M E T U

Northern Cyprus Campus

Math 219 **Differential Equations** **Midterm Exam II** **23.07.2014**								
Last Name: Solutions Name : Student No:				Dept./Sec. : Time : 17: 40 Duration : 110 *minutes*			Signature	
6 QUESTIONS							TOTAL 100 POINTS	
1	2	3	4	5	6			

Q1 (5+5+5=15 pts.) Consider the differential equation $y'' - 6y' + 9y = 0$. Find the related fundamental matrix $\Psi(t)$, then the matrix P, and finally the fundamental matrix $\Phi(t)$.

Characteristic equation: $\lambda^2 - 6\lambda + 9 = 0 \Rightarrow$

$\Rightarrow (\lambda - 3)^2 = 0 \Rightarrow \sigma(A) = \{3^{(2)}\}$, where

$A = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix} \Rightarrow y = c_1 t e^{3t} + c_2 e^{3t}$ is the gen. sol. $\Rightarrow$

$$\Rightarrow \Psi(t) = \begin{bmatrix} t e^{3t} & e^{3t} \\ (1+3t) e^{3t} & 3 e^{3t} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}}_{P} \begin{bmatrix} e^{3t} & 0 \\ t e^{3t} & e^{3t} \end{bmatrix} \Rightarrow$$

$$P^{-1} = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \Phi(t) = \Psi(t) \cdot P^{-1} =$$

$$= \begin{bmatrix} t e^{3t} & e^{3t} \\ (1+3t) e^{3t} & 3 e^{3t} \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} (1-3t) e^{3t} & t e^{3t} \\ -9t e^{3t} & (1+3t) e^{3t} \end{bmatrix}$$

Q2 (25 pts.) Find the fundamental matrix $\Psi(t)$ of the linear system $\mathbf{x}'(t) = A\mathbf{x}(t)$ with the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$. **(Bonus 5 pts.)** Find the fundamental matrix $\Phi(t)$.

$$\Delta(\lambda) = \det \begin{bmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix} = -(\lambda^3 - 3\lambda^2 + 4) = -(\lambda+1)(\lambda-2)^2$$

$$\sigma(A) = \{-1^{(1)}, 2^{(2)}\}$$

$$\lambda = -1 \Rightarrow A+1 = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V_{-1} = \ker(A+1) = \{2x + 3z = 0,\ y = 2z\},\ \vec{f}_1 = \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix}$$

$$\lambda = 2 \Rightarrow A-2 = \begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V_{-2,1} = \ker(A-2) = \{x = 0,\ y + z = 0\},\ (A-2)^2 = \begin{bmatrix} 3 & -3 & -3 \\ -4 & 4 & 4 \\ -2 & 2 & 2 \end{bmatrix}$$

$$V_{-2,2} = \ker(A-2)^2 = \{x = y + z\};\quad V_{-2,1} \subsetneq V_{-2,2}$$

$$\{x = y + z = 0\} \subsetneq \{x = y + z\}$$

$$\vec{f}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ \vec{f}_3 = (A-2)\vec{f}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} -3 & 1 & 0 \\ 4 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix},\ e^{Jt} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & te^{2t} & e^{2t} \end{bmatrix},\ \Psi(t) = \begin{bmatrix} -3e^{-t} & e^{2t} & 0 \\ 4e^{-t} & te^{2t} & e^{2t} \\ 2e^{-t} & (1-t)e^{2t} & -e^{2t} \end{bmatrix}$$

Bonus: $P^{-1} = \begin{bmatrix} -1/9 & 1/9 & 1/9 \\ 2/3 & 1/3 & 1/3 \\ 4/9 & 5/9 & -4/9 \end{bmatrix}$

$$\Phi(t) = \begin{bmatrix} 1/3\,e^{-t} + 2/3\,e^{2t} & -1/3\,e^{-t} + 1/3\,e^{2t} & -1/3\,e^{-t} + 1/3\,e^{2t} \\ -4/9\,e^{-t} + 2/3\,te^{2t} + 4/9\,e^{2t} & 4/9\,e^{-t} + 1/3\,te^{2t} + 5/9\,e^{2t} & 4/9\,e^{-t} + 1/3\,te^{2t} - 4/9\,e^{2t} \\ -2/9\,e^{-t} + 2/3(1-t)e^{2t} - 4/9\,e^{2t} & 2/9\,e^{-t} + 1/3(1-t)e^{2t} - 5/9\,e^{2t} & 2/9\,e^{-t} + 1/3(1-t)e^{2t} - 4/9\,e^{2t} \end{bmatrix}$$

