4. *(4× 5pts)* Evaluate the following integrals.

(a) $\int (x^{-\pi} + \frac{2}{\sqrt{\pi}})dx = \frac{x^{-\pi+1}}{-\pi+1} + \frac{2x}{\sqrt{\pi}} + C_1$

(b) $\int \cos^2 x \sin x dx = \int u^2 \cdot -du = -\frac{u^3}{3} + C_1 = -\frac{\cos^3 x}{3} + C_1$

Say $\cos x = u$

$-\sin x \, dx = du$

(c) $\int x^7 \sqrt{x^4 - 1} dx = \int u . \frac{2(u^2+1) 2u}{8} du = \frac{1}{2} \int u^4 + u^2 du$

Say $\sqrt{x^4-1} = u$

$x^4 = u^2+1$

$x^8 = (u^2+1)^2$

$8x^7 dx = 2(u^2+1).2u du$

$= \frac{u^5}{10} + \frac{u^3}{6} + C$

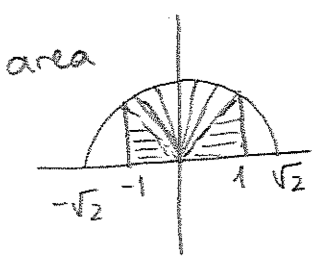
$= \frac{(x^4-1)^{5/2}}{10} + \frac{(x^4-1)^{3/2}}{6} + C_1$

(d) $\int_{-1}^{1} (x^2 \sin(x) + \sqrt{2 - x^2})dx = \underbrace{\int_{-1}^{1} x^2 \sin x \, dx}_{0} + \int_{-1}^{1} \sqrt{2-x^2} \, dx$

$x^2 \sin x = -((-x)^2 \sin(-x))$

So, it is an odd function and over a symmetric interval its integral is 0.

$\int_{-1}^{1} \sqrt{2-x^2} \, dx$ || Shaded area



So, area is;

$\frac{\pi . (\sqrt{2})^2}{4} + 2 . \frac{1.1}{2} = \frac{\pi}{2} + 1$

METU - NCC

CALCULUS WITH ANALYTIC GEOMETRY MIDTERM 2

Code : *MAT 119*	Last Name: KEY
Acad. Year: *2012-2013*	Name :
Semester : *SPRING*	Department:
Date : *27.04.2013*	Signature:
Time : *14:40*	Student No.:
Duration : *90 minutes*	Section:

5 QUESTIONS ON 6 PAGES
TOTAL 100 POINTS

1. (20)	2. (25)	3. (15)	4. (20)	5. (20)	Bonus	

Show your work! Please draw a $\boxed{\text{box}}$ around your answers!

1. *(20pts)* Let $f(x) = x\sqrt{x}$ on $(0, \infty)$. Find the point on the graph of $f(x)$ closest to the point $(1/2, 0)$. **JUSTIFY YOUR ANSWER!**

Distance of any point on the curve $(x, x\sqrt{x})$ to $(\frac{1}{2}, 0)$ is;

$$d^2(x) = (x - \frac{1}{2})^2 + (x\sqrt{x} - 0)^2$$

Since $a < b \Rightarrow a^2 < b^2$, we will try to minimize $d^2(x)$ funct

$$(d^2(x))' = 2(x - \frac{1}{2}) + 3x^2 = 0 \Rightarrow 3x^2 + 2x - 1 = 0$$

to find critical points

$(3x-1)(x+1) = 0$

$x = \frac{1}{3}$; $\cancel{x = -1}$ not in (

x	0		$1/3$	
$(d^2(x))'$		$-$		$+$
			local min	

Since $d^2(x)$ is decreasing before $x = 1/3$ and increasing after $x=$

actuall local min at $x = 1/3$ is global min

Closest distance; $d^2(1/3) = (\frac{1}{3} - \frac{1}{2})^2 + (\frac{1}{3}\sqrt{\frac{1}{3}})^2$

$= \frac{1}{36} + \frac{1}{27}$

$= \frac{7}{108}$

$d(1/3) = \sqrt{\frac{7}{108}}$

2. *(3+4+13+5=25pts)* Let $f(x) = \frac{x^3}{x^3-1}$.

