If we let $\theta = 7x$, then $\theta \to 0$ as $x \to 0$, so by Equation 2 we have

$$\lim_{x\to 0}\frac{\sin 7x}{4x} = \frac{7}{4}\lim_{x\to 0}\left(\frac{\sin 7x}{7x}\right)$$

$$= \frac{7}{4}\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \frac{7}{4}\cdot 1 = \frac{7}{4}$$

**V EXAMPLE 6** Calculate $\lim_{x\to 0} x\cot x$.

SOLUTION Here we divide numerator and denominator by $x$:

$$\lim_{x\to 0} x\cot x = \lim_{x\to 0}\frac{x\cos x}{\sin x}$$

$$= \lim_{x\to 0}\frac{\cos x}{\dfrac{\sin x}{x}} = \frac{\lim_{x\to 0}\cos x}{\lim_{x\to 0}\dfrac{\sin x}{x}}$$

$$= \frac{\cos 0}{1} \qquad \text{(by the continuity of cosine and Equation 2)}$$

$$= 1$$

## 2.4 Exercises

**1–16** Differentiate.

**1.** $f(x) = 3x^2 - 2\cos x$

**2.** $f(x) = \sqrt{x}\sin x$

**3.** $f(x) = \sin x + \frac{1}{2}\cot x$

**4.** $y = 2\sec x - \csc x$

**5.** $g(t) = t^3\cos t$

**6.** $g(t) = 4\sec t + \tan t$

**7.** $y = c\cos t + t^2\sin t$

**8.** $y = u(a\cos u + b\cot u)$

**9.** $y = \dfrac{x}{2 - \tan x}$

**10.** $y = \sin\theta\cos\theta$

**11.** $f(\theta) = \dfrac{\sec\theta}{1 + \sec\theta}$

**12.** $y = \dfrac{\cos x}{1 - \sin x}$

**13.** $y = \dfrac{t\sin t}{1 + t}$

**14.** $y = \dfrac{1 - \sec x}{\tan x}$

**15.** $h(\theta) = \theta\csc\theta - \cot\theta$

**16.** $y = x^2\sin x\tan x$

**17.** Prove that $\dfrac{d}{dx}(\csc x) = -\csc x\cot x$.

**18.** Prove that $\dfrac{d}{dx}(\sec x) = \sec x\tan x$.

**19.** Prove that $\dfrac{d}{dx}(\cot x) = -\csc^2 x$.

**20.** Prove, using the definition of derivative, that if $f(x) = \cos x$, then $f'(x) = -\sin x$.

**21–24** Find an equation of the tangent line to the curve at the given point.

**21.** $y = \sec x$, $(\pi/3, 2)$

**22.** $y = (1 + x)\cos x$, $(0, 1)$

**23.** $y = \cos x - \sin x$, $(\pi, -1)$

**24.** $y = x + \tan x$, $(\pi, \pi)$

**25.** (a) Find an equation of the tangent line to the curve $y = 2x\sin x$ at the point $(\pi/2, \pi)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

**26.** (a) Find an equation of the tangent line to the curve $y = 3x + 6\cos x$ at the point $(\pi/3, \pi + 3)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

**27.** (a) If $f(x) = \sec x - x$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by graphing both $f$ and $f'$ for $|x| < \pi/2$.

**28.** (a) If $f(x) = \sqrt{x}\sin x$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by graphing both $f$ and $f'$ for $0 \leqslant x \leqslant 2\pi$.

Graphing calculator or computer required

**1.** Homework Hints available at stewartcalculus.com

**29.** If $H(\theta) = \theta \sin \theta$, find $H'(\theta)$ and $H''(\theta)$.

**30.** If $f(t) = \csc t$, find $f''(\pi/6)$.

**31.** (a) Use the Quotient Rule to differentiate the function

$$f(x) = \frac{\tan x - 1}{\sec x}$$

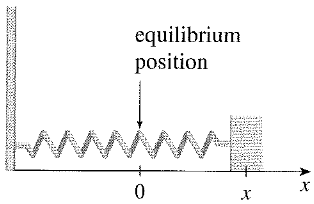
(b) Simplify the expression for $f(x)$ by writing it in terms of $\sin x$ and $\cos x$, and then find $f'(x)$.
(c) Show that your answers to parts (a) and (b) are equivalent.

**32.** Suppose $f(\pi/3) = 4$ and $f'(\pi/3) = -2$, and let $g(x) = f(x) \sin x$ and $h(x) = (\cos x)/f(x)$. Find
(a) $g'(\pi/3)$ (b) $h'(\pi/3)$

**33.** For what values of $x$ does the graph of $f(x) = x + 2 \sin x$ have a horizontal tangent?

**34.** Find the points on the curve $y = (\cos x)/(2 + \sin x)$ at which the tangent is horizontal.

**35.** A mass on a spring vibrates horizontally on a smooth level surface (see the figure). Its equation of motion is $x(t) = 8 \sin t$, where $t$ is in seconds and $x$ in centimeters.
(a) Find the velocity and acceleration at time $t$.
(b) Find the position, velocity, and acceleration of the mass at time $t = 2\pi/3$. In what direction is it moving at that time?




**36.** An elastic band is hung on a hook and a mass is hung on the lower end of the band. When the mass is pulled downward and then released, it vibrates vertically. The equation of motion is $s = 2 \cos t + 3 \sin t$, $t \geqslant 0$, where $s$ is measured in centimeters and $t$ in seconds. (Take the positive direction to be downward.)
(a) Find the velocity and acceleration at time $t$.
(b) Graph the velocity and acceleration functions.
(c) When does the mass pass through the equilibrium position for the first time?
(d) How far from its equilibrium position does the mass travel?
(e) When is the speed the greatest?

