M E T U
Northern Cyprus Campus

	Calculus for Functions of Several Variables Final Exam	

Code : *Math 120* Acad.Year: *2012-2013* Semester : *Summer* Date : *05.8.2013* Time : *09:00* Duration : *180 minutes*	Last Name: Name: KEY Department: Signature: Student No: Section:
	11 QUESTIONS ON 8 PAGES TOTAL 120 POINTS

1 (13)	2 (12)	3 (10)	4 (8)	5 (8)	6 (9)	7 (10)	8 (12)	9 (16)	10 (10)	11 (12)		

Show your work! No calculators! Please draw a $\boxed{\text{box}}$ **around your answers!**

Please do not write on your desk!

1.(6+7 pts) Given two lines $r_1(t) =< 1-t, 3+t, t >$ and $r_2(s) =< 2+s, 3s-2, 2s-4 >$

(a) Show that r_1 and r_2 are intersecting.

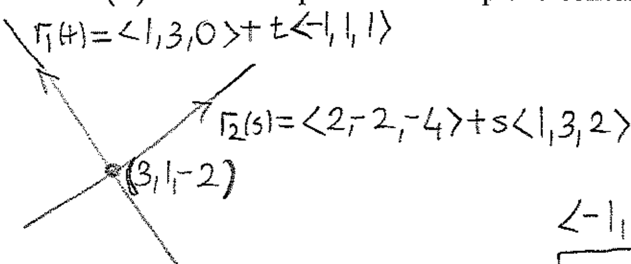
① $1-t = 2+s$ ② $3+t = 3s-2$ } $s+t=-1$, $3s-t=5$ $\Rightarrow$ $4s=4$, $s=1 \Rightarrow t=-2$

$r_1(-2) = \langle 3, 1, -2 \rangle$

$r_2(1) = \langle 3, 1, -2 \rangle$

③ $t = 2s-4$ (Check: $-2 = 2\cdot 1 - 4$ ✓)

(b) Find the equation of the plane containing the lines r_1 and r_2.



$$\vec{n} = \begin{vmatrix} i & j & k \\ -1 & 1 & 1 \\ 1 & 3 & 2 \end{vmatrix} = -i + 3j - 4k = \langle -1, 3, -4 \rangle$$

$$\langle -1, 3, -4 \rangle \cdot \langle x-3, y-1, z+2 \rangle = 0$$

$$\boxed{-x + 3y - 4z - 8 = 0}$$

2.(12 pts) Find and classify all critical points of $f(x, y) = 6xy - 2x^3 + 3x^2 + 3y^2 + 6y - 10$.

$f_x = 6y - 6x^2 + 6x = 6(y - x^2 + x) = 0 \Rightarrow y = x^2 - x$

$f_y = 6x + 6y + 6 = 6(x + y - 1) = 0 \Rightarrow y = 1 - x$

So, $1 - x = x^2 - x \Rightarrow x^2 - 1 = 0 \Rightarrow x = 1, y = 0$ $(1,0)$

OR

$x = -1, y = 2$ $(-1,2)$

$$H(f) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -12x+6 & 6 \\ 6 & 6 \end{bmatrix}$$

$(1,0)$

$$H(f) = \begin{bmatrix} -6 & 6 \\ 6 & 6 \end{bmatrix}$$ $D = -72 < 0$ Saddle Point

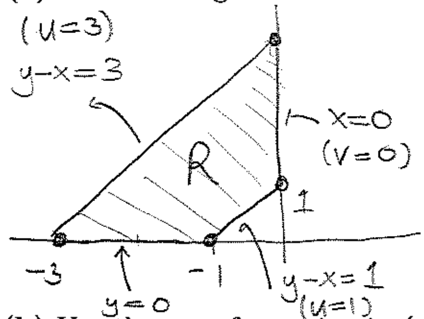
$(-1,2)$

$$H(f) = \begin{bmatrix} 18 & 6 \\ 6 & 6 \end{bmatrix}$$ $D = 72 > 0$, $f_{xx} = 18 > 0$

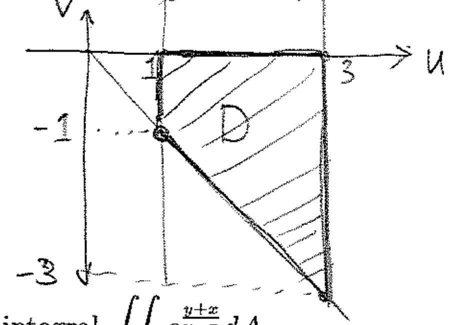
Local Minimum

3. (4+6 pts) Let R be the trapezoid with vertices (-3,0),(-1,0),(0,1) and (0,3).

