M E T U – N C C

	Applied Mathematics for Engineers Midterm	

Code : *Math 210* Acad.Year : *2013-2014* Semester : *Spring* Date : *05.04.2014* Time : *09:40* Duration : *120 minutes*	Last Name: Name: Student No: Department: Section: Signature: 9 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1 (5)	2 (5)	3 (5)	4 (5)	5 (20)	6 (25)	7 (15)	8 (10)	9 (10)	

Show your work! No calculators! Please draw a $\boxed{\textbf{box}}$ **around your answers!**
Please do not write on your desk!

1.(5pts) When is a diagonal matrix positive-definite?

If all of its diagonal entries are >0, a diagonal matrix is positive-definite.

2.(5pts) Find a vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ so that the energy function $E(x) = 2(x_1^2+x_2^2+x_3^2-x_1x_2-x_2x_3) < 0$.

$E(x)$ is the energy function of the matrix $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$. This matrix is positive definite hence $E(x) \geq 0$ for all x.

There is no x so that $E(x) < 0$!

3.(1pt each) Fill in the blanks with the Matlab/Octave equivalent of each mathematical statement or provide solution to the mathematical problem in Matlab/Octave.

Mathematical Statement	Matlab/Octave equivalent or solution in Matlab/Octave
$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	>> A = [1 2 ; 0 1]
$K = AA^T$	>> K = A * A'
Solve $A\mathbf{x} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$	>> x = A \ [7 ; 8]
Diagonalize K as $K = SDS^{-1}$	>> [S,D] = eig (K)
Second column of S	>> S (: , 2)

4.(1pt each) Fill in the blanks.

(a) Eigenvalues of a symmetric matrix are real. .

(b) The determinant of an upper triangular matrix is the product of its diagonal entries.

(c) The determinant of a positive definite matrix is positive.

(d) The inverse of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

(e) If a matrix A has an eigenvector $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$ with eigenvalue 2, the matrix A^2+3A-I has an eigenvector $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$ associated to eigenvalue 9.

5.(4+5+5+6=20pts) Convert each differential equation to a matrix equation $A\mathbf{u} = \mathbf{b}$ by discretizing on $[4, 5]$ with mesh size $h = 1/3$. For first derivatives use centered difference. (DO NOT SOLVE the equation $A\mathbf{u} = \mathbf{b}$).

(a) $\quad -u''(x) = x^2$ with $u(4) = 0$ and $u(5) = 0$.

Our mesh :

$$\begin{array}{cccc} u_0 & u_1 & u_2 & u_3 \\ \vdash & + & + & \dashv \\ 4 & \frac{13}{3} & \frac{14}{3} & 5 \\ \parallel & \parallel & \parallel & \parallel \\ x_0 & x_1 & x_2 & x_3 \end{array}$$

$$x=x_1: \; -\left(\frac{\overset{=0}{u_0} - 2u_1 + u_2}{h^2}\right) = x_1^2$$

$$x=x_2: \; -\left(\frac{u_1 - 2u_2 + \overset{=0}{u_3}}{h^2}\right) = x_2^2$$

$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} u = \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} \quad \text{OR} \quad 9\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} u = \begin{bmatrix} (13/3)^2 \\ (14/3)^2 \end{bmatrix}$$

(b) $\quad -u''(x) + \delta(x - \frac{13}{3})u'(x) = x^2$ with $u(4) = 0$ and $u(5) = 0$.

$$x=x_1: \; -\left(\frac{\overset{=0}{u_0} - 2u_1 + u_2}{h^2}\right) + \overset{=\delta(x-13/3)}{\frac{1}{h}}\left(\frac{u_2 - \overset{=0}{u_0}}{2h}\right) = x_1^2$$

$$x=x_2: \; -\left(\frac{u_1 - 2u_2 + \overset{=0}{u_3}}{h^2}\right) + 0\left(\frac{\overset{=0}{u_3} - u_1}{2h}\right) = x_2^2$$

$$\left(\frac{1}{h^2}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 1/h & \\ & 0 \end{bmatrix}\frac{1}{2h}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) u = \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix}$$

(c) $\quad -u''(x) + \delta(x - \frac{13}{3})u'(x) = x^2$ with $u(4) = a$ and $u(5) = b$.

$$x=x_1: \; -\left(\frac{\overset{=a}{u_0} - 2u_1 + u_2}{h^2}\right) + \overset{=\delta(x-13/3)}{\frac{1}{h}}\left(\frac{u_2 - \overset{=a}{u_0}}{2h}\right) = x_1^2$$

$$x=x_2: \; -\left(\frac{u_1 - 2u_2 + \overset{=b}{u_3}}{h^2}\right) + 0\left(\frac{\overset{=b}{u_3} - u_1}{2h}\right) = x_2^2$$

$$\left(\frac{1}{h^2}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 1/h & \\ & 0 \end{bmatrix}\frac{1}{2h}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) u = \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix} + \begin{pmatrix} \frac{a}{h^2} + \frac{a}{2h^2} \\ \frac{b}{h^2} \end{pmatrix}$$

