M E T U – N C C

	Applied Mathematics for Engineers Final Exam	
Code : *Math 210* Acad.Year: *2013-2014* Semester : *Spring* Date : *05.28.2014* Time : *09:00* Duration : *120 minutes*	Last Name: Name: Solution Department: Signature:	Student No: Section:
	6 PAGES TOTAL 110 POINTS (PAR=48)	

1 (5)	2 (5)	3 (10)	4 (20)	5 (10)	6 (20)	7 (20)	8 (5)	9 (10)	10 (4)	11 (1)	B (5)	

Show your work! No calculators! Please draw a box around your answers! Please do not write on your desk!

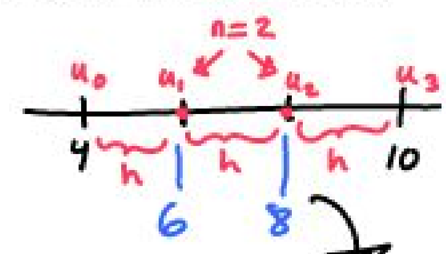
1. (5 pts) Let $A = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Solve the equation $A\mathbf{x} = 0$.

$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_2 - x_3 = 0 \\ x_4 = 0 \\ (x_1 \text{ is free}) \end{matrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_3 \\ 0 \end{bmatrix} = \boxed{x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}}$$

2. (5 pts) Provide a boundary value problem (a differential equation with boundary conditions) for an unknown function $u(x)$ over the interval $[4, 10]$ whose discrete/matrix version is the following matrix equation:

fixed/fixed 2^{nd} der. matrix

$$\overbrace{\frac{1}{h^2}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}^{-u''} u + \overbrace{\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}}^{5u} u = \overbrace{\begin{bmatrix} \frac{7}{h} \\ 0 \end{bmatrix}}^{7\cdot\delta(x-6)} \begin{matrix} \leftarrow x=6 \\ \leftarrow x=8 \end{matrix}$$

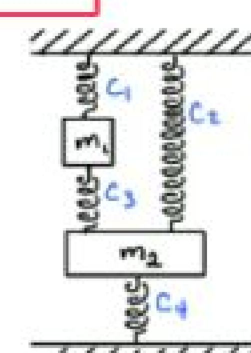


Assume the interval $[4, 10]$ is divided into equal subintervals of length h. What is h?

$$\boxed{\text{Differential equation: } -u'' + 5u = 7\cdot\delta(x-6) \quad u(4)=0,\ u(10)=0}$$

$$\boxed{h = \frac{10-4}{3} = 2}$$

3. (10 pts) The spring-mass system to the right has $c_1 = 1$, $c_2 = 3$, $c_3 = 2$, and $c_4 = 5$. If the system balances with the displacement vector $\mathbf{u} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ then what is the external force $\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ being applied at the masses?



Elongation matrix: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$

Spring constant matrix: $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

Stiffness matrix: $K = \begin{bmatrix} 1+2 & -2 \\ -2 & 3+2+5 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -2 & 10 \end{bmatrix}$

$K\underline{u} = \underline{f}$

$\begin{bmatrix} 3 & -2 \\ -2 & 10 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \underline{f}$

$\begin{bmatrix} 9-14 \\ -6+70 \end{bmatrix} =$

$$\boxed{\underline{f} = \begin{bmatrix} -5 \\ 64 \end{bmatrix}}$$

4.(1+2+2+3+4+4+4=20pts) Given the functions $f(x)$ and $g(x)$ with the complex Fourier series

$$\begin{aligned} f(x) &= -5 + (1+2i)e^{ix} - 3e^{i2x} + 4ie^{i3x} \\ g(x) &= 7 - e^{ix} - e^{-ix}. \end{aligned}$$

Calculate the Fourier series for the following:

(a) $f(x) + g(x) = \overbrace{-}^{c_{-1}}e^{-ix} + \overbrace{2}^{c_0} + \overbrace{2i}^{c_1}e^{ix} \overbrace{-3}^{c_2}e^{i2x} + \overbrace{4i}^{c_3}e^{i3x}$

(b) $g(2x) = \overbrace{-}^{c_{-2}}e^{-i2x} + \overbrace{7}^{c_0} \overbrace{-}^{c_2}e^{i2x}$

(c) $f'(x) = 0 + (1+2i)ie^{ix} - 3\cdot 2ie^{i2x} + 4i\cdot 3ie^{i3x} = \overbrace{(-2+i)}^{c_1}e^{ix} \overbrace{-6i}^{c_2}e^{i2x} \overbrace{-12}^{c_3}e^{i3x}$

(d) The anti-derivative $F(x)$ of $f(x)+5$ whose constant Fourier coefficient is 14.

$C = 14$ also, $\frac{1}{i} = -i$

$$\int f(x) + 5\,dx = \int (1+2i)e^{ix} - 3e^{i2x} + 4ie^{i3x}\,dx$$

$$= \frac{1+2i}{i}e^{ix} - \frac{3}{2i}e^{i2x} + \frac{4i}{3i}e^{i3x} + C = \underbrace{14}_{c_0} + \underbrace{(2-i)}_{c_1}e^{ix} + \overbrace{\tfrac{3}{2}i}^{c_2}e^{i2x} + \underbrace{\tfrac{4}{3}}_{c_3}e^{i3x}$$

(e) The convolution $g * g$.

Fourier coefficients of g are $c_{-1} = -1$, $c_0 = 7$, $c_1 = 1$

$\Rightarrow$ Fourier coeff. of $g*g$ are $c_{-1} = (-1)^2 = 1$, $c_0 = 7^2 = 49$, $c_1 = 1^2 = 1$

i.e.

$$g*g = e^{-ix} + 49 + e^{ix}$$

(f) The square g^2.

$$\begin{aligned} g^2 &= (7 - e^{ix} - e^{-ix})^2 \\ &= 49 + e^{2ix} + e^{-2ix} - 14e^{ix} - 14e^{-ix} + 2e^0 \\ &= \underbrace{e^{-2ix}}_{c_{-2}=1} \underbrace{-14}_{c_{-1}}e^{-ix} + \underbrace{51}_{c_0} \underbrace{-14}_{c_1}e^{ix} + \underbrace{e^{2ix}}_{c_2=1} \end{aligned}$$

(g) Are the functions $f(x)$ or $g(x)$ real-valued functions? That is, do these functions produce real values if x is real? Explain your answer.

f does not have the property that $c_{-n} = \overline{c_n}$ so it cannot be real valued.

$f = -5 + (1+2i)(\cos x + i\sin x) - 3(\cos 2x + i\sin 2x) + 4i(\cos 3x + i\sin 3x)$ ← imaginary parts don't cancel.

g has the property that $c_{-n} = \overline{c_n}$ so it is a real valued function.

$g = 7 - (\cos x + i\cancel{\sin x}) - (\cos(-x) + i\cancel{\sin(-x)})$ ← imaginary parts cancel.

5. (10pts) A blue jeans company's unit cost for a pair of blue jeans is 9 TL when company produces 1 million units. The unit price drops to 7 TL when 2 million units are produced. Suppose the unit price is modelled by the function a/x where x represents the units produced (in millions). What is the least squares best value of a?

formula: price $= a/x$

equations: $\begin{cases} 9 = a/1 \\ 7 = a/2 \end{cases} \rightarrow \begin{bmatrix} 9 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} a$

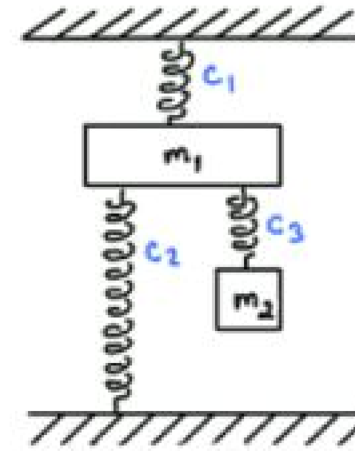
Matrix eqn:

Normal equation:

$$\begin{bmatrix} 1 & 1/2 \end{bmatrix}\begin{bmatrix} 9 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 1/2 \end{bmatrix}\hat{a}$$

$$\left[\tfrac{25}{2}\right] = \left[\tfrac{5}{4}\right]\hat{a} \qquad \boxed{\hat{a} = 10}$$

6. (8+12=20pts) The spring-mass system to the right oscillates as



$$\mathbf{u}(t) = \big(a_1\cos(\omega_1 t) + b_1 \sin(\omega_1 t)\big)\mathbf{x}_1 + \big(a_2\cos(\omega_2 t) + b_2 \sin(\omega_2 t)\big)\mathbf{x}_2,$$

where $\mathbf{u}(t)$ gives displacement of the masses from equilibrium.

Suppose $\omega_1 = \sqrt{3}$, $\quad \omega_2 = \sqrt{14}$, $\quad \mathbf{x}_1 = \begin{bmatrix}1\\2\end{bmatrix}$ $\quad$ and $\quad \mathbf{x}_2 = \begin{bmatrix}4\\-3\end{bmatrix}$.

(a) Figure out a_1, a_2, b_1, b_2 if $\mathbf{u}(0) = \begin{bmatrix}-1\\1\end{bmatrix}$ and $\mathbf{u}'(0) = \begin{bmatrix}0\\0\end{bmatrix}$.

$$\underline{u} = \left(a_1\cos(\sqrt{3}t) + b_1\sin(\sqrt{3}t)\right)\begin{bmatrix}1\\2\end{bmatrix} + \left(a_2\cos(\sqrt{14}t) + b_2\sin(\sqrt{14}t)\right)\begin{bmatrix}4\\-3\end{bmatrix}$$

$$\begin{bmatrix}-1\\1\end{bmatrix} = \underline{u}(0) = \left(a_1\cdot 1 + b_1\cdot 0\right)\begin{bmatrix}1\\2\end{bmatrix} + \left(a_2\cdot 1 + b_2 \cdot 0\right)\begin{bmatrix}4\\-3\end{bmatrix}$$

$$\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1 & 4\\2 & -3\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix} \rightsquigarrow \begin{bmatrix}a_1\\a_2\end{bmatrix} = \frac{1}{-11}\begin{bmatrix}-3 & -4\\-2 & 1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} = -\frac{1}{11}\begin{bmatrix}-1\\3\end{bmatrix}$$

$$\begin{bmatrix}0\\0\end{bmatrix} = \underline{u}'(0) = \left(a_1\sqrt{3}\cdot 0 + b_1\sqrt{3}\cdot 1\right)\begin{bmatrix}1\\2\end{bmatrix} + \left(a_2\sqrt{11}\cdot 0 + b_2\sqrt{11}\cdot 1\right)\begin{bmatrix}4\\-3\end{bmatrix}$$

$$\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}\sqrt{3} & 4\sqrt{11}\\2\sqrt{3} & -3\sqrt{11}\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix} \rightsquigarrow \begin{bmatrix}b_1\\b_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\boxed{a_1 = 1/11 \quad a_2 = -3/11 \quad b_1 = 0 \quad b_2 = 0}$$

(b) If $m_1 = 3$, $m_2 = 2$ and $c_1 = 17$, find the spring constants c_2 and c_3.