(a) Draw the region R and its image under the transfomation $y-x=u, x=v$



$y=0$
$-x=u$
$\boxed{-v=u}$



$x=v$
$y=u+v$

(b) Use the transformation in **(a)** to evaluate the integral $\iint_R e^{\frac{y+x}{y-x}} dA$

$$\frac{\partial(x,y)}{\partial(u,v)} = \left|\begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix}\right| = |-1| = 1$$

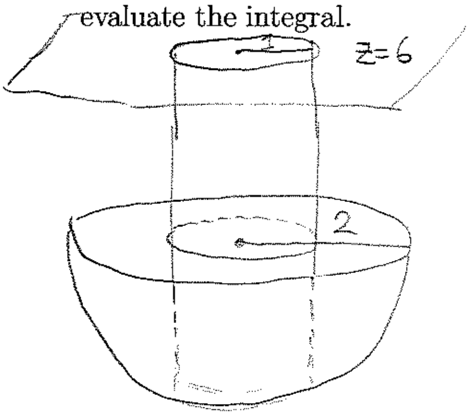
$$\iint_R e^{\frac{y+x}{y-x}} dA = \int_1^3 \int_{-u}^0 e^{\frac{u+2v}{u}} \cdot 1 \cdot dv\, du = \int_1^3 \int_{-u}^0 e \cdot e^{2\frac{v}{u}} dv\, du$$

$$= e \cdot \int_1^3 \left(\frac{u}{2} e^{2\frac{v}{u}} \Big|_{-u}^{0} \right) du = e \int_1^3 \left(\frac{u}{2} e^0 - \frac{u}{2} e^{-2} \right) du$$

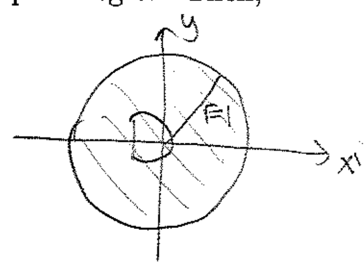
$$= e \int_1^3 (1-e^{-2}) \frac{u}{2} du = e(1-e^{-2}) \frac{u^2}{4} \Big|_1^3 = e(1-e^{-2})\left(\frac{9}{4} - \frac{1}{4}\right)$$

$$= \boxed{e(1-e^{-2}) \cdot 2}$$

4. (8 pts) Express the volume of the solid inside the circular cylinder $x^2+y^2=1$, bounded by the half sphere $\{x^2+y^2+z^2=4, z \le 0\}$ and the plane $z=6$ as a triple integral. Then, evaluate the integral.



$$\iiint_D \int_{-\sqrt{4-x^2-y^2}}^{6} 1\, dz\, dA$$



$$= \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{6} dz \cdot r\, dr\, d\theta = \int_0^{2\pi} \int_0^1 \left((z) \Big|_{-\sqrt{4-r^2}}^{6} \right) r\, d$$

$$= \int_0^{2\pi} \int_0^1 \left(6 + \sqrt{(4-r^2)}\right) r\, dr\, d\theta = \int_0^{2\pi} d\theta \cdot \int_0^1 6r + \sqrt{4-r^2}\, r\, d\theta$$

$$= \theta \Big|_0^{2\pi} \cdot \left(3r^2 + \frac{(4-r^2)^{3/2}}{-3} \right) \Big|_0^1 = 2\pi \cdot \left(\left(3 + \frac{3\sqrt{3}}{-3}\right) - \left(0 + \frac{8}{-3}\right) \right)$$

5. (4+4 pts) Evaluate the line integrals below.

