METU - NCC

DIFFERENTIAL EQUATIONS FINAL EXAM	
Code : *MAT 219* Acad.Year : *2013-2014* Semester : *SUMMER* Date : *15.08.2014* Time : *9:00* Duration : *120 min*	Last Name: Name : Solutions Student # : Signature :
	7 QUESTIONS ON 5 PAGES TOTAL 100 POINTS

1. (12)	2. (12)	3. (16)	4. (10)	5. (10)	6. (14)	7. (26)		

Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

1. *(2+4+6=12pts)* Compute Laplace transforms of the following functions.

(A) $f(t) = t^3 + \cos(2t)$

$$\mathcal{L}\{t^3\} + \mathcal{L}\{\cos 2t\}$$

$$\boxed{3!/s^4 + \frac{s}{s^2+2^2}}$$

(B) $f(t) = \begin{cases} t^2, & t < 2 \\ 4, & 2 \leqslant t \end{cases}$

$$= t^2 + (4-t^2)\cdot u(t-2)$$

$$\mathcal{L}\{t^2\} + \mathcal{L}\{(4-t^2)\cdot \underline{u(t-2)}\}$$

(pull out step funct.)

$$2/s^3 + e^{-2s}\mathcal{L}\{4-(t+2)^2\}$$

Note: t shifts to t+2

$$2/s^3 + e^{-2s}\mathcal{L}\{\cancel{4} - (t^2+4t+\cancel{4})\}$$

$$\boxed{2/s^3 + e^{-2s}\left(-2/s^3 - 4/s^2\right)}$$

(C) $f(t) = u(t-2)\, e^{3t+1}\, t^3$

$$\mathcal{L}\{\underline{u(t-2)}\cdot e^{3t+1}\cdot t^3\}$$

(pull out step function & shift t to t+2)

$$e^{-2s}\mathcal{L}\{e^{3(t+2)+1}\cdot(t+2)^3\}$$

$$e^{-2s}\cdot e^7\,\mathcal{L}\{\underline{e^{3t}}\cdot(t^3+6t^2+12t+8)\}$$

(exponential shift)

$$e^{-2s+7}\underbrace{\mathcal{L}\{t^3+6t^2+12t+8\}}_{\text{shift } s \mapsto s-3}$$

$$\boxed{e^{-2s+7}\cdot\left(\frac{3!}{(s-3)^4} + 6\frac{2}{(s-3)^3} + 12\frac{1}{(s-3)^2} + 8\frac{1}{(s-3)}\right)}$$

2. *(2+4+6=12pts)* Compute inverse Laplace transforms of the following functions.

(A) $F(s) = \dfrac{s-1}{s^2-2s+5}$

$$\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\}$$ (undo exponential shift)

$$e^t\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$\boxed{e^t\cos(2t)}$$

(B) $F(s) = \dfrac{2(s-1)\,e^{-2s}}{s^2-2s+2}$

$$\mathcal{L}^{-1}\left\{\frac{2(s-1)\cdot e^{-2s}}{(s-1)^2+1}\right\}$$ (undo exponential shift) $\Rightarrow$ s becomes s+1

$$e^t\,\mathcal{L}^{-1}\left\{\frac{2s\cdot e^{-2(s+1)}}{s^2+1}\right\}$$

$$e^t\cdot e^{-2}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\,e^{-2s}\right\}$$ (pull out step function) shift answer $t \mapsto t-2$

$$e^{t-2}\cdot u(t-2)\cdot \underbrace{\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}}_{\text{shift } t\mapsto t-2}$$

$$\boxed{e^{t-2}\cdot u(t-2)\cdot\cos(t-2)}$$

(C) $F(s) = \dfrac{2s^2+3s-1}{(s-1)^3}\,e^{(-3s+2)}$

$$\mathcal{L}^{-1}\left\{\frac{2s^2+3s-1}{(s-1)^3}\,e^{-3s+2}\right\}$$ (undo exponential shift) $\Rightarrow$ s becomes s+1

$$e^t\,\mathcal{L}^{-1}\left\{\frac{2(s+1)^2+3(s+1)-1}{s^3}\,e^{-3(s+1)+2}\right\}$$

$$e^t\,\mathcal{L}^{-1}\left\{\frac{2s^2+7s+6}{s^3}\,e^{-3s}\cdot e^{-1}\right\}$$