(a) Find the domain of $f(x)$, x intercepts and y intercepts.

$D_f = \mathbb{R} - \{1\}$; $y_{int} = \frac{0^3}{0^3-1} = 0$; $0 = \frac{x_{int}^3}{x_{int}^3 - 1} \Rightarrow x_{int} = 0$.

(b) Find the asymptotes of $f(x)$.

<u>Horizontal Asymptotes</u>

$$\lim_{x\to-\infty} \frac{x^3}{x^3-1} = \lim_{x\to-\infty} \frac{\cancel{x^3}}{\cancel{x^3}(1-\frac{1}{x^3})} = 1$$

$$\lim_{x\to+\infty} \frac{x^3}{x^3-1} = \lim_{x\to+\infty} \frac{\cancel{x^3}}{\cancel{x^3}(1-\frac{1}{x^3})} = 1$$

$y=1$ is horizontal asymptote

<u>Vertical Asymptote</u>

$$\lim_{x\to1^-} \frac{x^3}{x^3-1} = -\infty$$

$$\lim_{x\to1^+} \frac{x^3}{x^3-1} = +\infty$$

$x=1$ is vertical asymptote.

(c) Find the intervals of increase/decrease, intervals of concavity, local max/min and inflection points of $f(x)$.

$$f'(x) = \frac{3x^2(x^3-1) - 3x^2 \cdot x^3}{(x^3-1)^2} = -\frac{3x^2}{(x^3-1)^2}$$

$$f''(x) = -\left(\frac{6x \cdot (x^3-1)^{\cancel{2}\,1} - 2\cancel{(x^3-1)} \cdot 3x^2 \cdot 3x^2}{(x^3-1)^{\cancel{4}\,3}}\right) = \frac{12x^4+6x}{(x^3-1)^3} = \frac{6x(2x^3+1)}{(x^3-1)^3}$$

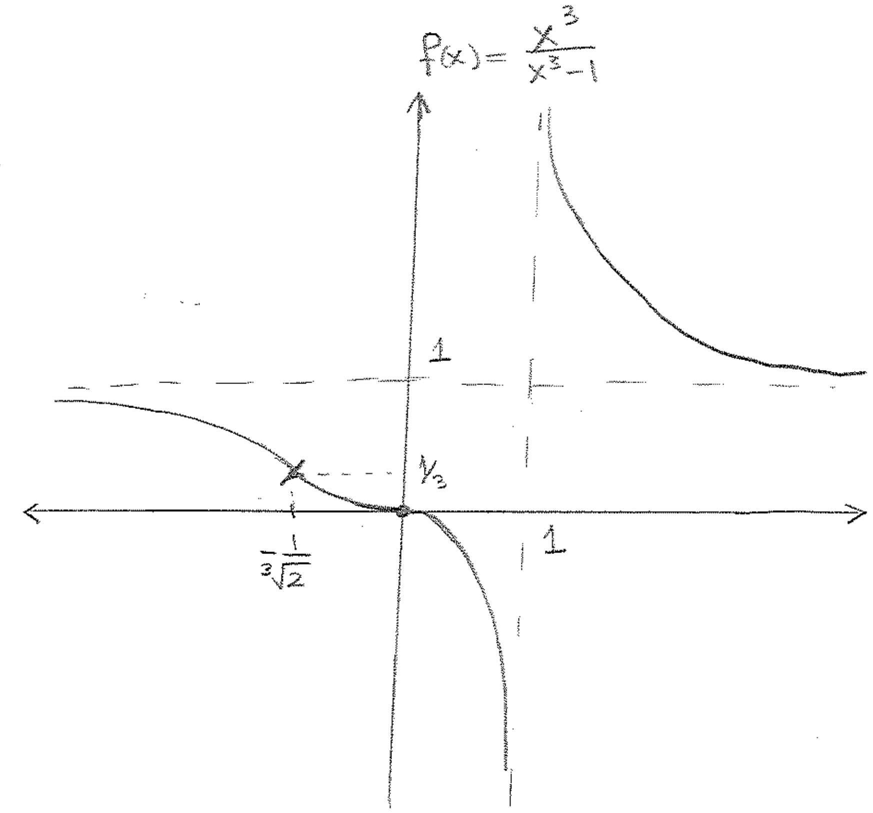
x		$-\frac{1}{\sqrt[3]{2}}$		0		1	
$f'(x)$	$-$		$-$	0	$-$		$-$
$f''(x)$	$-$		$+$		$-$		$+$

