M E T U
Northern Cyprus Campus

Math 260	Linear Algebra	Midterm Exam I	31.10.2013
Last Name Name : Student No	Dept./Sec.: Time : 17: 40 Duration : 70 *minutes*	Signature	
5 QUESTIONS ON 4 PAGES			TOTAL 100 POINTS

1	2	3	4	5				

Q1 (10 p.) Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$), and let $U \leq V$ be a subspace with its complement $W = V \backslash U = \{v \in V : v \notin U\} \subseteq V$.

a) Is it true that W is a subspace of V as well? Explain your answer.

Since $0_V \in \mathcal{U}$, we conclude that $0_V \notin W$, thereby W can not be a subspace of V.

b) If $U \neq V$, construct a nonzero subspace $X \leq V$ such that $U \cap X = \{0\}$.

Take $v \in V \setminus \mathcal{U}$ and put $X = \text{Span}\{v\} \leq V$.
Then $X \neq \{0_V\}$ and $X \cap \mathcal{U} = \{0_V\}$, for
$u \in X \cap \mathcal{U} \Rightarrow u = \lambda v$ for some $\lambda \in \mathbb{R}$ (or $\mathbb{C}$);
If $\lambda \neq 0$ then $v = \lambda^{-1} u \in \mathcal{U}$, a contradiction.

Q2 (15 p.) Consider the vector space $C[a,b]$ of all continuous real functions over the closed interval $[a,b]$. For each n, the set $\mathcal{P}_n$ of all polynomials on $[a,b]$ is a subspace in $C[a,b]$. Is true that their union $X = \bigcup_n \mathcal{P}_n$ is a subspace in $C[a,b]$? Explain your answer. What about their intersection $\bigcap_n \mathcal{P}_n$?

Note that $P_n + P_m \subseteq P_{\max\{n,m\}} + P_{\max\{n,m\}}$ $\subseteq P_{\max\{n,m\}} \subseteq X$ for all n, m, that is, $X + X \subseteq X$ or X is a subspace.

Finally, $\bigcap_n P_n = P_1 \leq C[a,b]$.

Q3 (25 p.) Consider the following vectors $\mathbf{a}_1 = (-1,2,0)$, $\mathbf{a}_2 = (-4,0,-1)$, $\mathbf{a}_3 = (2,-3,-1)$, $\mathbf{a}_4 = (2,-9,3)$ from the vector space $\mathbb{R}^3$. Find $\dim(\mathrm{Span}\{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4\}) =?$. Explain your answer.

If $\lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2 + \lambda_3 \vec{a}_3 = \vec{0}$ then we have

$$\begin{cases} -\lambda_1 - 4\lambda_2 + 2\lambda_3 = 0 \\ 2\lambda_1 - 3\lambda_3 = 0 \\ -\lambda_2 - \lambda_3 = 0 \end{cases} \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0.$$

Thus $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ are linearly ndependent vectors, and

$$3 = \dim(\mathbb{R}^3) \geqslant \dim(\mathrm{Span}\{\vec{a}_1,\vec{a}_2,\vec{a}_3,\vec{a}_4\}) \geqslant 3.$$

Whence $\dim(\mathrm{Span}\{\vec{a}_1,\vec{a}_2,\vec{a}_3,\vec{a}_4\}) = 3$.

Q4 (25 p.) Let $S = \{1,2,3,4,5\}$, and consider the vector subspace $V = \{f \in \mathrm{Fun}(S) : f(1) = 2f(5), 2f(1) = -f(4), 2f(2) = 3f(3), f(2) = 4f(4)\}$ in $\mathrm{Fun}(S)$. Find a basis B for V, and $\dim(V) = ?$. Explain what does V geometrically mean?

Put $f(5) = \lambda$. Then $f(1) = 2\lambda$, $f(4) = -2f(1) = = -4\lambda$, $f(2) = 4f(4) = -16\lambda$, $f(3) = \frac{2}{3} f(2) = = -\frac{32}{3}\lambda$. Then

$$f = \sum_{i=1}^{5} f(i)\, \chi_i = 2\lambda\chi_1 - 16\lambda\chi_2 - \frac{32}{3}\lambda\chi_3 - 4\lambda\chi_4 + \lambda\chi_5 =$$

$$= \lambda\underbrace{\left(2\chi_1 - 16\chi_2 - \frac{32}{3}\chi_3 - 4\chi_4 + \chi_5\right)}_{g}$$

$$= \lambda g.$$

Thus, for every $f \in V$ we have

$f = \lambda g \in \mathrm{Span}\{g\}$ with $\lambda = f(5)$,

that is, $V = \mathrm{Span}\{g\}$, $\dim(V) = 1$

and V is a line geometrically.

Q5 (25 p.) Let $\mathcal{P}_3(\mathbb{R})$ be the vector space of all polynomials of degrees at most 3, and $p_1(x) = 3 + x(9 - 4x - 2x^2)$, $p_2(x) = 2 + 3x - x^3$ and $p_3(x) = 2 + x(x^2 + 2x - 1)$ the elements of $\mathcal{P}_3(\mathbb{R})$. Test whether $q(x) = (3x^2 - 2)x \in \operatorname{Span}\{p_1(x), p_2(x), p_3(x)\}$ or not? Explain your answer.

Put $q(x) = \lambda p_1(x) + \mu p_2(x) + \theta p_3(x)$ or

$$-2x + 3x^3 = \lambda(3 + 9x - 4x^2 - 2x^3) + \mu(2 + 3x - x^3) + \theta(2 - x + 2x^2 + x^3)$$

$$\begin{cases} 3\lambda + 2\mu + 2\theta = 0 \\ 9\lambda + 3\mu - \theta = -2 \quad \longrightarrow \lambda = 1. \\ \left.\begin{array}{l} -4\lambda + 2\theta = 0 \\ -2\lambda - \mu + \theta = 3 \end{array}\right\} \Rightarrow \theta = 2\lambda,\ \mu = -3 \end{cases}$$

But there is an extra equation

$3\lambda + 2\mu + 2\theta = 0$ and

$3 \cdot 1 + 2 \cdot (-3) + 2 \cdot 2 = 1 \neq 0.$

Whence $q(x) \notin \operatorname{Span}\{p_1(x), p_2(x), p_3(x)\}$.