METU
Northern Cyprus Campus

Math 219	Differential Equations	Midterm Exam I	10.07.2014
Last Name: Name : Student No:	Dept./Sec. : Time : 17: 40 Duration : 90 *minutes*	Signature	
5 QUESTIONS		TOTAL 20 POINTS	

1	2	3	4	5			

Q1 (15 pts.) Consider IVP $\begin{cases} \ln(t)\, y' + 2y = \tan(t), \\ y(1.5) = -2. \end{cases}$. Find the longest open interval where the unique solution to IVP is certain to exist. Explain your answer. **Don't solve the equation.**

Put $p(t) = \dfrac{2}{\ln(t)}$, $q(t) = \dfrac{\tan(t)}{\ln(t)}$

10 Both are continuous on $I = (1, \pi/2)$

Note that $1.5 \in I$ and $f(t,y) = -p(t)y + q(t)$ is continuous over $I \times R$, and

5 $\dfrac{\partial f}{\partial y} = -p(t) \in C(I \times R)$

By E.U.T, I.V.P. has a unique solution over I (globally).

Q2 (25+5 pts.) A tank with a capacity of 1000 gal originally contains 400 gal of water with 200 oz of salt dissolved in. Water containing 1/4 oz per gallon is entering at a rate of 5 gal/min, and the mixture is allowed to flow out at a rate of 3 gal/min. Find the amount of salt in the tank at any time and decide when the mixture begins to overflow. (**5 pts**) Find also the concentration of salt in the tank when it is on the point of overflowing.

Let $Q(t)$ be the amount of salt in the tank at any time t. We have

10 $$\begin{cases} Q'(t) = \frac{5}{4} - 3\,\frac{Q(t)}{400+(5-3)t} \\ Q(0) = 200 \end{cases} ; \quad p(t) = \frac{3}{400+2t}, \quad q(t) = \frac{5}{4}$$

5 Integrating factor: $\mu(t) = e^{\int p(t)dt} = e^{\frac{3}{2}\ln(2t+400)} = (2t+400)^{3/2}$

Then $(\mu \cdot Q)' = \frac{5}{4}(2t+400)^{3/2} \Rightarrow \mu \cdot Q = C + \frac{1}{4}(2t+400)^{5/2} \Rightarrow$

$\Rightarrow Q(t) = C(2t+400)^{-3/2} + \frac{1}{4}(2t+400)$.

But $200 = Q(0) = C\,400^{-3/2} + 100 \Rightarrow C = 100 \cdot 400^{3/2} = 8 \cdot 10^{5}$.

10 Hence $Q(t) = 8 \cdot 10^{5}(2t+400)^{-3/2} + \frac{1}{4}(2t+400)$.

The overflowing time T: $400 + 2T = 1000 \Rightarrow T = 300$ min.

So, $Q(300) = 8 \cdot 10^{5} \cdot 1000^{-3/2} + \frac{1}{4} \cdot 1000 = 8 \cdot \sqrt{10} + 250$ oz.

Therefore the concentration of salt in the tank when it is on the point of overflowing is

5 $$\frac{Q(300)}{1000} = \frac{8}{10^{5/2}} + 0.25 \text{ oz/gal}.$$

Q3 (20 pts.) Solve the differential equation $(t^2+3ty+y^2)\,dt = t^2 dy$.

5 Put $v=\frac{y}{t}$. Then $y=t\cdot v$, $y'=v+tv'=1+3v+v^2$

$\Rightarrow tv'=1+2v+v^2=(v+1)^2 \Rightarrow \frac{dv}{(v+1)^2}=\frac{dt}{t}$ $(v\neq -1)$.

Hence $-\frac{1}{v+1}=\ln(C|t|)$, $C>0$ or just $-\frac{1}{v+1}=\ln|t|+C$

10 $\Rightarrow -\frac{1}{C+\ln|t|}=v+1 \Rightarrow -1-\frac{1}{\ln(C|t|)}=v$, $C>0 \Rightarrow$

$\Rightarrow y=-t-\frac{t}{\ln(C|t|)}$, $C>0$. But $v=-1 \Leftrightarrow$

5 $y=-t$ is a solution as well lost in the separation process. Hence

$$y=-t-\frac{t}{\ln(C|t|)}, \quad y=-t$$

is the general solution.

Q4 (10 pts.) Consider the matrix-valued function $A(t)=\begin{bmatrix}\cos^2(t) & \ln(t)\\ te^{2t} & \sec^2(t)\end{bmatrix}$. Find the matrix $\int_1^2 A(t)\,dt$.

$\int_1^2 \cos^2(t)\,dt=\int_1^2 \frac{1+\cos(2t)}{2}\,dt=\frac{1}{2}+\frac{1}{4}(\sin(4)-\sin(2))$;

5 $\int_1^2 \ln(t)\,dt=(t\ln(t)-t)\Big|_1^2=2\ln(2)-1$.

$\int_1^2 te^{2t}\,dt=\left(\frac{1}{2}te^{2t}-\frac{1}{4}e^{2t}\right)\Big|_1^2=\frac{1}{2}(2e^4-e^2)-\frac{1}{4}(e^4-e^2)$

5 $\int_1^2 \sec^2(t)\,dt=\tan(t)\Big|_1^2=\tan(2)-\tan(1)$. Whence

$$\int_1^2 A(t)\,dt=\begin{bmatrix}\frac{1}{2}+\frac{1}{4}(\sin(4)-\sin(2)) & 2\ln(2)-1\\ \frac{1}{2}(2e^4-e^2)-\frac{1}{4}(e^4-e^2) & \tan(2)-\tan(1)\end{bmatrix}$$

Q5 (25 pts.) Solve the differential equation $ydt + (2ty - e^{-2y})\, dy = 0$.

5 Suppose $\mu = \mu(y)$

$$\underbrace{\mu y}_{M}\, dt + \underbrace{\mu(2ty - e^{-2y})}_{N}\, dy = 0$$

$$M_y = \mu' y + \mu = N_t = 2y\mu \Rightarrow y \cdot \mu' = (2y-1)\mu.$$

$$\frac{d\mu}{\mu} = \left(\frac{2y-1}{y}\right) dy = \left(2 - \frac{1}{y}\right) dy$$

10 $\ln|\mu| = 2y - \ln|y| + c \Rightarrow \boxed{\mu = e^{2y} \cdot y^{-1}}$ is an integrating factor.

$$e^{2y} dt + y^{-1} e^{2y} (2ty - e^{-2y})\, dy = 0$$

$$\underbrace{e^{2y}}_{M}\, dt + \underbrace{(2te^{2y} - y^{-1})}_{N}\, dy = 0$$

$M_y = 2e^{2y} = N_t = 2e^{2y}$. So it is an exact dif. equation.

Find a potential function ψ!

10 $\begin{cases} \psi_t = e^{2y} \Rightarrow \psi = e^{2y} t + c(y) \\ \psi_y = 2te^{2y} - \frac{1}{y} = 2te^{2y} + c'(y) \Rightarrow c'(y) = -\frac{1}{y} \Rightarrow \end{cases}$

$\Rightarrow c(y) = -\ln|y| \Rightarrow e^{2y} t - \ln|y| = c$ is the general solution in the implicit form.