METU - NCC

LINEAR ALGEBRA MIDTERM 2	
Code : *MAT 260*	Last Name:
Acad.Year: *2012-2013*	Name : Student No.:
Semester : *Spring*	Department: Section:
Date : *24.04.2013*	Signature :
Time : *17:40*	6 QUESTIONS ON 4 PAGES
Duration : *100 min*	TOTAL 100 POINTS

1. (15)	2. (10)	3. (20)	4. (15)	5. (15)	6. (25)	Bonus		

1. *(15pts)* Find a basis for $\mathcal{W} = \{p \in \mathcal{P}_3(\mathbb{R}) \mid p(2) = p(5) = 0\}$.

$$W = \left\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \;\middle|\; \begin{array}{l} a_0 + a_1 \cdot 2 + a_2 \cdot 4 + a_3 \cdot 8 = 0 \\ - \; a_0 + a_1 \cdot 5 + a_2 \cdot 25 + a_3 \cdot 125 = 0 \end{array} \right\}$$

$$3a_1 + 21 a_2 + 117 a_3 = 0$$

$$\hookrightarrow a_1 = -7a_2 - 39 a_3$$

plug into equation:

$$a_0 + (-7a_2 - 39a_3) \cdot 2 + 4a_2 + 8a_3 = 0$$

$$\hookrightarrow a_0 = 10 a_2 + 70 a_3$$

Fast solution:

$$W = \{(x-2)(x-5)(a_0 + a_1 x)\}$$

$$= \{a_0 (x-2)(x-5) + a_1 x (x-2)(x-5)\}$$

$$\boxed{\text{Basis} = \{(x-2)(x-5),\ x(x-2)(x-5)\}}$$

$$W = \left\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \;\middle|\; \begin{array}{l} a_0 = 10 a_2 + 70 a_3 \\ a_1 = -7 a_2 - 39 a_3 \end{array} \right\}$$

$$= \{(10a_2 + 70 a_3) + (-7a_2 - 39a_3)x + a_2 x^2 + a_3 x^3\}$$

$$\boxed{\text{Basis} = \{10 - 7x + x^2,\ 70 - 39x + x^3\}}$$

2. *(10pts)* Show that if $\{v_1, v_2, ..., v_n\}$ is a linearly independent set in $\mathcal{V}$, then $\dim(\mathcal{V}) \geqslant n$.

Suppose $\{v_1, v_2, \dots, v_n\} \subset V$ is linearly independent.

By the basis extension theorem, we can extend to $\{v_1, v_2, \dots, v_n, w_1, \dots, w_m\}$ a basis of V.

Then $\dim(V) = n + m \geq n$. ▨

3. *(12+13pts)* In the parts below, $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\mathbf{T}(x, y) = (ax + by, cx + dy)$, where a, b, c, d are fixed real numbers.

(A) Show that $\mathbf{T}$ is a linear transformation.

$$\begin{aligned}T((x_1, y_1) + (x_2, y_2)) &= T(x_1 + x_2,\ y_1 + y_2) \\ &= (a(x_1+x_2) + b(y_1+y_2),\ c(x_1+x_2) + d(y_1+y_2)) \\ &= ((ax_1 + by_1) + (ax_2 + by_2),\ (cx_1 + y_1) + d(x_2 + y_2)) \\ &= (ax_1 + by_1,\ cx_1 + dy_1) + (ax_2 + by_2,\ cx_2 + dy_2) \\ &= T(x_1, y_1) + T(x_2, y_2)\end{aligned}$$

$$\begin{aligned}T(k \cdot (x, y)) &= T(kx, ky) \\ &= (akx + bky,\ ckx + dky) \\ &= k(ax + by,\ cx + dy) \\ &= k \cdot T(x, y)\end{aligned}$$

(B) Show that $\mathbf{T}$ is an isomorphism if and only if $ad \neq bc$.