(a) $\int_C yz\cos x\,ds$ where C has a parametrization $\mathbf{r}(t) =< t, 3\cos t, 3\sin t >$, $0 \le t \le \pi$.

$$\int_0^{\pi} 3\cos t\cdot 3\sin t\cos(t)\sqrt{1+9\sin^2 t+9\cos^2 t}\,dt = \int_0^{\pi} 9\cos^2(t)\cdot\sin t\cdot\sqrt{10}\,dt$$

$u=\cos t$
$du=-\sin t\,dt$

$$= -\int_1^{-1} 9\sqrt{10}\,u^2\,du = \int_{-1}^{1} 9\sqrt{10}\,u^2\,du = 9\sqrt{10}\,\frac{u^3}{3}\Big|_{-1}^{1}$$

$$= 9\sqrt{10}\left(\frac{1}{3}-\left(\frac{-1}{3}\right)\right) = \boxed{\frac{2\sqrt{10}}{3}}$$

(b) $\int_C xy\,dx - y^2\,dy + xz\,dz$ where $C$is the line segment from $(1,1,2)$ to $(0,0,0)$.

$$r(t) = \langle 1,1,2\rangle + t\langle -1,-1,-2\rangle = \langle 1-t, 1-t, 2-2t\rangle \quad 0\le t\le 1$$

$$\int_0^1 -(\cancel{1-t})^2\,dt + (\cancel{1-t})^2\,dt - 2\cdot(1-t)(2-2t)\,dt = -2\int_0^1 2t^2-4t+2\,dt$$

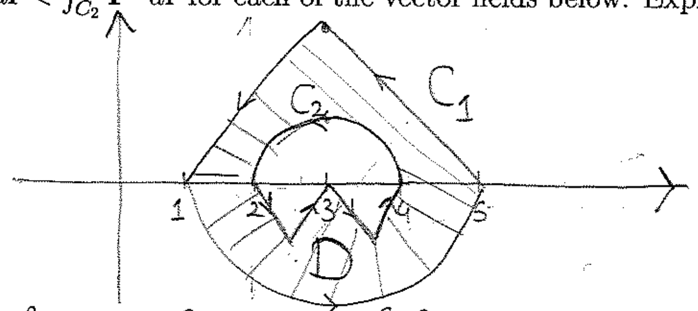
$$= -2\left(2\frac{t^3}{3} - 2t^2 + 2t\right)\Big|_0^1 = -2\left(\frac{2}{3}-2+2\right) = \boxed{-\frac{4}{3}}$$

6.(9 pts) Consider the curves C_1 and C_2 given in the figure. Decide whether $\int_{C_1}\mathbf{F}\cdot d\mathbf{r} > \int_{C_2}\mathbf{F}\cdot d\mathbf{r}$, $\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_{C_2}\mathbf{F}\cdot d\mathbf{r}$ or $\int_{C_1}\mathbf{F}\cdot d\mathbf{r} < \int_{C_2}\mathbf{F}\cdot d\mathbf{r}$ for each of the vector fields below. Explain your answer.

(a) $\mathbf{F}(x,y) = y\mathbf{i} + 3\mathbf{j}$

(b) $\mathbf{F}(x,y) = x^2\mathbf{i} + y^2\mathbf{j}$

(c) $\mathbf{F}(x,y) = x\mathbf{i} + x^2\mathbf{j}$



By Green's Theorem, $\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_{C_2}\mathbf{F}\cdot d\mathbf{r} + \iint_D (Q_x - P_y)\,dA$

a) $\int_{C_1} F\cdot dr = \int_{C_2} F\cdot dr + \boxed{\iint_D (-1)\cdot dA} \Rightarrow \int_{C_1} F\cdot dr < \int_{C_2} F\cdot dr$

b) $\int_{C_1} F\cdot dr = \int_{C_2} F\cdot dr + \iint_D (0-0)\,dA \Rightarrow \int_{C_1} F\cdot dr = \int_{C_2} F\cdot dr$

c) $\int_{C_1} F\cdot dr = \int_{C_2} F\cdot dr + \iint_D 2x\cdot dA \Rightarrow \int_{C_1} F\cdot dr > \int_{C_2} F\cdot dr$