Q3 (25 pts.) Find the general solution to the folllowing nonhomogeneous linear system $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{b}(t)$ with $A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$ and $\mathbf{b}(t) = \begin{bmatrix} 0 \\ \sin(t) \end{bmatrix}$.

$$\Delta(\lambda) = \begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = -(2-\lambda)(2+\lambda)+5 = (\lambda-2)(\lambda+2)+5 = \lambda^2+1=0$$

$$\sigma(A) = \{i, -i\}, \quad A - i = \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \sim \begin{bmatrix} 2-i & 5 \\ 0 & 0 \end{bmatrix}, \quad \vec{f} = \begin{bmatrix} 1 \\ \frac{2-i}{5} \end{bmatrix}$$

$$\vec{x}(t) = \vec{f} e^{it} = \begin{bmatrix} 1 \\ \frac{2-i}{5} \end{bmatrix} (\cos(t) + i\sin(t)) = \left(\begin{bmatrix} 1 \\ 2/5 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1/5 \end{bmatrix} \right) (\cos(t) + i\sin(t))$$

$$= \left(\begin{bmatrix} \cos(t) \\ 2/5\cos(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/5\sin(t) \end{bmatrix} + i \left(\begin{bmatrix} \sin(t) \\ 2/5\sin(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -1/5\cos(t) \end{bmatrix} \right) \right) =$$

$$= \begin{bmatrix} \cos(t) \\ 2/5\cos(t) + 1/5\sin(t) \end{bmatrix} + i \begin{bmatrix} \sin(t) \\ 2/5\sin(t) - \frac{1}{5}\cos(t) \end{bmatrix}$$

$$\Psi(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ 2/5\cos(t) + 1/5\sin(t) & 2/5\sin t - 1/5\cos(t) \end{bmatrix},$$

$$W(t) = -\frac{1}{5}. \qquad \vec{Y} = \Psi(t)\vec{c}(t), \quad \Psi(t)\vec{c}'(t) = \vec{b}(t)$$

$$c_1' = -5 \begin{vmatrix} 0 & \sin(t) \\ \sin(t) & 2/5\sin(t) - 1/5\cos(t) \end{vmatrix} = 5\sin^2(t) = 5/2\,(1-\cos(2t))$$

$$c_1 = 5/2\,t - 5/4\sin(2t)$$

$$c_2' = -5 \begin{vmatrix} \cos(t) & 0 \\ 2/5\cos(t) + 1/5\sin(t) & \sin(t) \end{vmatrix} = 5/2\sin(2t)$$

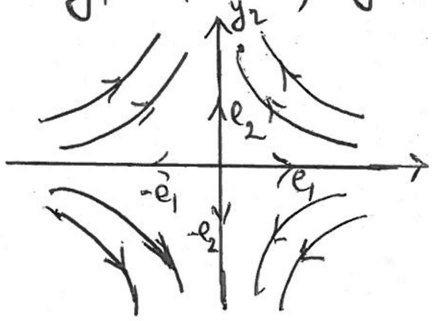
$$c_2 = 5/4\cos(2t). \qquad \vec{Y}(t) = \Psi(t) \begin{bmatrix} 5/2\,t - 5/4\sin(2t) \\ 5/4\cos(2t) \end{bmatrix}$$