If $ad \neq bc$ then T^{-1} is

$$T^{-1} = S(x, y) = \frac{1}{ad - bc}(dx - by,\ -cx + ay)$$

$$\begin{aligned}\to TS(x, y) &= \frac{1}{ad-bc} T(dx - by,\ -cx + ay) \\ &= \frac{1}{ad-bc}\big(a(dx - by) + b(-cx + ay),\ c(dx - by) + d(-cx + ay)\big) \\ &= \frac{1}{ad-bc}\big((ad - bc)x,\ (ad - bc)y\big) = (x, y)\end{aligned}$$

$$\begin{aligned}\to ST(x, y) &= S(ax + by,\ cx + dy) \\ &= \frac{1}{ad-bc}\big(d(ax + by) - b(cx + dy),\ -c(ax + by) + a(cx + dy)\big) \\ &= \frac{1}{ad-bc}\big((ad - bc)x,\ (ad - bc)y\big) = (x, y)\end{aligned}$$

Note that T is an isomorphism if and only if $\ker(T) = \{0\}$.

Suppose $(x, y) \in \ker(T)$ then $(0, 0) = T(x, y) = (ax + by,\ cx + dy)$

$$\begin{aligned}&(0 = ax + by) \cdot (c) \\ -&(0 = cx + dy) \cdot (a) \\ \hline &0 = (bc - ad)y\end{aligned}$$

$(y = 0$ if $ad - bc \neq 0)$

$\Rightarrow x = 0$

So T is an isomorphism if and only if $ad - bc \neq 0$.

Note that this could also be done with matrices

4. *(5+10pts)* The parts below use the linear transformation $\mathrm{T} : \mathbb{R}^3 \to \mathrm{Fun}(\{s,t\})$ with $\mathrm{T}(1,0,2) = \chi_s$ and $\mathrm{T}(0,2,1) = \chi_t$ and $\mathrm{T}(0,1,0) = \chi_s + \chi_t$.

(A) What are the dimensions of $\mathrm{im}(\mathrm{T})$ and $\ker(\mathrm{T})$? (Justify your answer.)

Hint: this can be answered without any serious computation.

$\chi_s, \chi_t \in \mathrm{im}(T)$. So $\underline{2} = \dim(\mathrm{Span}\{\chi_s, \chi_t\}) \leq \underline{\dim(\mathrm{im}(T))} \leq \dim(\mathrm{Fun}(\{s,t\})) = \underline{2}$

So $\dim(\mathrm{im}(T)) = 2$.

Thus $\dim(\ker(T)) = \dim(\mathbb{R}^3) - \dim(\mathrm{im}(T)) = 3 - 2 = 1$.

(B) $\mathrm{T}(1,0,0)$ is a function, $\mathrm{T}(1,0,0) : \{s,t\} \to \mathbb{R}$. What is the value of $\left(\mathrm{T}(1,0,0)\right)(s)$?

$$(1,0,0) = a(1,0,2) + b(0,2,1) + c(0,1,0)$$

$$\hookrightarrow \begin{cases} 1 = a \\ 0 = 2b + c \\ 0 = 2a + b \end{cases} \Longrightarrow \quad \begin{aligned} a &= 1 \\ b &= -2 \\ c &= 4 \end{aligned}$$

$$T(1,0,0) = T(1,0,2) - 2\,T(0,2,1) + 4\,T(0,1,0)$$
$$= \chi_s - 2\,\chi_t + 4\,(\chi_s + \chi_t)$$

$$(T(1,0,0))(s) = 1 - 2 \cdot 0 + 4\,(1 + 0) = 5$$

5. *(10+5pts)* In the parts below T is a linear transformation $\mathrm{T} : \mathcal{V} \to \mathcal{V}$ and $\mathrm{T}^2 = \mathrm{T} \circ \mathrm{T}$.

(A) Show that $\ker(\mathrm{T}) = \{0\}$ if and only if $\ker(\mathrm{T}^2) = \{0\}$.

($\Longrightarrow$) If $\ker(T) = \{\underline{0}\}$ then $\mathrm{im}(T) = \mathcal{V}$ (because $\dim(\mathrm{im}(T)) = \dim(V) - 0$)

(*) Thus $\mathrm{im}(T \circ T) = \mathcal{V}$. So $\ker(T \circ T) = \{\underline{0}\}$.

