes;

$$\int_0^1\int_{-v}^{v} e^{\frac{u}{v}}\cdot\frac{1}{2}\,du\,dv = \int_0^1 \frac{e^{\frac{u}{v}}\cdot v}{2}\Bigg|_{-v}^{v} dv = \int_0^1 \frac{e^1 - e^{-1}}{2}\, v\, dv = \frac{e^1-e^{-1}}{2}\cdot\frac{v^2}{2}\Bigg|_0^1 = \frac{e^1-e^{-1}}{4}$$

(B) Use a suitable substitution to evaluate $\displaystyle\iint_D \frac{x}{\text{[redacted]}}\,dA$ where D is the region bounded by $y = \frac{1}{x^2},\ y = \frac{2}{x^2},\ y = \frac{1}{2x},\ y = \frac{1}{x}$.

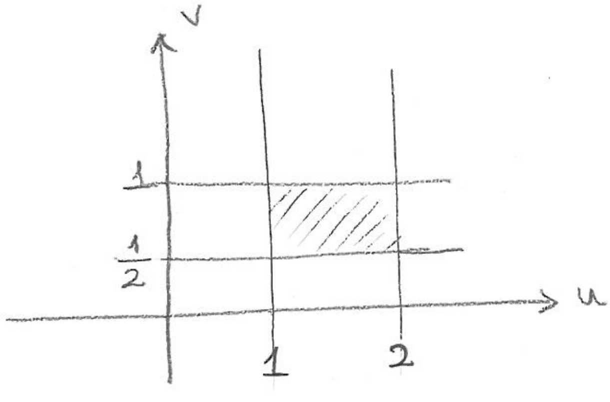
Let $\left.\begin{array}{l} u = x^2y \\ v = xy \end{array}\right\} \Rightarrow \left.\begin{array}{l} x = \frac{u}{v} \\ y = \frac{v^2}{u} \end{array}\right\} \Rightarrow J = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ -\frac{v^2}{u^2} & \frac{2v}{u} \end{vmatrix} = \frac{2}{u} - \frac{1}{u} = \frac{1}{u}$

$y = \frac{1}{x^2} \Rightarrow x^2y = 1 \Rightarrow u = 1$

$y = \frac{2}{x^2} \Rightarrow x^2y = 2 \Rightarrow u = 2$

$y = \frac{1}{2x} \Rightarrow xy = \frac{1}{2} \Rightarrow v = \frac{1}{2}$

$y = \frac{1}{x} \Rightarrow xy = 1 \Rightarrow v = 1$



Integral becomes;

$$\int_{\frac{1}{2}}^{1}\int_1^2 \cancel{\frac{u}{v}}\cdot\frac{1}{\cancel{u}}\,du\,dv = \int_{\frac{1}{2}}^{1} \frac{u}{v}\Bigg|_1^2 dv = \int_{\frac{1}{2}}^{1} \frac{1}{v}\,dv = \ln v\Big|_{\frac{1}{2}}^{1} = \ln 2.$$