M E T U
Northern Cyprus Campus

Math 260 Linear Algebra Midterm Exam II 11.12.2012		
Last Name Name : Student No	Dept./Sec.: Time : 17: 40 Duration : 80 *minutes*	Signature

5 QUESTIONS ON 4 PAGES							TOTAL 100 POINTS	
1	2	3	4	5				

Q1 (20=5+15p.) Let $T \in \mathcal{L}(\mathbb{R}^3)$, $T(x,y,z) = (2x+z, z-x, y+x)$ be a linear trasnformation.

(*a*) First find the matrix $M_{e,e}(T)$ of T with respect to the standard basis e for $\mathbb{R}^3$.

$$M_{e,e}(T) = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

(*b*) Use the Change of Bases Theorem to find $M_{f,f}(T)$ with respect to the basis $f = ((1,0,1),(0,1,1),(1,0,0))$ for $\mathbb{R}^3$.

Note that $e_1 = f_3$, $e_2 = -f_1 + f_2 + f_3$, $e_3 = f_1 - f_3$.

By Change of Bases Theorem, we have

$$M_{f,f}(T) = M_{e,f}(I)\, M_{e,e}(T)\, M_{f,e}(I) =$$

$$= \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

Q2 (15 p.) Let $T : \mathcal{P}_4(\mathbb{R}) \to \mathbb{R}^3$, $T(p(x)) = \left(p'(0), p(-1), \int_0^1 2p(x)\,dx\right)$ be a linear trasnformation. Find its matrix relative to standard bases $(1, x, x^2, x^3, x^4)$ and $e = ((1,0,0),(0,1,0),(0,0,1))$, respectively.

Put $f = (1, x, x^2, x^3, x^4)$. Then

$$M_{f,e}(T) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 \\ 2 & 1 & 2/3 & 1/2 & 2/5 \end{bmatrix}$$

Q3 (15 p.) Let $T \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}))$, $T(p(x)) = xp''(x)$. Show that T is a nilpotent transformation and find a basis f for $\mathcal{P}_3(\mathbb{R})$ such that $M_{f,f}(T) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

First note that $T(a_0 + a_1 x + a_2 x^2 + a_3 x^3) =$
$= 2a_2 x + 6a_3 x^2$, that is, $T(p(x)) = 2a_2 x + 6a_3 x^2$.
Then $T^2(p(x)) = 12 a_3 x$ and $T^3(p(x)) = 0$, that is, T is nilpotent of index 3.
Put $f = (x^3, 6x^2, 12x, 1)$ which is a basis for $P_3(\mathbb{R})$, and $T(x^3) = 6x^2$, $T(6x^2) = 12x$, $T(12x) = 0$, $T(1) = 0$. Whence

$$M_{f,f}(T) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Q4 (25=15+10 p.) Consider the plane $\mathcal{W} = \{2x - y + 5z = 0\}$ in $\mathbb{R}^3$.

(*a*) Find the matrix $M_{e,e}(P)$ of the orthogonal projection $P \in \mathcal{L}(\mathbb{R}^3)$ onto $\mathcal{W}$ with respect to the standard basis e for $\mathbb{R}^3$.

Note that $f_1 = (1,2,0)$, $f_2 = (0,5,1)$ is a basis for W, and $f_3 = (2,-1,5)$ is the normal vector to W.
Moreover, $e_1 = {}^{13}/_{15}\, f_1 - {}^{1}/_{3}\, f_2 + {}^{1}/_{15}\, f_3$,
$e_2 = {}^{1}/_{15}\, f_1 + {}^{1}/_{6}\, f_2 - {}^{1}/_{30}\, f_3$, $e_3 = -\frac{1}{3} f_1 + \frac{1}{6} f_2 + \frac{1}{6} f_3$.
Therefore $P(e_1) = {}^{13}/_{15}\, f_1 - {}^{1}/_{3}\, f_2 = ({}^{13}/_{15}, {}^{1}/_{15}, -{}^{1}/_{3})$,
$P(e_2) = {}^{1}/_{15}\, f_1 + {}^{1}/_{6}\, f_2 = ({}^{1}/_{15}, {}^{29}/_{30}, {}^{1}/_{6})$, $P(e_3) =$
$= -{}^{1}/_{3}\, f_1 + {}^{1}/_{6}\, f_2 = (-{}^{1}/_{3}, {}^{1}/_{6}, {}^{1}/_{6})$. Hence

$$M_{e,e}(P) = \begin{bmatrix} {}^{13}/_{15} & {}^{1}/_{15} & -{}^{1}/_{3} \\ {}^{1}/_{15} & {}^{29}/_{30} & {}^{1}/_{6} \\ -{}^{1}/_{3} & {}^{1}/_{6} & {}^{1}/_{6} \end{bmatrix}$$