(Proof of (*):
If $\underline{v} \in \mathcal{V}$ then there is $\underline{w} \in \mathcal{V}$ with $T(\underline{w}) = \underline{v}$ because $\mathrm{im}(T) = \mathcal{V}$.
Similarly there is $\underline{x} \in \mathcal{V}$ with $T(\underline{x}) = \underline{w}$. $T(T(\underline{x})) = \underline{v}$.)

($\Longleftarrow$) Note that if $\underline{v} \in \ker(T)$ then $T(T(\underline{v})) = T(\underline{0}) = \underline{0}$

So $\ker(T) \subset \ker(T^2)$. Thus if $\ker(T^2) = \{\underline{0}\}$

then $\ker(T) = \{\underline{0}\}$.

(B) Is it always true that $\ker(\mathrm{T}) = \ker(\mathrm{T}^2)$? (Give proof or counter-example.)

<u>No</u>. If T is nilpotent (for example) this will usually not be true.... say

$$T(x,y) = (0,x) \longrightarrow \begin{aligned} \ker(T) &= \mathrm{Span}\{(0,1)\} \\ \ker(T^2) &= \mathbb{R}^2. \end{aligned}$$

6. *(16+9pts)* Let $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation given by $\mathrm{T}(x, y, z, w) = (w, z, 0, y)$.

(A) Find the matrices $[\mathrm{T}]$ and $[\mathrm{T} \circ \mathrm{T}]$ (with respect to the standard basis for $\mathbb{R}^4$).

$$[T] = \begin{bmatrix} | & | & | & | \\ T(1,0,0,0) & T(0,1,0,0) & T(0,0,1,0) & T(0,0,0,1) \\ | & | & | & | \end{bmatrix} \qquad \begin{pmatrix} T(1,0,0,0) = (0,0,0,0) \\ T(0,1,0,0) = (0,0,0,1) \\ T(0,0,1,0) = (0,1,0,0) \\ T(0,0,0,1) = (1,0,0,0) \end{pmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$[T \circ T] = \begin{bmatrix} | & | & | & | \\ T^2(1,0,0,0) & T^2(0,1,0,0) & T^2(0,0,1,0) & T^2(0,0,0,1) \\ | & | & | & | \end{bmatrix} \qquad \begin{pmatrix} T^2(x,y,z,w) \\ \parallel \\ T(w,z,0,y) \\ \parallel \\ (y,0,0,z) \end{pmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(B) Is the matrix $[\mathrm{T}]$ idempotent? Nilpotent? Singular? Explain your answer for each question.

- $[T]$ is <u>not</u> idempotent, because $[T^2] = [T]^2 \neq [T]$.
- $[T]$ <u>is</u> nilpotent, because
$$\begin{aligned} T(x,y,z,w) &= (w,z,0,y) \\ T^2(x,y,z,w) &= (y,0,0,z) \\ T^3(x,y,z,w) &= (z,0,0,0) \\ T^4(x,y,z,w) &= (0,0,0,0) \end{aligned}$$
So $[T]^4 = \underline{0}$
- $[T]$ <u>is</u> singular, because $[T]$ is nilpotent (nilpotent matrices cannot have inverse)

BONUS

If $\mathcal{V}$ is not finite dimensional, then $\mathcal{V}^* = \mathscr{L}(\mathcal{V}, \mathbb{R})$ is not isomorphic to $\mathcal{V}$. Why not?

If $\{\underline{b}_1, \ldots\}$ is a basis for $\mathcal{V}$ then

$\mathcal{V} = \mathrm{Span}\{\underline{b}_1, \ldots, \underline{b}_n\} = \{$ finite sums of $\underline{b}_i \}$

<u>But</u> $\mathcal{V}^*$ has <u>infinite</u> sums of $\underline{b}_i^*$.

→ Actually things are a bit more complicated, but this is the idea.