$\Rightarrow f$ is alway decreasing excep $x=0$.

$\Rightarrow f$ is concave up on $(-\frac{1}{\sqrt[3]{2}}, 0) \cup (1, \infty)$

f is concave down on $(-\infty, -\frac{1}{\sqrt[3]{2}}) \cup (0, 1)$

(d) Sketch the graph of $f(x)$.



3. *(8+7=15pts)*

(a) Let $F(x) = \int_{\cos x}^{x^3} \sec^5(t)dt$. Find $F'(x)$.

Using F.T.C – I; $F'(x) = \sec^5(x^3) \cdot 3x^2 - \sec^5(\cos x) \cdot -\sin x$

$F'(x) = 3x^2 \sec^5(x^3) + \sin x \sec^5(\cos x)$

(b) Suppose that $f(x)$ is a differentiable function with $f(-1) = 1, f(0) = 4$ and $f(2) = -3$. Find $\int_0^2 f'(x)dx$.

Using F.T.C – II; $\int_0^2 f'(x)dx = f(2) - f(0)$

$= -3 - 4 = -7$

5. *(20pts)* Find the area between the curves $y = \sqrt{3}\cos x$ and $y = \sin x$ on the interval $[0, \pi/2]$.

Let's check the intersection points: $\sqrt{3}\cos x = \sin x$

$\Rightarrow \tan x = \sqrt{3} \Rightarrow x = \frac{\pi}{3}$ is the only solution in $[0, \pi/2]$

$$\text{Area} = \int_0^{\pi/2} \left| \sqrt{3}\cos x - \sin x \right| dx$$

$$= \int_0^{\pi/3} \sqrt{3}\cos x - \sin x \, dx + \int_{\pi/3}^{\pi/2} \sin x - \sqrt{3}\cos x \, dx$$

$$= \sqrt{3}\sin x + \cos x \Big|_0^{\pi/3} + -\cos x - \sqrt{3}\sin x \Big|_{\pi/3}^{\pi/2}$$

$$= \left(\sqrt{3}.\frac{\sqrt{3}}{2} + \frac{1}{2}\right) - \left(\sqrt{3}.0 + 1\right) + \left(-0 - \sqrt{3}.1\right) - \left(-\frac{1}{2} - \sqrt{3}\frac{\sqrt{3}}{2}\right)$$

$$= 3 - \sqrt{3}$$

Bonus.

Suppose that $f(x)$ is a differentiable function with $f''(x) < 0$, $f(0) = 5, f'(0) = 1$. Show that $\int_{-1}^{3} f(x)dx < 24$.

Since $f(x)$ is concave down, its graph stays under the graph of any tangent line.

Tangent line eqn at $x=0$; $y = f'(0)x + f(0) \Rightarrow y = x + 5$

So, $f(x) \leqslant x + 5 \Rightarrow \int_{-1}^{3} f(x)dx < \int_{-1}^{3} x + 5 \, dx = \frac{x^2}{2} + 5x \Big|_{-1}^{3} = 24$

$$\Rightarrow \int_{-1}^{3} f(x)dx < 24.$$