(*b*) Find the matrix $M_{e,e}(P)$ of the projection $P \in \mathcal{L}(\mathbb{R}^3)$ onto $\mathcal{W}$ parallel to the vector $\mathbf{v} = (1,0,1)$ ($\mathbf{v} \notin \mathcal{W}$) with respect to the standard basis e for $\mathbb{R}^3$.

As above $e_1 = {}^{5}/_{7}\, f_1 - {}^{2}/_{7}\, f_2 + {}^{2}/_{7}\, f_3$,
$e_2 = {}^{1}/_{7}\, f_1 + {}^{1}/_{7}\, f_2 - {}^{1}/_{7}\, f_3$, $e_3 = -{}^{5}/_{7}\, f_1 + {}^{2}/_{7}\, f_2 + {}^{5}/_{7}\, f_3$,
where $f_3 = v = (1,0,1)$. Therefore
$P(e_1) = {}^{5}/_{7}\, f_1 - {}^{2}/_{7}\, f_2 = ({}^{5}/_{7}, 0, -{}^{2}/_{7})$, $P(e_3) =$
$= -{}^{5}/_{7}\, f_1 + {}^{2}/_{7}\, f_2 = (-{}^{5}/_{7}, 0, {}^{2}/_{7})$ and $P(e_2) =$
$= {}^{1}/_{7}\, f_1 + {}^{1}/_{7}\, f_2 = ({}^{1}/_{7}, 1, {}^{1}/_{7})$. Whence

$$M_{e,e}(P) = \begin{bmatrix} {}^{5}/_{7} & {}^{1}/_{7} & -{}^{5}/_{7} \\ 0 & 1 & 0 \\ -{}^{2}/_{7} & {}^{1}/_{7} & {}^{2}/_{7} \end{bmatrix}$$

Q5 (25 p.) Find the general solution Sol_{nh} (an affine subspace in $\mathbb{R}^5$) to the following nonhomogeneous linear system

$$\begin{cases} x_1 + x_2 - x_3 + x_5 = -1 \\ x_1 - x_2 + x_3 + 2x_4 - x_5 = 1 \\ x_1 + x_3 - 3x_4 - 2x_5 = -1 \\ x_2 + 2x_3 - x_4 + x_5 = 1 \end{cases}$$

Express Sol_{nh} as a sum $Sol_h + \mathbf{x}_0$ of the general solution Sol_h (a subspace in $\mathbb{R}^5$) to the related homogeneous system and a special solution $\mathbf{x}_0$ to the nonhomogeneous one. Write down the dimension of Sol_h.

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 0 & 1 & -1 \\ 1 & -1 & 1 & 2 & -1 & 1 \\ 1 & 0 & 1 & -3 & -2 & -1 \\ 0 & 1 & 2 & -1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 0 & 1 & -1 \\ 0 & -2 & 2 & 2 & -2 & 2 \\ 0 & -1 & 2 & -3 & -3 & 0 \\ 0 & 1 & 2 & -1 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 & 3 & 0 \\ 0 & 1 & 2 & -1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -5 & -1 & -2 \\ 0 & 0 & 1 & -4 & -2 & -1 \\ 0 & 0 & 0 & 12 & 6 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 12 & 0 & 0 & 0 & -6 & -5 \\ 0 & 12 & 0 & 0 & 18 & 1 \\ 0 & 0 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 12 & 6 & 5 \end{array}\right]$$ the reduced echelon form

Then $Sol_h = \{ (\frac{\lambda}{2}, -\frac{18\lambda}{12}, 0, -\frac{\lambda}{2}, \lambda) : \lambda \in \mathbb{R} \} =$

$= \mathrm{Span}\{ (1/2, -3/2, 0, -1/2, 1) \} \Rightarrow \dim(Sol_h) = 1$, and

$\vec{x}_0 = (-5/12, 1/12, 2/3, 5/12, 0)$. Thus

$Sol_{nh} = Sol_h + \vec{x}_0$.