METU
Northern Cyprus Campus

Math 219 Differential Equations Midterm Exam I 15.07.2012		
Last Name : Name : Student No :	Dept./Sec.: Time : 19: 00 Duration : 90 *minutes*	Signature
6 QUESTIONS ON 4 PAGES	KEY	TOTAL 100 POINTS

1	2	3	4	5	6		

Q1 (20 p.) A tank contains 10 gal of water and 5 lb of salt. Water containing a salt concentration of $\frac{1}{2}\left(1+\frac{1}{2}\cos(t)\right)$ lb/gal flows into the tank at a rate of 2 lb/min, and the well-stirred mixture in the tank flows out at the same rate. Find the amount of salt $Q(t)$ in the tank at any time. Finally, **(bonus 10 p.)** predict the time when amount of salt will be at most 5.7 lb.

We have $Q'(t) = \text{rate in} - \text{rate out} = \frac{1}{2}\left(1+\frac{1}{2}\cos(t)\right)\cdot 2 - \frac{Q}{10}\cdot 2$, or

$Q'(t) + \frac{1}{5}Q(t) = 1 + \frac{1}{2}\cos(t)$, $Q(0)=5$.

The integrating factor $\mu(t) = e^{t/5} \Rightarrow (\mu Q)' = e^{t/5} + \frac{1}{2}e^{t/5}\cos(t) \Rightarrow$

$\Rightarrow \mu Q = 5e^{t/5} + \frac{1}{2}\int e^{t/5}\cos(t)\,dt + C = 5e^{t/5} + \frac{e^{t/5}}{2}\frac{25}{26}\left(\frac{1}{5}\cos(t)+\sin(t)\right) + C$

$\Rightarrow Q = 5 + \frac{1}{2}\frac{25}{26}\left(\frac{1}{5}\cos(t)+\sin(t)\right) + Ce^{-t/5}$

But $5 = Q(0) = 5 + \frac{1}{2}\frac{25}{26}\frac{1}{5} + C \Rightarrow C = -\frac{5}{52}$

Consequently, $Q(t) = 5 + \frac{25}{52}\left(\frac{1}{5}\cos(t)+\sin(t)\right) - \frac{5}{52}e^{-t/5}$

Finally, let's estimate

$|Q(t)| \le 5 + \frac{25}{52}\left(\frac{1}{5}+1\right) + \frac{5}{52}e^{-t/5} = 5 + \frac{15}{26} + \frac{5}{52}e^{-t/5}$

$< 5 + \frac{2}{3} + \frac{5}{52}e^{-t/5} = \frac{17}{3} + \frac{5}{52}e^{-t/5}$

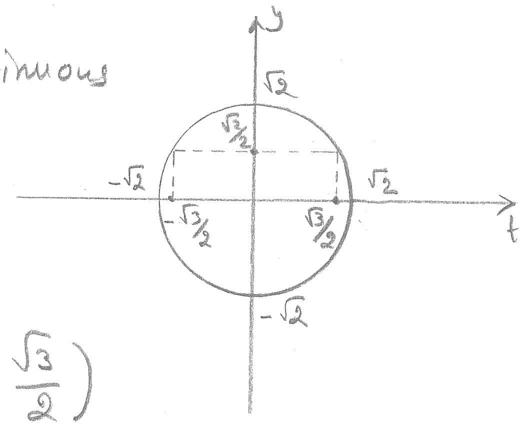
Put $\frac{5}{52}e^{-t/5} \le 0.01 \Leftrightarrow e^{-t/5} \le \frac{52}{5}\cdot 0.01 \Leftrightarrow \frac{5}{0.01\cdot 52} \le e^{t/5}$

$\Leftrightarrow t/5 \ge \ln\left(\frac{5}{0.01\cdot 52}\right) \Leftrightarrow t \ge 5\ln(9.61538462)$. Then

$|Q(t)| < 5.66667 + 0.01 < 5.7$

Q2 (15 p.) Consider the following IVP $\begin{cases} y' = 3t\sqrt{2-t^2-y^2} \\ y(0) = \dfrac{\sqrt{2}}{2} \end{cases}$. Based on the Existence and Uniqueness Theorem, find a largest possible (open) interval about the origin where the unique solution to IVP could exist in.

The function $f(t,y) = 3t\sqrt{2-t^2-y^2}$ is continuous on the region $t^2+y^2 \leq 2$, whereas $\frac{\partial f}{\partial y} = \frac{-3ty}{\sqrt{2-t^2-y^2}}$ is continuous on the region $t^2+y^2<2$. By E-U-T, $I = \left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$ is the longest possible interval.



Q3 (20 p.) Find the solution to the initial value problem $\begin{cases} ty' + (t+1)\,y = 2t, & t>0 \\ y(\ln(3)) = 2 \end{cases}$ using Variation of Parameters Method. (*Do not use the integrating factors*).