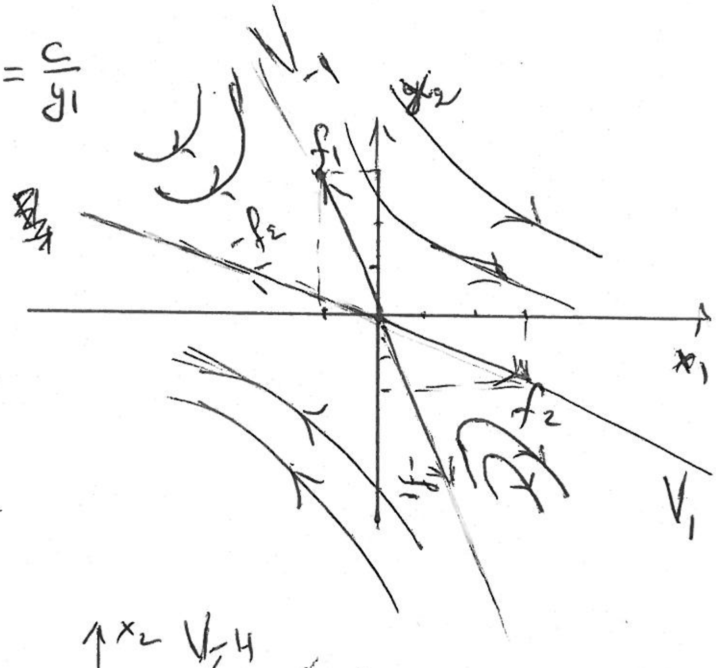
$$\vec{X}(t) = \Psi(t)\vec{c} + \vec{Y}(t)$$

Q4 (5+5+10=20 pts.) Sketch the phase portrait of 2×2-linear system $\mathbf{x}'(t) = A\mathbf{x}(t)$ with constant matrix $A \in \mathbb{M}_2(\mathbb{R})$ whose Jordan matrix J and the matrix P of (generalized) eigenvectors are given below:

a) $J = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} -1 & 3 \\ 3 & -2 \end{bmatrix}$.

$y_1 = c_1 e^{-t}$, $y_2 = c_2 e^{t} = c_2 (e^{-t})^{-1} = \frac{c}{y_1}$

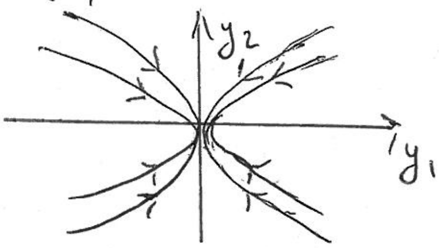


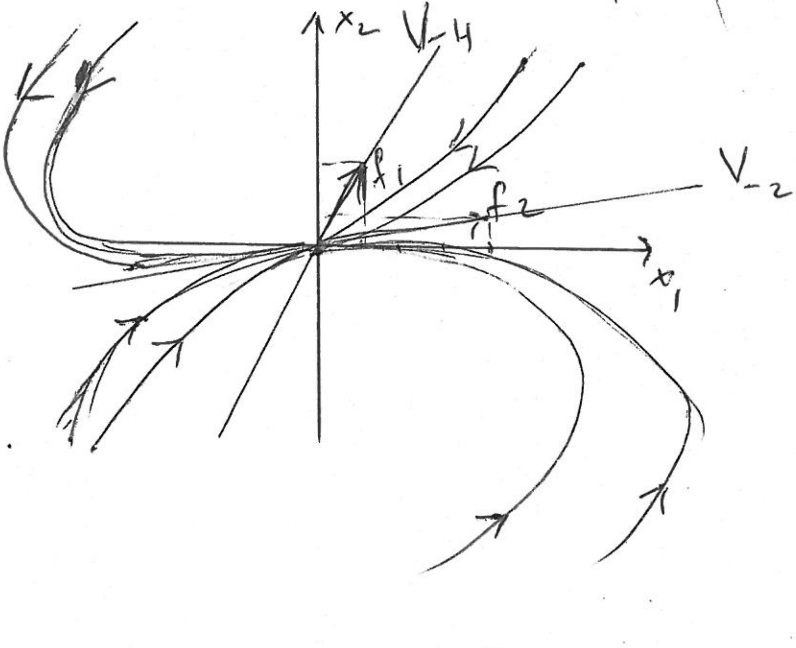


b) $J = \begin{bmatrix} -4 & 0 \\ 0 & -2 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$.