Stiffness matrix: $K = \begin{bmatrix}c_1+c_2+c_3 & -c_3\\-c_3 & c_3\end{bmatrix} = \begin{bmatrix}17+c_2+c_3 & -c_3\\-c_3 & c_3\end{bmatrix}$

Mass matrix: $M = \begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}$

$$M^{-1}K = \begin{bmatrix}\frac{17+c_2+c_3}{3} & -\frac{c_3}{3}\\-\frac{c_3}{2} & \frac{c_3}{2}\end{bmatrix}$$

eigenvectors should be
- $\begin{bmatrix}1\\2\end{bmatrix}$ with $\lambda = 3$
- $\begin{bmatrix}4\\-3\end{bmatrix}$ with $\lambda = 14$

$$(M^{-1}K)\begin{bmatrix}1\\2\end{bmatrix} = 3\cdot\begin{bmatrix}1\\2\end{bmatrix} \implies \frac{17+c_2+c_3}{3} + 2\left(\frac{-c_3}{3}\right) = 3 \longrightarrow c_2 = 9+12-17 = 4$$

$$-\frac{c_3}{2} + 2\left(\frac{c_3}{2}\right) = 6 \longrightarrow c_3 = 12$$

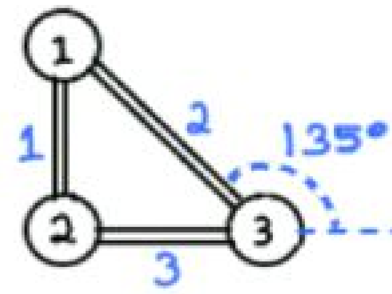
Check: $\begin{bmatrix}\frac{17+12+4}{3} & -\frac{12}{3}\\-\frac{12}{2} & \frac{12}{2}\end{bmatrix}\begin{bmatrix}4\\-3\end{bmatrix} \overset{??}{=} 14\cdot\begin{bmatrix}4\\-3\end{bmatrix}$

$$\begin{bmatrix}11 & -4\\-6 & 6\end{bmatrix}\begin{bmatrix}4\\-3\end{bmatrix} = \begin{bmatrix}44+12\\-24-18\end{bmatrix} = \begin{bmatrix}56\\-42\end{bmatrix} \checkmark$$

$$\boxed{\begin{matrix}c_2 = 4\\c_3 = 12\end{matrix}}$$

7. (8+2+10=20pts) The following parts are about truss systems.

(a) Write the elongation matrix for the truss system to the right.



$$A = \begin{array}{c} \begin{array}{cccccc} \multicolumn{2}{c}{\text{Node 1}} & \multicolumn{2}{c}{\text{Node 2}} & \multicolumn{2}{c}{\text{Node 3}} \\ h & v & h & v & h & v \end{array} \\ \left[\begin{array}{cc|cc|cc} 0 & 1 & 0 & -1 & 0 & 0 \\ -\cos 45^\circ & \sin 45^\circ & 0 & 0 & \cos 45^\circ & -\sin 45^\circ \\ 0 & 0 & -1 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$$\boxed{= \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}}$$

(b) Is the truss system in part **(a)** stable?
Explain why or why not using the elongation matrix you wrote down in **(a)**.

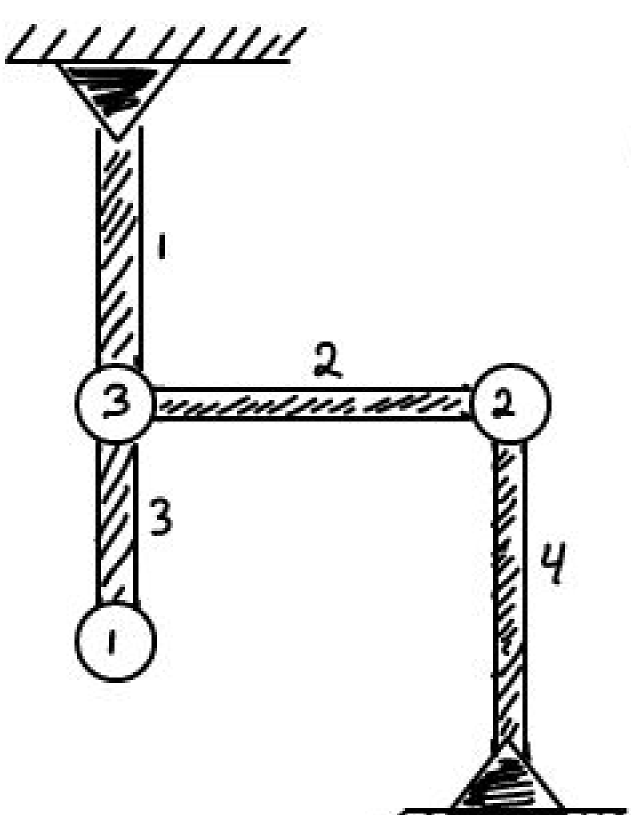
It is not stable.