7.(3+7 pts) (a) Find the value of the constant a for which the vector field $\mathbf{F}(x,y) = (ax^3y^2 - 2xy^3)\mathbf{i} + (2x^4y - 3x^2y^2 + 4y^3)\mathbf{j}$ is conservative.

$$\frac{\partial}{\partial x}\left(2x^4y - 3x^2y^2 + 4y^3\right) = \frac{\partial}{\partial y}\left(ax^3y^2 - 2xy^3\right)$$

$$8x^3y - 6xy^2 + 0 = 2ax^3y - 6xy^2$$

$$2a = 8 \Rightarrow \boxed{a=4}$$

(b) For the value of a found above, find $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the union of the line segment from $(-5,0)$ to $(3,7)$, the line segment from $(3,7)$ to $(-2,-2)$ and the counterclockwise circular arc centered at the origin from $(-2,-2)$ to $(2,2)$.

$$F(x,y) = \overbrace{(4x^3y^2 - 2xy^3)}^{P}\,i + \overbrace{(2x^4y - 3x^2y^2 + 4y^3)}^{Q}\,j$$ is cont. and partial derivatives of P & Q are also cont. on $\mathbb{R}^2$

Since, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ and $\mathbb{R}^2$ is simply connected, F is conserva

So, there exists $f(x,y)$ on $\mathbb{R}^2$ such that $\nabla f = F$

$$f_x = 4x^3y^2 - 2xy^3 \Rightarrow f(x,y) = \int 4x^3y^2 - 2xy^3\,dx = x^4y^2 - x^2y^3 + C(y)$$

$$f_y = 2x^4y - 3x^2y^2 + 4y^3 = 2x^4y - 3x^2y^2 + C'(y)$$

$$4y^3 = C'(y)$$

$$y^4 + K = C(y)$$

$$\therefore f(x,y) = x^4y^2 - x^2y^3 + y^4 + K$$

By Fundamental Theorem of Line Integrals,

$$\int_C F\cdot dr = \int_C \nabla f\cdot dr = f(2,2) - f(-5,0)$$

$$= (2^4\cdot 2^2 - 2^2\cdot 2^3 + 2^4 + K) - (0 + K)$$

$$= 64 - 32 + 16 = \boxed{48}$$

8.(4 pts each) For each of the sequences below, state whether it is (i) bounded above / bounded below or neither, (ii) increasing, decreasing or neither. Then state whether or not the sequence has a limit, and find its value if so. Make sure to include the necessary explanation in your answers. In each question, $n = 1, 2, \ldots$

(a) $a_n = (-1)^n = \{-1, 1, -1, 1, \cdots\}$

$|a_n| \leq 1$ It's bounded

$a_{2n-1} < a_{2n}$ but $a_{2n+1} < a_{2n}$
So, not monotonic.

$\lim_{n\to\infty} (-1)^n =$ Limit doesn't exist, since a_n is alternating between 1

(b) $a_n = -\frac{1}{n}$

$\left|\frac{-1}{n}\right| \leq 1$ It's bounded

$\frac{-1}{n+1} > \frac{-1}{n}$, so increasing

$\lim_{n\to\infty} -\frac{1}{n} = 0$

(c) a_n is the nth digit of $5/24$ after the decimal point.

$\frac{5}{24} = 0.208\overline{3}$ So, $a_n = \{2, 0, 8, 3, 3, 3, \cdots\}$

$|a_n| \leq 8$ Bounded. $n \geq 3$ $a_{n+1} = a_n$, so constant

$\lim_{n\to\infty} a_n = 3$

(d) a_n is the sequence defined recursively by $a_1 = 2$, and $a_{n+1} = 1 - 2a_n$ for $n \geq 1$.