Homg. dif. equation related to nonhomg: $y' + \frac{t+1}{t}y = 0 \Rightarrow$

$\Rightarrow y(t) = C\,e^{-\int \frac{t+1}{t}dt} = C\,e^{-t-\ln(t)} = \frac{C}{te^t}$

Put $Y(t) = C(t)\frac{1}{te^t}$. Then $Y'(t) = \frac{C'(t)}{te^t} - C(t)\frac{e^t(t+1)}{t^2e^{2t}} =$

$= \frac{C'(t)}{te^t} - \frac{t+1}{t^2e^t}C(t) \Rightarrow Y'(t) + \frac{t+1}{t}Y(t) = \frac{C'(t)}{te^t} - \frac{t+1}{t^2e^t}C(t) +$

$+ \frac{t+1}{t}\frac{C(t)}{te^t} = \frac{C'(t)}{te^t} = 2 \Rightarrow C'(t) = 2te^t \Rightarrow C(t) = 2\int te^t dt =$

$= 2(t-1)e^t + c$. Hence $y(t) = \frac{2(t-1)}{t} + \frac{c}{te^t}$

IVP: $2 = y(\ln(3)) = 2 - \frac{2}{\ln(3)} + \frac{c}{\ln(3)\cdot 3} \Rightarrow \frac{c}{3} = 2 \Rightarrow c = 6$

$y(t) = \frac{2(t-1)+6e^{-t}}{t}$ is the solution to IVP.

Q4 (20 p.) Show that the given differential equation is not exact, whereas $\mu(y) = ye^y$ is an integrating factor. Using this fact find the general solution to the differential equation $y\cos(x)\,dx + (y+2)\sin(x)\,dy = 0$.

$y^2e^y\cos(x)dx + y(y+2)e^y\sin(x)dy = 0$ is exact:

$M_y = 2ye^y\cos(x) + y^2e^y\cos(x) = N_x = y(y+2)e^y\cos(x)$

Thereby, there is a potential function $\Psi(x,y)$:

$$\begin{cases} \Psi_x = y^2e^y\cos(x) \Rightarrow \Psi = y^2e^y\sin(x) + C(y) \\ \Psi_y = y(y+2)e^y\sin(x) \Rightarrow y(y+2)e^y\sin(x) = 2ye^y\sin(x) + \end{cases}$$

$+\, y^2e^y\sin(x) + C'(y) \Rightarrow C'(y) = 0 \Rightarrow C(y) = \text{const}$

Hence $y^2e^y\sin(x) = C$ is the general solution.

Q5 (10 p.) Let $\mathbf{x}^{(1)}(t) = \begin{bmatrix} e^t\sin(t) \\ e^t\cos(t) \end{bmatrix}$ and $\mathbf{x}^{(2)}(t) = \begin{bmatrix} 3\sin(t) \\ 3\cos(t) \end{bmatrix}$ be the vector-valued functions defined on the real line. Show that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly depended vectors at each point $t \in \mathbb{R}$. But show that as a vector-valued functions they are linear independent on $\mathbb{R}$.

Indeed, at each point $t \in \mathbb{R}$, we have $e^{-t}\vec{x}^{(1)}(t) - \frac{1}{3}\vec{x}^{(2)}(t) = \vec{0}$

or $\vec{x}^{(2)}(t) = 3e^{-t}\vec{x}^{(1)}(t)$, $3e^{-t} \neq 0$.

Put $c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t) = \vec{0}$, $\forall t \in \mathbb{R}$. So,

$$\begin{cases} e^t\sin(t)\,c_1 + 3\sin(t)\,c_2 = 0 \\ e^t\cos(t)\,c_1 + 3\cos(t)\,c_2 = 0 \end{cases}$$

$$\left.\begin{aligned} \text{Put } t = 0 &\Rightarrow c_1 + 3c_2 = 0 \\ t = \pi/2 &\Rightarrow e^{\pi/2}c_1 + 3c_2 = 0 \end{aligned}\right\} \quad \begin{vmatrix} 1 & 3 \\ e^{\pi/2} & 3 \end{vmatrix} = 3 - 3e^{\pi/2} \neq 0$$

Whence $c_1 = c_2 = 0$. Thus $\vec{x}^{(1)}(t), \vec{x}^{(2)}(t)$ are linearly independent functions on $\mathbb{R}$.

Q6 (15 p.) Find the inverse of 3×3-matrix $A = \begin{bmatrix} 2 & -2 & -2 \\ 4 & 2 & 0 \\ 6 & -4 & 2 \end{bmatrix}$. Use the row reduction technique.

$$2\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \sim 2\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 2 & -2 & 1 & 0 \\ 0 & 1 & 4 & -3 & 0 & 1 \end{array}\right]$$

$$\sim 2\left[\begin{array}{ccc|ccc} 3 & 0 & -1 & 1 & 1 & 0 \\ 0 & 3 & 2 & -2 & 1 & 0 \\ 0 & 0 & -10 & 7 & 1 & -3 \end{array}\right] \sim 2\left[\begin{array}{ccc|ccc} -30 & 0 & 0 & -3 & -9 & -3 \\ 0 & 15 & 0 & -3 & 6 & -3 \\ 0 & 0 & -10 & 7 & 1 & -3 \end{array}\right]$$

$$\sim 2\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/10 & 3/10 & 1/10 \\ 0 & 1 & 0 & -1/5 & 2/5 & -1/5 \\ 0 & 0 & 0 & -7/10 & -1/10 & 3/10 \end{array}\right]$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 1/10 & 3/10 & 1/10 \\ -1/5 & 2/5 & -1/5 \\ -7/10 & -1/10 & 3/10 \end{bmatrix}$$