The elongation matrix has only three rows so $\text{rank}(A) \leq 3$ (In fact, $\text{rank}(A) = 3$)
Thus $\text{rank}(A) < 6 = \#\text{columns}$.

⇝ In particular, note that $[1\ 0\ 1\ 0\ 1\ 0]^T$ (horiz. shift) and $[0\ 1\ 0\ 1\ 0\ 1]^T$ (vert. shift) are both in nullspace(A).

(c) Draw a truss system which has the following elongation matrix.

$$\begin{array}{cl} A = \left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] & \begin{array}{l} \text{bar 1} \\ \text{bar 2} \\ \text{bar 3} \\ \text{bar 4} \end{array} \\ \begin{array}{cccccc} h & v & h & v & h & v \\ \multicolumn{2}{c}{\text{Node 1}} & \multicolumn{2}{c}{\text{Node 2}} & \multicolumn{2}{c}{\text{Node 3}} \end{array} & \end{array}$$



According to A, the truss system has the following bars:

bar 1: connects to node 3 from above, other end is fixed

bar 2: connects node 2 to node 3 horizontally (node 3 is left of node 2)

bar 3: connects node 1 to node 3 vertically (node 1 is below node 3)

bar 4: connects to node 2 from below other end is fixed

8.(3+2=5pts) The following parts are about the function

$$f(x)=\delta\left(x-\tfrac{\pi}{2}\right)-\delta\left(x+\tfrac{\pi}{2}\right) \quad \text{for} \quad -\pi \leqslant x < \pi.$$

(a) Calculate the **complex** Fourier coefficients of f.

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\delta(x-\pi/2)e^{-inx} - \delta(x+\pi/2)e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left(e^{-in\pi/2} - e^{-in(-\pi/2)}\right)$$

$$= \frac{1}{2\pi}\left(\left(\cancel{\cos n\pi/2} - i\sin\frac{n\pi}{2}\right) - \left(\cancel{\cos n\pi/2} + i\sin n\pi/2\right)\right)$$

$$= -\frac{i}{\pi}\sin\left(\frac{n\pi}{2}\right)$$

$c_0=0 \quad c_1=-i/\pi \quad c_2=0 \quad c_3=i/\pi \quad c_4=0$ etc

$$\boxed{\begin{array}{l} c_{2n}=0 \\ c_{2n+1}=(-1)^{n+1}\frac{i}{\pi} \end{array}}$$

(b) What can you say about the **real** Fourier series of f?

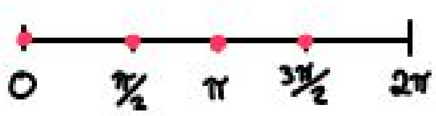
Since the complex Fourier series is pure imaginary the real Fourier series will have only "sin" terms (i.e. all $a_n=0$)

(You also know this because f is odd.)