$a_1 = 2$ $a_{n+1} = 1 - 2a_n$ $a_n = \{2, -3, 7, -13, 27, \ldots\}$

a_n is unbounded; actually $|a_n| > n$.

proof: $n=1$ $|a_1| = 2 > 1$ ✓

Suppose it's true for $n=k$ ie $|a_k| > k$, then $|a_{k+1}| = |1 - 2a_k|$

$\geq 2|a_k| - 1 > 2k - 1 \geq (k+2) - 1 = k+1$ ✓

Hence, by induction true for all n.

a_n is alternating: Let a_n be positive (without loss of generality)

$a_{n+1} = 1 - 2 \cdot a_n < 0$ since $|a_n| > n$ (Similarly, if a_n is negative a_{n+1} is positive

(Not Monotonic)

$\lim_{n\to\infty} a_n =$ Limit doesn't exist.

9.(4 pts each) Determine if the given series are convergent or divergent. To recieve credit, explain the tests used and give all necessary details.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}+\sqrt{n}}{n^2+e^n}$

$\frac{(-1)^{n+1}+\sqrt{n}}{n^2+e^n} > 0$

$$\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}+\sqrt{n}}{n^2+e^n}}_{\text{Convergent by Comparison Test}} < \sum_{n=1}^{\infty} \frac{2\cdot\sqrt{n}}{n^2} = 2\cdot\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}}_{\text{convergent by p-test } p=3/2>}$$

$\frac{5}{2}+\frac{1}{3}$

$(5)\ \ (\frac{2}{6})+1\ \ \frac{17}{3}$

(b) $\sum_{n=1}^{\infty} \frac{3\sqrt{n}+2\sqrt[3]{n}}{n^3+4n^2-\pi}$ $\approx \sum_{n=1}^{\infty} \frac{3\sqrt{n}}{n^3} \approx 3\cdot\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ is convergent by p-test $p=$

$$\lim_{n\to\infty} \frac{\frac{3\sqrt{n}+2\sqrt[3]{n}}{n^3+4n^2-\pi}}{\frac{1}{n^{5/2}}} = \lim_{n\to\infty} \frac{3n^3+2n^{17/6}}{n^3+4n^2-\pi} = \lim_{n\to\infty} \frac{\cancel{n^3}(3+2\cdot\frac{1}{n^{1/6}})}{\cancel{n^3}(1+\frac{4}{n}-\frac{\pi}{n^3})} = 3 > 0$$

By Limit Comparison it's also conver

(c) $\sum_{n=2}^{\infty}(-1)^{n-1}\frac{n}{(\ln n)^2}$ $\lim_{n\to\infty} \frac{n}{(\ln n)^2} = \lim_{x\to\infty} \frac{x}{(\ln x)^2} \left(\frac{\infty}{\infty}\right)$

$\lim_{n\to\infty} (-1)^{n-1}\frac{n}{(\ln n)^2} =$ Limit Doesn't Exist

$$\overset{\text{L'Hospital}}{=} \lim_{x\to\infty} \frac{1}{2\ln x\cdot\frac{1}{x}} = \lim_{x\to\infty} \frac{x}{2\ln x} \left(\frac{\infty}{\infty}\right)$$

$$\overset{\text{L'Hospital}}{=} \lim_{x\to\infty} \frac{1}{2\cdot\frac{1}{x}} = \lim_{x\to\infty} \frac{x}{2} = \infty.$$

$\therefore$ Divergent by Test for Divegence

(d) $\sum_{n=3}^{\infty} \frac{1}{n\ln n\sqrt{\ln(\ln n)}}$

$\frac{1}{x\ln x\sqrt{\ln(\ln x)}}$ is positive and obviously decreasing and cont.

We can use Integral Test

$$\int_3^{\infty} \frac{1}{x\ln x\sqrt{\ln(\ln x)}}\,dx \underset{\substack{u=\ln x\\ du=\frac{1}{x}dx}}{=} \int_{\ln 3}^{\infty} \frac{1}{u\sqrt{\ln(u)}}\,du \underset{\substack{t=\ln(u)\\ dt=\frac{1}{u}du}}{=} \int_{\ln(\ln 3)}^{\infty} \frac{1}{\sqrt{t}}\,dt$$ is divergen by p-tes

$\therefore \sum^{\infty} \frac{1}{n\ln n\sqrt{\ln(\ln n)}}$ is divergent by Integral Test.