(d) $\quad -u''(x) + \delta(x - \frac{13}{3})u'(x) = x^2$ with $u'(4) = 0$ and $u(5) = b$. To estimate $u'(4)$, use a forward difference.

$$u'(4) \approx \frac{u_1 - u_0}{h} = 0 \Rightarrow u_1 = u_0$$

$$x=x_1: \; -\left(\frac{\overset{=u_1}{u_0} - 2u_1 + u_2}{h^2}\right) + \overset{=\delta(x-13/3)}{\frac{1}{h}}\left(\frac{u_2 - \overset{=u_1}{u_0}}{2h}\right) = x_1^2$$

$$x=x_2: \; -\left(\frac{u_1 - 2u_2 + \overset{=b}{u_3}}{h^2}\right) + 0\left(\frac{\overset{=b}{u_3} - u_1}{2h}\right) = x_2^2$$

$$\left(\frac{1}{h^2}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 1/h & \\ & 0 \end{bmatrix}\frac{1}{2h}\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\right) u = \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{b}{h^2} \end{pmatrix}$$

6.(15+4+6=25pts) Consider the matrix $A = \begin{pmatrix} -4 & 3 & 0 \\ -6 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) Calculate the eigenvalues and eigenvectors of A.

EIGENVALUES : $2, -1, 1$

EXPAND WITH RESPECT TO 3RD ROW

$$\det(A-\lambda I) = \det\begin{bmatrix} -4-\lambda & 3 & 0 \\ -6 & 5-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)\det\begin{vmatrix} -4-\lambda & 3 \\ -6 & 5-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(\lambda+4)(\lambda-5) - 6(\lambda-3)\right] = (1-\lambda)\left[\lambda^2-\lambda-20+18\right] = (1-\lambda)(\lambda^2-\lambda-2)$$

$$= (1-\lambda)(\lambda+1)(\lambda-2) \Rightarrow \text{EIGENVALUES : } 2, -1, 1.$$

$\boxed{\lambda=2}$

$$A-2I: \begin{bmatrix} -6 & 3 & 0 \\ -6 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-R_1+R_2\to R_2} \begin{bmatrix} -6 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2\leftrightarrow R_3]{\frac{1}{3}R_1\to R_1} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \begin{bmatrix} -2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$-2x_1+x_2=0$

$x_3=0$

$\Rightarrow$ EIGENVECTOR FOR $\lambda=2$: $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$\boxed{\lambda=-1}$

$$A-(-1)I: \begin{vmatrix} -3 & 3 & 0 \\ -6 & 6 & 0 \\ 0 & 0 & 1 \end{vmatrix} \xrightarrow{2R_1+R_2\to R_2} \begin{vmatrix} -3 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \xrightarrow[R_2\leftrightarrow R_3]{\frac{1}{3}R_1\to R_1} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$-x_1+x_2=0 \Rightarrow$

$x_3 = 0$

FOR $\lambda=-1$: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\boxed{\lambda=1}$

$$A-1\cdot I = \begin{vmatrix} -5 & 3 & 0 \\ -6 & 4 & 0 \\ 0 & 0 & 0 \end{vmatrix} \xrightarrow[-1/6 R_2\to R_2]{-\frac{1}{5}R_1\to R_1} \begin{bmatrix} 1 & -\frac{3}{5} & 0 \\ 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-R_1+R_2\to R_2} \begin{bmatrix} 1 & -\frac{3}{5} & 0 \\ 0 & -\frac{1}{15} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - \frac{3}{5}x_2 = 0 \Rightarrow x_1 = 0$

$-\frac{1}{15}x_2 = 0$

$x_3 = \text{free}$

$\boxed{\lambda=+1}$: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(b) Diagonalize A as $A = S\Lambda S^{-1}$, where Λ is a diagonal matrix. DO NOT CALCULATE S^{-1}.

$$A = \underbrace{\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{= S} \begin{bmatrix} 2 & & \\ & -1 & \\ & & +1 \end{bmatrix} \underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}_{S^{-1}}$$

(c) Matrix C diagonalizes as $C = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{=Q} \underbrace{\begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}}_{\Lambda} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{Q^T}$. Find a positive definite matrix X so that $X^2 = C$.

C is a symmetric matrix because $C = Q\Lambda Q^T$ and $QQ^T = I$.