$$e^{t-1}\,\mathcal{L}^{-1}\left\{\left(2\cdot\tfrac{1}{s}+7\tfrac{1}{s^2}+6\tfrac{1}{s^3}\right)e^{-3s}\right\}$$ pull out step function $\Rightarrow$ changes t to $t-3$

$$e^{t-1}\cdot u(t-3)\cdot \underbrace{\mathcal{L}^{-1}\left\{2\cdot\tfrac{1}{s}+7\cdot\tfrac{1}{s^2}+6\cdot\tfrac{1}{s^3}\right\}}_{\text{shift } t\mapsto t-3}$$

$$\boxed{e^{t-1}\cdot u(t-3)\cdot\left(2+7(t-3)+3(t-3)^2\right)}$$

3. *(16pts)* Solve the following differential equation using Laplace transforms.

$$y''' + y' = \delta(t-1) \qquad \text{with} \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0.$$

$$\Big\downarrow \mathcal{L}$$

$$\left(s^3Y - s^2\cancel{y(0)}^{0} - s\cancel{y'(0)}^{1} - \cancel{y''(0)}^{0}\right) + \left(sY - \cancel{y(0)}^{0}\right) = e^{-s}$$

$$s^3Y - s + sY = e^{-s}$$

$$(s^3+s)Y = s + e^{-s}$$

$$Y = \frac{s}{s^3+s} + \left(\frac{1}{s^3+s}\right)e^{-s}$$

$$Y = \frac{1}{s^2+1} + \left(\frac{A}{s} + \frac{Bs+C}{s^2+1}\right)e^{-s}$$

$$Y = \frac{1}{s^2+1} + \left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-s}$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$\boxed{y = \sin(t) + u(t-1)\cdot\left(1 - \cos(t-1)\right)}$$

Partial Fractions:

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$1 = A(s^2+1) + (Bs+C)s$$

$$\left.\begin{aligned} (1)\quad & 1 = A \\ (s)\quad & 0 = C \\ (s^2)\quad & 0 = A + B \end{aligned}\right\} \Rightarrow \begin{aligned} A&=1 \\ B&=-1 \\ C&=0 \end{aligned}$$

4. *(10pts)* Use convolutions (and the convolution theorem) to write the inverse Laplace transform:

$$F(s) = \frac{1}{(s+1)^2\,(s^2+4)}$$

$$F(s) = \left(\frac{1}{s+1}\right)\cdot\left(\frac{1}{s+1}\right)\cdot\left(\frac{1}{s^2+4}\right)$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$\boxed{f(t) = e^{-t} * e^{-t} * \tfrac{1}{2}\sin(2t)}$$

Alternate solution:

$$F(s) = \left(\frac{1}{(s+1)^2}\right)\cdot\left(\frac{1}{s^2+4}\right)$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$\boxed{f(t) = te^{-t} * \tfrac{1}{2}\sin(2t)}$$

5. *(10pts)* Use separation of variables to convert the following partial differential equation of $u(x,t)$ into two separate differential equations (one involving only x and one involving only t).

$$u_{xx} + x\,u_{tt} + x\,t^2\,u_t = 0$$

Suppose $u(x,t) = X(x)\cdot T(t)$ → plug into PDE:

Then $u_x = X'\cdot T$
$u_{xx} = X''\cdot T$
$u_t = X\cdot T'$
$u_{tt} = X\cdot T''$

$$X''T + xXT'' + xt^2XT' = 0$$

$$X''T + xX(T'' + t^2T') = 0$$

$$X''T = -xX(T'' + t^2T')$$

$$\frac{X''}{xX} = -\frac{T'' + t^2T'}{T}$$

function of x function of t

$\Longrightarrow$ These must both equal some constant λ. $\Rightarrow$

$$\frac{X''}{xX} = \lambda \quad \text{and} \quad -\frac{T'' + tT'}{T} = \lambda$$

$$\boxed{X'' - \lambda x X = 0 \quad \text{and} \quad T'' + tT' + \lambda T = 0}$$

6. *(7+7=14pts)* The boundary value problem $y'' + y = 0$ for $y(0) = 0$ and $y(\pi) = c$ can have either infinitely many solutions, or no solution (depending on c).

(A) Find a value for c so that the equation has no solution.