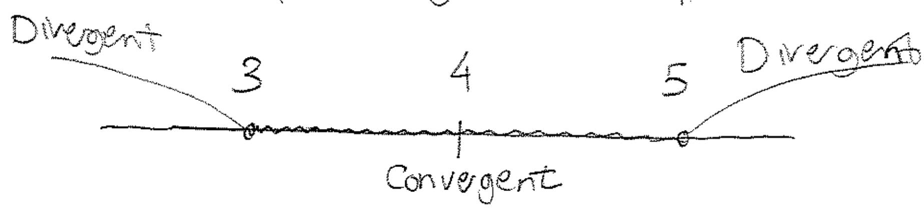
10.(10 pts) Find the radius of convergence and the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(x-4)^n}{3n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(x-4)^{n+1}}{3n+4}}{\dfrac{(x-4)^n}{3n+1}}\right| = \lim_{n\to\infty} |x-4|\cdot\frac{3n+1}{3n+4}$$

$$= |x-4|\cdot\lim_{n\to\infty}\frac{\cancel{n}(3+\frac{1}{n})}{\cancel{n}(3+\frac{4}{n})} = |x-4| < 1$$

Radius of Convergence is equal to $\underline{1}$.



We need to check boundaries.

$\underline{x=3}$ $\quad \sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}$ $\quad$ i) $b_n = \frac{1}{3n+1} > \frac{1}{3n+4} = b_{n+1}$ (b_n is decrea

ii) $\lim_{n\to\infty}\frac{1}{3n+1} = 0$

Convergent by Alternating Series Test.

$\underline{\underline{x=5}}$ $\quad \sum_{n=0}^{\infty}\frac{1}{3n+1}$ $\quad \lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{3n+1}} = \lim_{n\to\infty}\frac{3n+1}{n} = 3 > 0$

By Limit Comparison Test, they behave the same Since $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent by p-test ($p=1$), $\sum_{n=1}^{\infty}\frac{1}{3n+1}$ is divergent

Hence, Interval of Convergence $= [3, 5)$

11.(4 pts each) Use either a previously known power series representation, or the Taylor expansion of the function, or both, in order to obtain a power series representation of the given function centered around $a = 0$:

(a) $e^{3x} - \dfrac{1}{1-x^2}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad e^{3x} = \sum_{n=0}^{\infty} \frac{3^n \cdot x^n}{n!} \qquad R = \infty$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n} \qquad R = 1$$

$$e^{3x} - \frac{1}{1-x^2} = \sum_{n=0}^{\infty} \frac{3^n x^n}{n!} - \sum_{n=0}^{\infty} x^{2n} \qquad R = 1.$$

(b) $x\ln(1-x) + \displaystyle\int \arctan x\,dx$

$$R=1 \quad \ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \qquad \ln(1-x) = \sum_{n=0}^{\infty} (-1)^n (-1)^{n+1} \frac{x^{n+1}}{n+1} = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$R=1 \quad \int \arctan x\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+2}}{(2n+1)(2n+2)}$$

$$x\ln(1-x) + \int \arctan x\,dx = -\sum_{n=0}^{\infty} \frac{x^{n+2}}{n+1} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)(2n+2)} \qquad R = 1.$$

(c) $\displaystyle\int x^2 e^{-x^2}\,dx + \frac{d}{dx}\left(\frac{1}{1+x^8}\right)$

$$\int x^2 e^{-x^2}\,dx = \int x^2 \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{n!}\,dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n \cdot x^{2n+2}}{n!}\,dx$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+3}}{(2n+3)n!} \qquad \underline{R = \infty}$$

$$\frac{d}{dx}\left(\frac{1}{1+x^8}\right) = \frac{d}{dx}\left(\frac{1}{1-(-x^8)}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} (-1)^n \cdot x^{8n}\right)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{d}{dx}\left(x^{8n}\right) = \sum_{n=1}^{\infty} (-1)^n \cdot 8n\, x^{8n-1} \qquad R = 1$$