In fact, $a_n=0$ and $b_{2n}=0$, $b_{2n+1}=(-1)^n 2/\pi$

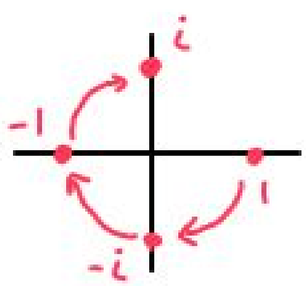
9.(2+8=10pts) The following parts are about the function $f(x)=\tan\left(x+\frac{\pi}{4}\right)$ for $0 \leqslant x < 2\pi$.

(a) Discretize f by dividing the interval $[0, 2\pi)$ into 4 pieces. Denote the resulting vector by $\mathbf{t}$.



$$\underline{t} = \begin{bmatrix} \tan(0+\pi/4) \\ \tan(\pi/2+\pi/4) \\ \tan(\pi+\pi/4) \\ \tan(3\pi/2+\pi/4) \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}}$$

(b) What is the discrete Fourier transform of $\mathbf{t}$?



$c_0 = \frac{1}{4}(1-1+1-1) = 0$

$c_1 = \frac{1}{4}(1\cdot 1 + (-1)(-i) + (1)(-1) + (-1)(i)) = 0$

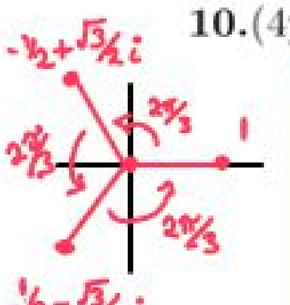
$c_2 = \frac{1}{4}(1\cdot 1 + (-1)(-1) + (1)(1) + (-1)(-1)) = \frac{1}{4}\cdot 4 = 1$

$c_3 = \bar{c}_1 = 0$

$$\boxed{\frac{1}{4}\bar{F}_4\,\underline{t} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}} \begin{array}{l} \leftarrow \text{freq } 0 \\ \leftarrow \text{freq } 1 \\ \leftarrow \text{freq } 2 \\ \leftarrow \text{freq } 3 \end{array}$$

<u>Note</u>: We could have guessed this because $\underline{t}$ is clearly periodic with frequency $=2$

10.(4pt) Write the Fourier matrices F_3 and F_3^{-1}.



$$\boxed{F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & (-\frac{1}{2}+\frac{\sqrt{3}}{2}i) & (-\frac{1}{2}-\frac{\sqrt{3}}{2}i) \\ 1 & (-\frac{1}{2}-\frac{\sqrt{3}}{2}i) & (-\frac{1}{2}+\frac{\sqrt{3}}{2}i) \end{bmatrix}}$$

$$\boxed{F_3^{-1} = \frac{1}{3}\bar{F}_3 = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & (-\frac{1}{2}-\frac{\sqrt{3}}{2}i) & (\frac{1}{2}+\frac{\sqrt{3}}{2}i) \\ 1 & (-\frac{1}{2}+\frac{\sqrt{3}}{2}i) & (\frac{1}{2}-\frac{\sqrt{3}}{2}i) \end{bmatrix}}$$

11.(1pt) What happens to a vertical 1-dimensional spring-mass system if a horizontal force is applied?

Our theory only allowed vertical forces

→ horizontal forces make the system 2-dimensional → you would need to add extra columns to A to allow horiz. movement of masses (like trusses)

B1. *(3pts to course grade)* A spring-mass system has the following stiffness matrix.

(0.5 pts)

$$K = \begin{bmatrix} 9 & \boxed{-1} & -5 \\ -1 & 5 & \boxed{-4} \\ \boxed{-5} & -4 & 11 \end{bmatrix}.$$

(Missing entries are determined by symmetry.)

Fill in the missing numbers **and draw** a spring system with this stiffness matrix (include values for all spring constants).