$y_1 = c_1 e^{-4t}$, $y_2 = c_2 e^{-2t}$

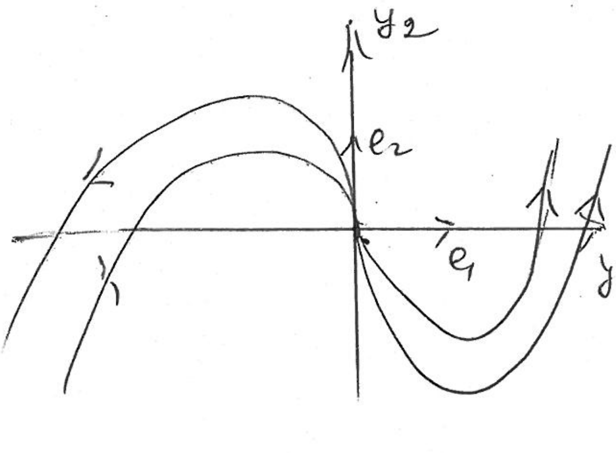
$y_2 = c_2 \left(\frac{e^{-4t}}{c_1}\right)^{1/2} = c y_1^{1/2}$

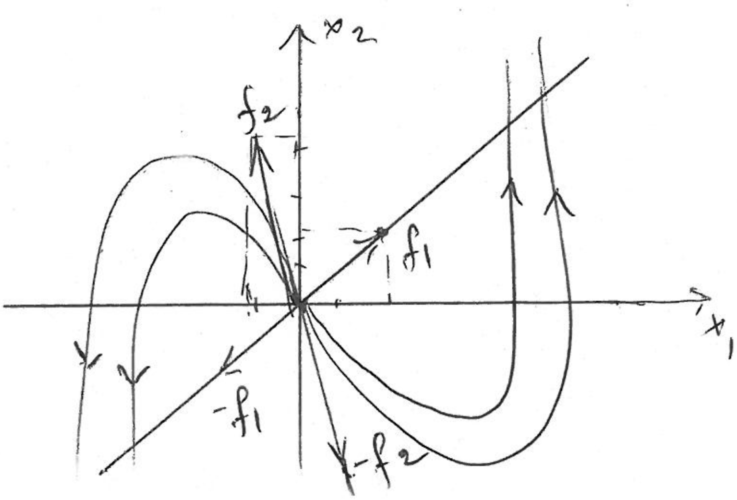




c) $J = \begin{bmatrix} 5 & 0 \\ 1 & 5 \end{bmatrix}$ and $P = \begin{bmatrix} 2 & -1 \\ 2 & 4 \end{bmatrix}$.

$y_2 = \left(c + \frac{1}{5}\ln|y_1|\right) y_1$





Q5 (15 pts.) Consider the following higher order linear differential equation $y^{(6)} - y^{(3)} = 0$. Find the general solution.

Characteristic equation: $\lambda^6 - \lambda^3 = 0 \Rightarrow$

$\Rightarrow \lambda^3(\lambda^3 - 1) = 0 \Rightarrow \lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = e^{i\,2\pi/3},$

$\lambda_4 = e^{i\,4\pi/3} = \bar{\lambda}_3$. But $\lambda_3 = \cos(2\pi/3) + i \sin(2\pi/3) =$

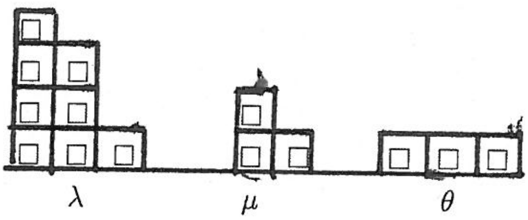
$= -1/2 + i\,\sqrt{3}/2 \Rightarrow \lambda_4 = -\frac{1}{2} - i\,\frac{\sqrt{3}}{2}$.

Therefore

$$y = c_1 + c_2 t + c_3 t^2 + c_4 e^t + c_5 e^{-t/2} \cos(\sqrt{3}/2\, t) + c_6 e^{-t/2} \sin(\sqrt{3}/2\, t)$$

is the general solution.

Q6 (Bonus 10 pts.) Write down the Jordan matrix which corresponds to the following diagram



$$J = \begin{bmatrix} J_\lambda & & 0 \\ & J_\mu & \\ 0 & & J_\theta \end{bmatrix},$$ where

$$J_\lambda = \begin{bmatrix} \begin{matrix} \lambda & 0 & 0 & 0 \\ 1 & \lambda & 0 & 0 \\ 0 & 1 & \lambda & 0 \\ 0 & 0 & 1 & \lambda \end{matrix} & & 0 \\ & \begin{matrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{matrix} & \\ 0 & & \lambda \end{bmatrix}, \quad J_\mu = \begin{bmatrix} \mu & 0 & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad J_\theta = \begin{bmatrix} \theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta \end{bmatrix}$$