$X = Q\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}Q^T$ is also a symmetric matrix.

with eigenvalues $2, 3 > 0$ AND

$$X^2 = \left(Q\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\underbrace{Q^T\right)\left(Q}_{=I}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}Q^T\right) = Q\begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}Q^T = C.$$

7.(5+8+2=15pts) In the following parts, let $A = \begin{pmatrix} 1 & 4 & 0 \\ 1 & 7 & 5 \\ 0 & 6 & 16 \end{pmatrix}$.

ANSWER

$$x = \begin{bmatrix} 7 \\ -5 \\ 3 \end{bmatrix}$$

(a) Solve $A\mathbf{x} = \begin{pmatrix} -13 \\ -13 \\ 18 \end{pmatrix}$.

We use (b) to solve (a): $A = LU$

Call $Ux = v$, solve $Lv = \begin{pmatrix} -13 \\ -13 \\ 18 \end{pmatrix}$ for v first

$Ux = v$ for x next.

$$Lv = \begin{pmatrix} -13 \\ -13 \\ 18 \end{pmatrix}$$

$$\begin{array}{ccc|c} v_1 & v_2 & v_3 & \\ \hline 1 & 0 & 0 & -13 \\ 1 & 1 & 0 & -13 \\ 0 & 2 & 1 & 18 \end{array}$$

$\Rightarrow v_1 = -13$

$\Rightarrow v_1 + v_2 = -13$

$\Rightarrow v_2 = 0$

$\Rightarrow 2v_2 + v_3 = 18 \Rightarrow v_3 = 18$

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & v \\ \hline 1 & 4 & 0 & -13 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 6 & 18 \end{array}$$

$U =$

$x_1 + 4x_2 = -13$

$\boxed{x_1 = -13 - 4(-5) = 7}$

$\Rightarrow 3x_2 + 5x_3 = 0$

$\boxed{x_2 = -5}$

$\Rightarrow 6x_3 = 18$

$\boxed{x_3 = 3}$

(b) Compute LU–decomposition of the matrix A.

$$\begin{pmatrix} 1 & 4 & 0 \\ 1 & 7 & 5 \\ 0 & 6 & 16 \end{pmatrix} \xrightarrow{R_2 - 1R_1 \to R_2} \begin{pmatrix} 1 & 4 & 0 \\ 0 & 3 & 5 \\ 0 & 6 & 16 \end{pmatrix} \xrightarrow{R_3 - 2R_2} \begin{pmatrix} 1 & 4 & 0 \\ 0 & 3 & 5 \\ 0 & 0 & 6 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

CHECK (b)

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 0 \\ 0 & 3 & 5 \\ 0 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 0 \\ 1 & 4+3 & 5 \\ 0 & 6 & 10+6 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 0 \\ 1 & 7 & 5 \\ 0 & 6 & 16 \end{pmatrix} \checkmark$$

CHECK

(c) Check your answers in (a) and (b)

(a)

$$\begin{pmatrix} 1 & 4 & 0 \\ 1 & 7 & 5 \\ 0 & 6 & 16 \end{pmatrix} \begin{pmatrix} 7 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} 7-20 \\ 7-35+15 \\ 0-30+48 \end{pmatrix} = \begin{pmatrix} -13 \\ -13 \\ 18 \end{pmatrix} \checkmark$$

8.(10pts) Use the LU–decomposition for the matrix T to solve the equation $T\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$$T = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}}_{L=} \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{U=}.$$

Solve $Lv = x$ first

$Ux = v$ second.

$$\begin{array}{ccc|c} v_1 & v_2 & v_3 & \\ \hline 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -1 \end{array}$$

$\Rightarrow v_1 = 0$

$-v_1 + v_2 = 1 \Rightarrow v_2 = 1$

$-v_2 + v_3 = -1 \Rightarrow v_3 = 0$

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline 1 & -1 & 0 & 0 \\ & 1 & -1 & 1 \\ & & 1 & 0 \end{array}$$

$\Rightarrow$

$x_1 - x_2 = 0 \Rightarrow x_1 = 1$

$x_2 - x_3 = 1 \Rightarrow x_2 = 1$

$x_3 = 0$

SOLUTION $= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

9.(5+4+1=10pts) The matrix A has eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ with corresponding eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 5$ and $\lambda_3 = 0$.

(a) Give a solution to $A\mathbf{x} = 6 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - 5 \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$?

$$Ax = 6v_1 - 5v_1 = 2(3v_1) - (5v_1) = 2(Av_1) - Av_2$$
$$= A(2v_1 - v_2)$$

So $x = 2v_1 - v_2$ is a solution.

(b) Give another solution to $A\mathbf{x} = 6 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - 5 \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$?

Since $Av_3 = 0v_3 = 0$, in fact for all $k \in \mathbb{R}$,

$x = 2v_1 - v_2 + k \cdot v_3$ is a solution of $Ax = 0$

(c) What is the determinant of A?

Determinat = product of eigenvalues

$= 3 \cdot 5 \cdot 0 = 0$