(2.5 pts)

spring with c=1 connects mass 1 & 2

$$K = \begin{bmatrix} 9 & -1 & -5 \\ -1 & 5 & -4 \\ -5 & -4 & 11 \end{bmatrix}$$

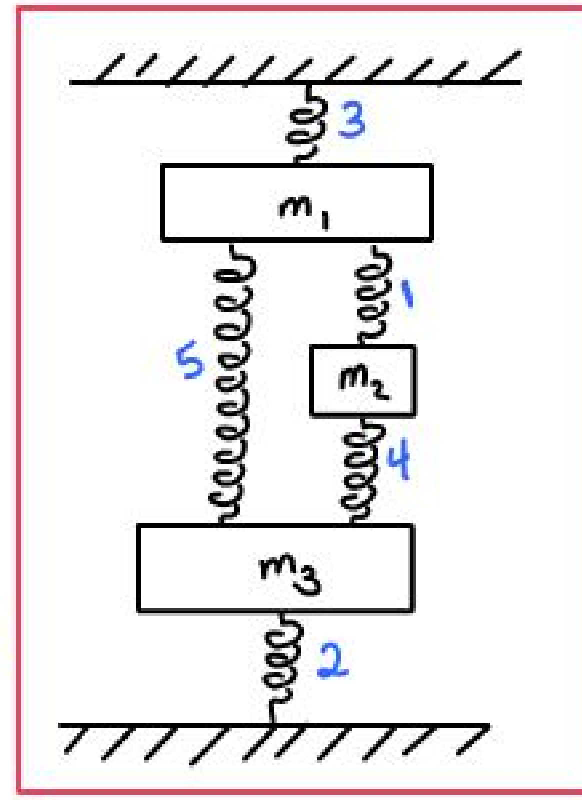
spring with c=5 connects mass 1 & 3

spring with c=4 connects mass 2 & 3

9-1-5=3 so a spring with c=3 connects mass 1 to outside

5-1-4=0 so mass 2 is not connected to outside

11-4-5=2 so a spring with c=2 connects mass 3 to outside



B2. *(2pts to course grade)* Find the vector **k** so that for any **f** with Fourier transform $\begin{bmatrix} c_0 & c_1 & c_2 & c_3 \end{bmatrix}^T$ the convolution $\mathbf{f} \circledast \mathbf{k}$ will have Fourier transform $\begin{bmatrix} 0 & c_1 & 0 & c_3 \end{bmatrix}^T$

Recall that, for any column vectors **f** *and* **g** *of length* N*, the following formula holds:*

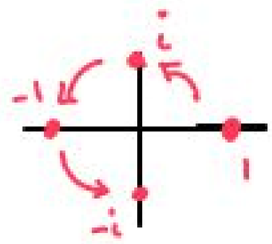
$$F_N^{-1}(\mathbf{f} \circledast \mathbf{g}) = N F_N^{-1}(\mathbf{f}) \cdot F_N^{-1}(\mathbf{g}),$$

where the product on the right multiplies corresponding components.

The Fourier transform of $\underline{f} \circledast \underline{k}$ is

$$\tfrac{1}{4}\bar{F}_4(f \circledast k) = \begin{bmatrix} 0 \\ c_1 \\ 0 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_0 \cdot 0 \\ c_1 \cdot 1 \\ c_2 \cdot 0 \\ c_3 \cdot 1 \end{bmatrix} = 4 \cdot \left(\tfrac{1}{4}\bar{F}_4 \underline{f}\right) \cdot \left(\tfrac{1}{4}\bar{F}_4 \underline{k}\right)$$

$\rightsquigarrow$ so $\underline{k}$ is the inverse Fourier transform of $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot \tfrac{1}{4}$



$$k_0 = \frac{0+1+0+1}{4} = \tfrac{1}{2}$$

$$k_1 = \frac{0+i+0-i}{4} = 0$$

$$k_2 = \frac{0-1+0-1}{4} = -\tfrac{1}{2}$$

$$k_3 = \frac{0-i+0+i}{4} = 0$$

$$\boxed{\underline{k} = \begin{bmatrix} \tfrac{1}{2} \\ 0 \\ -\tfrac{1}{2} \\ 0 \end{bmatrix}}$$