**37.** A ladder 6 m long rests against a vertical wall. Let $\theta$ be the angle between the top of the ladder and the wall and let $x$ be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does $x$ change with respect to $\theta$ when $\theta = \pi/3$?

**38.** An object with mass $m$ is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle $\theta$ with the plane, then the magnitude of the force is

$$F = \frac{\mu m g}{\mu \sin \theta + \cos \theta}$$

where $\mu$ is a constant called the *coefficient of friction*.
(a) Find the rate of change of $F$ with respect to $\theta$.
(b) When is this rate of change equal to 0?
(c) If $m = 20$ kg, $g = 9.8$ m/s$^2$, and $\mu = 0.6$, draw the graph of $F$ as a function of $\theta$ and use it to locate the value of $\theta$ for which $dF/d\theta = 0$. Is the value consistent with your answer to part (b)?

**39–48** Find the limit.

**39.** $\lim_{x \to 0} \dfrac{\sin 3x}{x}$

**40.** $\lim_{x \to 0} \dfrac{\sin 4x}{\sin 6x}$

**41.** $\lim_{t \to 0} \dfrac{\tan 6t}{\sin 2t}$

**42.** $\lim_{\theta \to 0} \dfrac{\cos \theta - 1}{\sin \theta}$

**43.** $\lim_{x \to 0} \dfrac{\sin 3x}{5x^3 - 4x}$

**44.** $\lim_{x \to 0} \dfrac{\sin 3x \sin 5x}{x^2}$

**45.** $\lim_{\theta \to 0} \dfrac{\sin \theta}{\theta + \tan \theta}$

**46.** $\lim_{x \to 0} \dfrac{\sin(x^2)}{x}$

**47.** $\lim_{x \to \pi/4} \dfrac{1 - \tan x}{\sin x - \cos x}$

**48.** $\lim_{x \to 1} \dfrac{\sin(x - 1)}{x^2 + x - 2}$

**49–50** Find the given derivative by finding the first few derivatives and observing the pattern that occurs.

**49.** $\dfrac{d^{99}}{dx^{99}} (\sin x)$

**50.** $\dfrac{d^{35}}{dx^{35}} (x \sin x)$

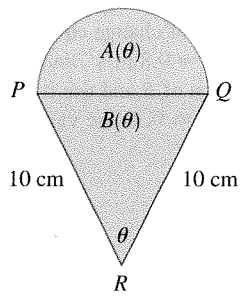
**51.** Find constants $A$ and $B$ such that the function $y = A \sin x + B \cos x$ satisfies the differential equation $y'' + y' - 2y = \sin x$.

**52.** (a) Evaluate $\lim_{x \to \infty} x \sin \dfrac{1}{x}$.
(b) Evaluate $\lim_{x \to 0} x \sin \dfrac{1}{x}$.
(c) Illustrate parts (a) and (b) by graphing $y = x \sin(1/x)$.

**53.** Differentiate each trigonometric identity to obtain a new (or familiar) identity.
(a) $\tan x = \dfrac{\sin x}{\cos x}$ (b) $\sec x = \dfrac{1}{\cos x}$
(c) $\sin x + \cos x = \dfrac{1 + \cot x}{\csc x}$

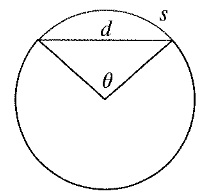
**54.** A semicircle with diameter $PQ$ sits on an isosceles triangle $PQR$ to form a region shaped like a two-dimensional ice-cream cone, as shown in the figure. If $A(\theta)$ is the area of the semicircle and $B(\theta)$ is the area of the triangle, find

$$\lim_{\theta \to 0^+} \frac{A(\theta)}{B(\theta)}$$




**55.** The figure shows a circular arc of length $s$ and a chord of length $d$, both subtended by a central angle $\theta$. Find

$$\lim_{\theta \to 0^+} \frac{s}{d}$$




**56.** Let $f(x) = \dfrac{x}{\sqrt{1 - \cos 2x}}$.

(a) Graph $f$. What type of discontinuity does it appear to have at 0?

(b) Calculate the left and right limits of $f$ at 0. Do these values confirm your answer to part (a)?

## 2.5 The Chain Rule

Suppose you are asked to differentiate the function

$$F(x) = \sqrt{x^2 + 1}$$

The differentiation formulas you learned in the previous sections of this chapter do not enable you to calculate $F'(x)$.

See Section 1.3 for a review of composite functions.

Observe that $F$ is a composite function. In fact, if we let $y = f(u) = \sqrt{u}$ and let $u = g(x) = x^2 + 1$, then we can write $y = F(x) = f(g(x))$, that is, $F = f \circ g$. We know how to differentiate both $f$ and $g$, so it would be useful to have a rule that tells us how to find the derivative of $F = f \circ g$ in terms of the derivatives of $f$ and $g$.

It turns out that the derivative of the composite function $f \circ g$ is the product of the derivatives of $f$ and $g$. This fact is one of the most important of the differentiation rules and is called the *Chain Rule*. It seems plausible if we interpret derivatives as rates of change. Regard $du/dx$ as the rate of change of $u$ with respect to $x$, $dy/du$ as the rate of change of $y$ with respect to $u$, and $dy/dx$ as the rate of change of $y$ with respect to $x$. If $u$ changes twice as fast as $x$ and $y$ changes three times as fast as $u$, then it seems reasonable that $y$ changes six times as fast as $x$, and so we expect that

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

**The Chain Rule** If $g$ is differentiable at $x$ and $f$ is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at $x$ and $F'$ is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$