$y'' + y = 0$ has general solution: $y = a\cos t + b\sin t$

Plug in the boundary values: $0 = y(0) = a\cdot 1 + b\cdot 0 \Rightarrow a = 0$
$c = y(\pi) = a\cdot(-1) + b\cdot 0 \Rightarrow a = -c$

This has no solution if $c \neq 0$

(B) Find a value for c so that the equation has infinitely many solutions.

If $c = 0$ then the work above gives

solutions $\begin{cases} a = 0 \\ b = \text{anything} \end{cases}$

i.e. the differential equation will have infinitely many solutions: $y = b\sin t$ for any b

7. *(10+10+6=26pts)* The following parts involve $f(x) = \begin{cases} 0, & 0 \leqslant x < 1 \\ 1, & 1 \leqslant x < 2. \end{cases}$ $\rightsquigarrow L=2$

(A) Give the Fourier **cosine** series for $f(x)$.

$$a_0 = \frac{2}{L}\int_0^L f\,dx = \frac{2}{2}\int_0^2 f\,dx = 1\cdot\left(\int_0^1 0\,dx + \int_1^2 1\,dx\right) = 0 + x\Big|_1^2 = \underline{1}$$

$$a_m = \frac{2}{L}\int_0^L f\cdot\cos\left(\frac{m\pi}{L}x\right)dx = 1\cdot\int_1^2 1\cdot\cos\left(\frac{m\pi}{2}x\right)dx$$

$$= \frac{2}{m\pi}\sin\left(\frac{m\pi}{2}x\right)\Big|_{x=1}^{x=2} = \frac{2}{m\pi}\left(\cancel{\sin(m\pi)} \to 0 - \sin\left(\frac{m\pi}{2}\right)\right)$$

$$\boxed{\text{So}\quad f(x) = \frac{1}{2} + \sum_{m=1}^{\infty} \frac{-2}{m\pi}\sin\left(\frac{m\pi}{2}\right)\cdot\cos\left(\frac{m\pi}{2}x\right)}$$

Fancy people will rewrite this with $m = 2n-1$

$$= \frac{1}{2} + \sum_{n=1}^{\infty}\frac{-2}{(2n-1)\pi}(-1)^n\cdot\cos\left(\frac{(2n-1)\pi}{2}x\right)$$

Formula: $f = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m\cos\left(\frac{m\pi}{L}x\right)$

(B) Give the Fourier **sine** series for $f(x)$.

$$b_m = \frac{2}{L}\int_0^L f\cdot\sin\left(\frac{m\pi}{L}x\right)dx = 1\cdot\int_1^2 1\cdot\sin\left(\frac{m\pi}{2}x\right)dx$$

$$= -\frac{2}{m\pi}\cos\left(\frac{m\pi}{2}x\right)\Big|_{x=1}^{x=2} = -\frac{2}{m\pi}\left(\cancel{\cos(m\pi)} \to (-1)^m - \cos\left(\frac{m\pi}{2}\right)\right)$$

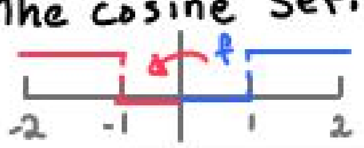
$$\text{So}\quad \boxed{f(x) = \sum_{m=1}^{\infty} -\frac{2}{m\pi}\left((-1)^m - \cos\left(\frac{m\pi}{2}\right)\right)\cdot\sin\left(\frac{m\pi}{2}x\right)}$$

Formula: $f = \sum_{m=1}^{\infty} b_m\sin\left(\frac{m\pi}{L}x\right)$

You could try to rewrite this in a fancy way but it is very complicated...

(C) What are the values of the series you computed in **(A)** and **(B)** at the point $x = -\frac{3}{2}$? (Your answers to this part should be numbers (like "3" or "8") – not series or expressions.)

The cosine series is even so it is:



$= \boxed{1}$ at $x = -3/2$

The sine series is odd so it is:

$= \boxed{-1}$ at $x = -3/2$

BONUS Use youre answers from **(A)** and **(B)** to write the Fourier series for
$f(x) = \begin{cases} 0, & -2 \leqslant x < 1 \\ 1, & 1 \leqslant x < 2. \end{cases}$

From (A), (B), and (C) it is clear that $(A) + (B) = 2f$

So $\boxed{f = \frac{1}{2}\left((A) + (B)\right)}$